New Examples of Homogeneous Einstein Metrics

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1. Introduction

A Riemannian metric is said to be *Einstein* if the Ricci curvature is a constant multiple of the metric. Given a manifold M, one can ask whether M carries an Einstein metric and, if so, how many. This fundamental question in Riemannian geometry is for the most part unsolved (cf. [Bes]). As a global PDE or a variational problem, the question is intractible. It becomes more manageable in the homogeneous setting, and so many of the known examples of compact simply connected Einstein manifolds are homogeneous. In this paper we give a technique for finding and classifying all homogeneous metrics on any given homogeneous space, including those that are not diagonal with respect to the isotropy representation. We also examine some compact simply connected homogeneous spaces G/H, where G is simple and H is closed and connected. On each space we describe all G-invariant Einstein metrics. For such spaces, the normal homogeneous Einstein metrics were classified by Wang and Ziller [WZ1]. Among the metrics we shall find, there is only one normal metric: the metric on $S^7 \times S^7$ induced by the Killing form. In fact, apart from $S^7 \times S^7$, none of our examples of homogeneous Einstein metrics is even naturally reductive.

Each of our examples has G-invariant metrics that are not diagonal with respect to the isotropy representation of H. Few examples of this type have been previously examined. Some nondiagonal examples arise as fibrations with Riemannian submersion metrics, where the base and fibre are Einstein—for example, if the base and fibre are irreducible symmetric spaces. Using this method, we can expect a product Einstein metric on each of the examples to follow. Jensen does this to find a homogeneous Einstein metric on Stiefel manifolds $V_k\mathbb{R}^n$. He restricts to a two-parameter family of diagonal $\mathrm{SO}(n)$-invariant metrics on $V_k\mathbb{R}^n$ [J2]. Using very different methods, Sagle also considers Stiefel manifolds, showing that $V_k\mathbb{R}^n$ carries at least one Einstein metric [S]. Sagle first discovered the $\mathrm{SO}(n)$-invariant Einstein metric on $V_2\mathbb{R}^n$. Neither Sagle nor Jensen observes that the homogeneous Einstein metric on $V_2\mathbb{R}^n$ is unique. More recently, Arvanitoyeorgos looks at a special family of $\mathrm{SO}(n)$-invariant metrics on $V_k\mathbb{R}^n$ [A]. None of these methods exhausts all possible homogeneous Einstein metrics.

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Our examples consist of three symmetric spaces and the unit tangent bundle of the n-sphere. We shall establish the following result.

THEOREM 1.

(1) $S^7 \times S^7 = \mathrm{Spin}(8)/\mathrm{G}_2$ *carries exactly two distinct* $\mathrm{Spin}(8)$*-invariant Einstein metrics*: *the product metric and the metric induced by the Killing form*.

(2) $S^7 \times S^6 = \mathrm{Spin}(7)/\mathrm{SU}(3)$ *carries exactly three distinct* $\mathrm{Spin}(7)$*-invariant Einstein metrics*: *the product metric and two others.*

(3) $S^7 \times G_2^+(\mathbb{R}^8) = \mathrm{Spin}(8)/\mathrm{U}(3)$ *carries exactly two distinct* $\mathrm{Spin}(8)$*-invariant Einstein metrics*: *the product metric and one other*.

(4) $V_2(\mathbb{R}^{n+1}) = \mathrm{SO}(n+1)/\mathrm{SO}(n-1)$ *carries exactly one* $\mathrm{SO}(n+1)$*-invariant Einstein metric, inherited from* $G_2^+(\mathbb{R}^{n+1})$.

The first three examples involve the geometry of the Cayley numbers and the triality principle. The last example is perhaps the simplest setting in which the space of all homogeneous metrics includes many "off-diagonal" metrics. For our analysis it was necessary to develop a scalar curvature formula that does not depend on an orthonormal, or even orthogonal, basis.

This work extends the classification of invariant Einstein metrics on compact *irreducible* symmetric spaces (cf. [DZ; Z; K]) and the characterization of left-invariant metrics on Lie groups (cf. [J1]).

We want to consider products of compact irreducible symmetric spaces, and we require that a simple Lie group act transitively. We use Oniščik's classification of simple compact Lie algebras $\mathfrak{g}$ with Lie subalgebras $\mathfrak{g}'$ and $\mathfrak{g}''$, such that $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}''$. In terms of transitive group actions, let G be the simply connected compact Lie group corresponding to $\mathfrak{g}$ and let G', G'' be Lie subgroups corresponding to $\mathfrak{g}'$, $\mathfrak{g}''$, respectively. Then $G/(G' \cap G'') = G/G' \times G/G''$. Oniščik's result gives the following list of simple groups acting on compact reducible symmetric spaces [O]:

$$\mathrm{Spin}(8)/\mathrm{G}_2 = S^7 \times S^7, \tag{1}$$

$$\mathrm{Spin}(7)/\mathrm{SU}(3) = S^7 \times S^6, \tag{2}$$

$$\mathrm{Spin}(8)/\mathrm{U}(3) = S^7 \times G_2^+(\mathbb{R}^8), \tag{3}$$

$$\mathrm{Spin}(8)/\mathrm{SO}(4) = S^7 \times G_3^+(\mathbb{R}^8), \tag{4}$$

$$\mathrm{Spin}(7)/\mathrm{U}(2) = S^7 \times G_2^+(\mathbb{R}^7), \tag{5}$$

$$\mathrm{SU}(2n)/\mathrm{Sp}(n-1) = S^{4n-1} \times \mathrm{SU}(2n)/\mathrm{Sp}(n), \tag{6}$$

$$\mathrm{SU}(2n)/\mathrm{Sp}(n-1)\mathrm{U}(1) = \mathbb{C}P^{2n-1} \times \mathrm{SU}(2n)/\mathrm{Sp}(n), \tag{7}$$

$$\mathrm{SO}(2n+2)/\mathrm{U}(n) = S^{2n+1} \times \mathrm{SO}(2n+2)/\mathrm{U}(n+1). \tag{8}$$

To find the Einstein metrics on each symmetric space, we begin by parameterizing the space of G-invariant metrics using the isotropy representation of the space, which is well known for all the foregoing examples. The second step is to express the scalar curvature as a function of these parameters. Step three is to find

M	G/H	$\dim \mathcal{M}_G$	no. Einstein
$S^7 \times S^7$	$\mathrm{Spin}(8)/\mathrm{G}_2$	2	2
$S^7 \times S^6$	$\mathrm{Spin}^+(7)/\mathrm{SU}(2)$	4	3
$S^7 \times G_2^+(\mathbb{R}^8)$	$\mathrm{Spin}(8)/\mathrm{U}(3)$	3	2
$V_2(\mathbb{R}^{n+1})$	$\mathrm{SO}(n+1)/\mathrm{SO}(n-1)$	2	1

Table 1

the critical points of the scalar curvature functional, which correspond to Einstein metrics. With the help of Maple, we were able to carry out step three for the first three of the listed spaces. Given the computational limitations, we focused on the first five examples. The last three families of products of symmetric spaces are further complicated by the variable n. We summarize our results in Table 1, where $M = G/H$ is the homogeneous space and $\mathcal{M}_G$ is the moduli space of volume one G-invariant metrics on M.

In Sections 3–7 we prove Theorem 1. In the Appendix we discuss the geometry of the G-invariant spaces $S^7 \times G_3^+(\mathbb{R}^8)$ and $S^7 \times G_2^+(\mathbb{R}^7)$ for $G = \mathrm{Spin}(8)$ and $\mathrm{Spin}(7)$, respectively. We also describe their moduli spaces of G-invariant metrics.

2. Preliminaries

A Riemannian manifold M is defined to be *G-homogeneous* if the Lie group G acts transitively on M by isometries. That is, for any p and $q \in M$, there exists an isometry φ with $\varphi(p) = q$. We write $H_p = \{\varphi \in G \mid \varphi(p) = p\}$ for the isotropy subgroup corresponding to p. Via the map $\varphi \mapsto \varphi(p)$ we identify G/H_p and M.

We say (M, g) is *Einstein* if the Ricci curvature satisfies

$$\mathrm{Ric}_p(X, Y) = \lambda(p) g_p(X, Y)$$

for some function λ, for all $p \in M$, and for all $X, Y \in T_p M$. If $\dim(M) \geq 3$ then λ must be constant, so one says that Einstein spaces have constant Ricci curvature. Assume $M = G/H$ is compact, and let $S(g)$ denote the scalar curvature of g. Einstein metrics can also be characterized as the critical points of the total scalar curvature functional

$$T(g) = \int_M S(g)\, d\,\mathrm{vol}_g$$

on the space $\mathcal{M}$ of Riemannian metrics of volume 1 [Ber; H]. If we restrict to the G-invariant metrics in $\mathcal{M}$, denoted $\mathcal{M}_G$, then $T(g) = S(g)$. Critical points of $T|_{\mathcal{M}_G}$ are precisely the G-invariant Einstein metrics of volume 1 [Bes, p. 121]. The variational characterization of Einstein metrics is essential in what follows.

Consider the underlying manifold $M = G/H$, where G is compact and H is closed. Just as every left-invariant metric on G is uniquely determined by an inner product on $\mathfrak{g} = T_e G$, every G-invariant metric on G/H is uniquely determined by an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{g}/\mathfrak{h} \cong T_{[H]}G/H$. We can identify the quotient $\mathfrak{g}/\mathfrak{h}$ with an $\mathrm{Ad}(H)$-invariant complement $\mathfrak{p}$ to $\mathfrak{h}$ in $\mathfrak{g}$. For $\mathfrak{g}$ a semisimple Lie algebra, the Killing form κ is $\mathrm{Ad}(H)$-invariant and $-\kappa$ is positive definite. We use $-\kappa$ to choose $\mathfrak{p} = \mathfrak{h}^{\perp}$. To describe the moduli space of invariant metrics on G/H, we must describe the space of all $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{p}$.

For G a compact, simple matrix group and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ on the Lie algebra level, we will take as our comparison the $\mathrm{Ad}(H)$-invariant inner product $Q(X, Y) = -\frac{1}{2}\operatorname{tr}(XY)$, which is a multiple of κ. Any other $\mathrm{Ad}(H)$-invariant inner product satisfies $\langle\cdot,\cdot\rangle = Q(L\cdot,\cdot)$, where L is a positive-definite symmetric $\mathrm{Ad}(H)$-equivariant linear map $L\colon \mathfrak{p} \to \mathfrak{p}$. We parameterize the space of $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{p}$ by parameterizing the space of possible Ls. Decompose $\mathfrak{p}$ into orthogonal $\mathrm{Ad}(H)$-irreducible subrepresentations, $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k$. It is well known that if the $\mathfrak{p}_i$ are pairwise inequivalent representations then the decomposition is unique and $\langle\cdot,\cdot\rangle = x_1 Q|_{\mathfrak{p}_1} \perp \cdots \perp x_k Q|_{\mathfrak{p}_k}$ with $x_i > 0$ for all i. That is, L is scalar on each $\mathfrak{p}_i$. In this paper we discuss examples where $\mathfrak{p}_i \simeq \mathfrak{p}_j$ for some $i \neq j$, so that the decomposition of $\mathfrak{p}_i \oplus \mathfrak{p}_j$ is not unique and $\langle \mathfrak{p}_i, \mathfrak{p}_j\rangle$ does not necessarily vanish.

To parametrize the space of Ls (positive-definite symmetric $\mathrm{Ad}(H)$-equivariant linear maps), we need a positive variable (x_i) for each irreducible representation and a parameterization of the space of $\mathrm{Ad}(H)$-equivariant maps between each pair of equivalent representations. We use Schur's lemma, but with caution. Our representations are real; thus, we must first complexify them, and the complexification of a real irreducible representation need not be irreducible. If we begin with ψ and complexify, there are three possibilities. If $\psi \otimes \mathbb{C}$ is irreducible, we say ψ is orthogonal. Otherwise, $\psi \otimes \mathbb{C} = \varphi \oplus \bar{\varphi}$. If $\varphi \not\simeq \bar{\varphi}$, we say ψ is unitary. If $\varphi \simeq \bar{\varphi}$, we say ψ is symplectic. When ψ is an orthogonal representation, the space of intertwining operators ρ, $\psi \circ \rho = \rho \circ \psi$, is 1-dimensional. When ψ is a unitary representation, the space of intertwining operators is 2-dimensional; when ψ is symplectic, the space of intertwining operators is 4-dimensional.

3. The Scalar Curvature Functional

We will need a formula for the scalar curvature functional that does not assume we have an orthonormal basis to work with; this will allow us to fix a basis and vary the metric. Assume that we have a compact homogeneous space G/H with G semisimple, and that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. We choose a Q-orthogonal decomposition of $\mathfrak{p}$ into $\mathrm{Ad}(H)$-irreducible subspaces $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$, and we take a Q-orthonormal basis for $\mathfrak{p}$: $\{X_i\}$. We first rewrite the formula found in Besse [Bes, Sec. 7.39] so that we will see plainly the result of a change of coordinates (here $\pi_{\mathfrak{p}}$ denotes projection onto $\mathfrak{p}$ and $C_i = \mathrm{ad}_{\mathfrak{g}} X_i$, the structure constants):

$$\begin{aligned} S &= -\frac{1}{4}\sum_{i,j} |[X_i, X_j]_{\mathfrak{p}}|^2 - \frac{1}{2}\sum_i \operatorname{tr}(C_i \circ C_i) \\ &= -\frac{1}{4}\sum_{i,j} Q(C_i(X_j), \pi_{\mathfrak{p}} \circ C_i(X_j)) - \frac{1}{2}\sum_i \operatorname{tr}(C_i \circ C_i) \end{aligned}$$

$$= -\frac{1}{4}\sum_{i,j} Q(X_j, C_i^t \circ \pi_{\mathfrak{p}} \circ C_i(X_j)) - \frac{1}{2}\sum_i \operatorname{tr}(C_i \circ C_i)$$
$$= -\frac{1}{4}\sum_i \operatorname{tr}(\pi_{\mathfrak{p}} \circ C_i^t \circ \pi_{\mathfrak{p}} \circ C_i) - \frac{1}{2}\sum_i \operatorname{tr}(C_i \circ C_i).$$

If we complete our basis to unity for all of $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, then

$$\pi_{\mathfrak{p}} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \quad \text{and} \quad C_i = \begin{pmatrix} 0 & \alpha_i \\ -\alpha_i^t & \gamma_i \end{pmatrix}.$$

With respect to this Q-orthonormal basis, we have

$$S = -\frac{1}{2}\sum_i \operatorname{tr}(C_i \circ C_i) - \frac{1}{4}\sum_i \operatorname{tr}(\gamma_i^t \circ \gamma_i).$$

Now any other $\mathrm{Ad}(H)$-invariant inner product can be written $\langle \cdot, \cdot \rangle = Q(g\cdot, \cdot)$ for g a symmetric positive-definite $\mathrm{Ad}(H)$-equivariant map. Suppose we change coordinates to obtain a g-orthonormal basis $\{\tilde{X}_i\}$, where $\tilde{X}_i = AX_i$. This changes the matrices of structure constants in the following way: let

$$\tilde{A} = \begin{pmatrix} \mathrm{Id}_{\dim \mathfrak{h}} & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad \tilde{C}_k = \sum_i a_{ik}\tilde{A}^{-1}C_i\tilde{A}$$

(cf. [J1, p. 1127]). The change of basis matrix A satisfies $AA^t = g^{-1}$. We can now express the scalar curvature as a function of g (where $g^{jk} = (g^{-1})_{jk}$). We show how the first sum reduces; the second sum reduces similarly.

Because

$$\tilde{C}_i \circ \tilde{C}_i = \sum_{j,k} a_{ji}(\tilde{A}^{-1}C_j\tilde{A})a_{ki}(\tilde{A}^{-1}C_k\tilde{A}) = \sum_{j,k} a_{ji}a_{ki}\tilde{A}^{-1}C_jC_k\tilde{A},$$

it follows that

$$\sum_i \operatorname{tr}(\tilde{C}_i \circ \tilde{C}_i) = \sum_{i,j,k} a_{ji}a_{ki} \operatorname{tr}(\tilde{A}^{-1}C_jC_k\tilde{A})$$
$$= \sum_{j,k}(AA^t)_{jk} \operatorname{tr}(\tilde{A}^{-1}C_jC_k\tilde{A}) = \sum_{j,k} g^{jk} \operatorname{tr}(C_jC_k).$$

Hence

$$S(g) = -\frac{1}{2}\sum_i \operatorname{tr}(\tilde{C}_i \circ \tilde{C}_i) - \frac{1}{4}\sum_i \operatorname{tr}(\tilde{\gamma}_i \circ \tilde{\gamma}_i)$$
$$= -\frac{1}{2}\sum_{j,k} g^{jk} \operatorname{tr}(C_j \circ C_k) - \frac{1}{4}\sum_{j,k} g^{jk} \operatorname{tr}(\gamma_j^t \circ g \circ \gamma_k \circ g^{-1})$$
$$= -\frac{1}{2}\sum_{j,k} g^{jk}B(X_j, X_k) - \frac{1}{4}\sum_{j,k} g^{jk} \operatorname{tr}(\gamma_j^t \circ g \circ \gamma_k \circ g^{-1}). \qquad (*)$$

We will use this scalar curvature formula in the following examples.

4. The Manifold $V_2(\mathbf{R}^{n+1})$

The Stiefel manifold $V_2(\mathbb{R}^{n+1})$ of 2-flags in Euclidean $(n+1)$-space can be written homogeneously as $V_2(\mathbb{R}^{n+1}) = \mathrm{SO}(n+1)/\mathrm{SO}(n-1)$. Although it is not a symmetric space, $V_2(\mathbb{R}^{n+1})$ inherits an $\mathrm{SO}(n+1)$-invariant Einstein metric from the Grassmannian $G_2^+(\mathbb{R}^{n+1})$ of oriented 2-planes via the following fibration:

$$S^1 \to V_2(\mathbb{R}^{n+1}) \to G_2^+(\mathbb{R}^{n+1}).$$

Consider the 1-parameter family of submersion metrics $g_t = g_B + tg_F$ $(t > 0)$ on $V_2(\mathbb{R}^{n+1})$, where the base $B = G_2^+(\mathbb{R}^{n+1})$ with the symmetric metric, and the fibre $F = S^1$. By scaling the metric in the direction of the fibre, we find one Einstein metric [Bes, Sec. 9.77]. We show that, up to scaling, this is the only $\mathrm{SO}(n+1)$-invariant Einstein metric that $V_2(\mathbb{R}^{n+1})$ carries.

An element of $V_2(\mathbb{R}^{n+1})$ is a 2-flag: a choice of a line and a 2-plane containing that line, $\mathcal{F}\colon \operatorname{span}\{v\} \subset \operatorname{span}\{v, w\}$. We may assume that v and w are orthonormal. To see that $\mathrm{SO}(n+1)$ acts transitively, we will send $\mathcal{F}_0\colon \operatorname{span}\{e_1\} \subset \operatorname{span}\{e_1, e_2\}$ (in the standard basis) to $\mathcal{F}$. We use a matrix with v as the first column vector and w as the second column vector, then fill in the rest of the columns to complete v and w to an orthonormal basis for $\mathbb{R}^{n+1}$ with the same orientation as the standard basis. The isotropy subgroup H fixing the flag $\mathcal{F}_0$ is

$$\mathrm{SO}(n-1) \cong \begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & \mathrm{SO}(n-1) \end{pmatrix} \subset \mathrm{SO}(n+1).$$

On the Lie algebra level, we have

$$\mathfrak{h} \cong \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{so}(n-1) \end{pmatrix} \subset \mathfrak{so}(n+1).$$

Choose the $\operatorname{Ad}\mathrm{SO}(n-1)$-invariant complement $\mathfrak{p} = \mathfrak{so}(n-1)^\perp$ (with respect to the inner product Q). We decompose $\mathfrak{p}$ into its irreducible subrepresentations of $\mathrm{SO}(n-1)$, obtaining $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$. Let E_{ij} denote the matrix with 1 in the ijth entry and -1 in the jith entry. Then $\mathfrak{p}_0 = \operatorname{span}\{E_{12}\}$ and $\mathfrak{p}_j = \operatorname{span}\{ E_{j,2+i} \mid 1 \le i \le n-1\}$ for $j = 1, 2$. The decomposition is not unique: $\mathfrak{p}_1 \simeq \mathfrak{p}_2 \simeq \rho_{n-1}$, the standard $(n-1)$-dimensional representation of $\mathrm{SO}(n-1)$. This is an orthogonal representation, so the space of intertwining maps is 1-dimensional and generated by the isometry $I\colon \mathfrak{p} \to \mathfrak{p}$ in the form

$$I = \begin{pmatrix} 0 & \mathrm{Id}_{n-1} \\ \mathrm{Id}_{n-1} & 0 \end{pmatrix}$$

with respect to the foregoing natural ordered basis for $\mathfrak{p}_1 \oplus \mathfrak{p}_2$. This implies that every $\operatorname{Ad}\mathrm{SO}(n+1)$-invariant inner product on $\mathfrak{p}$ is parameterized by $\langle \cdot, \cdot \rangle = Q(g\cdot, \cdot)$ for some g of the form

$$(g) = \begin{pmatrix} x_0 & 0 & 0 \\ 0 & x_1\,\mathrm{Id}_{n-1} & \lambda\,\mathrm{Id}_{n-1} \\ 0 & \lambda\,\mathrm{Id}_{n-1} & x_2\,\mathrm{Id}_{n-1} \end{pmatrix} \quad \text{for } x_0, x_1, x_2 > 0,\ \lambda \in \mathbb{R}.$$

Before we compute the scalar curvature we can simplify. We have the following Lie bracket relations:

$$[E_{12}, E_{1,2+i}] = -E_{2,2+i}, \qquad [E_{12}, E_{2,2+i}] = E_{1,2+i},$$
$$[E_{1,2+i}, E_{2,2+j}] = -\delta_{ij} E_{12},$$
$$[E_{1,2+i}, E_{1,2+j}] = [E_{2,2+i}, E_{2,2+j}] = -E_{2+i,2+j} \in \mathfrak{so}(n-1).$$

Let $N(\mathrm{SO}(n-1))$ be the normalizer of $\mathrm{SO}(n-1)$ in $\mathrm{SO}(n+1)$. Observe that $N(\mathrm{SO}(n-1))/\mathrm{SO}(n-1) \cong \mathrm{SO}(2)$ with tangent algebra $\mathfrak{p}_0$. Conjugation by any element of this $\mathrm{SO}(2)$ is a diffeomorphism preserving $\mathfrak{p}$. This gives a 1-parameter subgroup of homotheties: $g \cong g(t) = \mathrm{Ad}(\exp tE_{12}) \cdot g$. We can find a t such that $g(t)$ is diagonal, so we may assume that $\lambda = 0$.

One can check that the Ricci tensor diagonalizes with the metric g. Then we use the scalar curvature functional in terms of x_0, x_1, x_2 from [WZ2, (1.3)]:

$$S = \frac{1}{2}\sum_i \frac{d_i b_i}{x_i} - \frac{1}{4}\sum_{i,j,k} \binom{k}{i\ j} \frac{x_k}{x_i x_j}.$$

Here $d_i = \dim(\mathfrak{p}_i)$; $-\kappa|_{p_i} = b_i Q|_{\mathfrak{p}_i}$ (κ denotes the Killing form); the triple $\binom{i}{j\,k} = \sum Q([X_\alpha, X_\beta], X_\gamma)^2$ summed over $\{X_\alpha\}, \{X_\beta\}, \{X_\gamma\}$, the Q-orthonormal bases for $\mathfrak{p}_i, \mathfrak{p}_j, \mathfrak{p}_k$, respectively. We have $d_1 = d_2 = n-1$, $d_0 = 1$, and $b_i = 2(n-1)$ for $i = 0, 1, 2$. From the Lie bracket relations we see $\binom{0}{1\,2} = 1$; all other triples (except rearrangements) are zero. Thus

$$S(g) = (n-1)\left(\frac{n-1}{x_1} + \frac{n-1}{x_2} + \frac{1}{x_0}\right) - \frac{n-1}{2}\left(\frac{x_1}{x_2 x_0} + \frac{x_2}{x_1 x_0} + \frac{x_0}{x_1 x_2}\right).$$

We normalize for volume 1, $\tilde{S} = S - k(x_1^{n-1} x_2^{n-1} x_0 - 1)$, where k is the Lagrange multiplier. Critical points are solutions to the following equations:

$$x_0^2 - x_1^2 + x_2^2 - 2(n-1)x_2 x_0 = (n-1)((x_1 - x_2)^2 - x_0^2),$$
$$(x_2 - x_1)(x_1 + x_2 - (n-1)x_0) = 0.$$

Solving, we conclude that if $x_1 = x_2$ then $x_0 = 2\left(\frac{n-1}{n}\right)x_1$. This was the original submersion metric. If $x_1 \neq x_2$, there are no solutions. Thus there is exactly one $\mathrm{SO}(n+1)$-invariant Einstein metric on $V_2(\mathbb{R}^{n+1})$.

5. The Product $S^7 \times S^7$

Just as we think of S^7 as the unit sphere in $\mathbb{R}^8 \cong \mathbb{O}$ (the Octonians, or Cayley numbers), the product of two 7-spheres is a natural submanifold of $\mathbb{O} \times \mathbb{O}$. Because it is a product of symmetric spaces, $S^7 \times S^7$ is homogeneous; the simple Lie group $\mathrm{Spin}(8)$ acts transitively on $S^7 \times S^7$ with isotropy subgroup G_2. We expect at least two distinct $\mathrm{Spin}(8)$-invariant Einstein metrics: one is the product metric, and the other is induced from the Killing form [WZ1, p. 575]. We find that it carries exactly these, and no others.

We begin by describing the homogeneous presentation of $S^7 \times S^7$. Then we can determine $\mathcal{M}_{\mathrm{Spin}(8)}$, the space of invariant metrics, and consider it for Einstein metrics. We have a natural matrix group representation for Spin(8):

$$\mathrm{Spin}(8) = \{\, (A, B, C) \in \mathrm{SO}(8)^3 \mid A(x)B(y) = C(xy)\ \forall x, y \in \mathbb{O}\,\}.$$

The triality principle gives us a way to see that Spin(8) is indeed a double cover of SO(8), since a choice of $A \in \mathrm{SO}(8)$ determines the corresponding B and C, up to sign [M]. The Moufang identities give us three families of triples generating Spin(8): $(R_z, L_z R_{\bar{z}}, R_z)$, $(L_z R_{\bar{z}}, L_z, L_z)$, and $(L_z, R_z, -L_z R_{\bar{z}})$, where L_z and R_z denote (respectively) left multiplication and right multiplication by z for each $z \in \mathrm{Im}(\mathbb{O})$ with $\|z\| = 1$. We note some of the subgroups of Spin(8):

$$\mathrm{Spin}^+(7) = \{\, (A, B, C) \in \mathrm{Spin}(8) \mid B = C\,\} \quad \text{generated by } \{(L_z R_{\bar{z}}, L_z, L_z)\},$$
$$\mathrm{Spin}^-(7) = \{\, (A, B, C) \in \mathrm{Spin}(8) \mid A = C\,\} \quad \text{generated by } \{(R_z, L_z R_{\bar{z}}, R_z)\},$$
$$\mathrm{G}_2 = \{\, (A, B, C) \in \mathrm{Spin}(8) \mid A = B = C\,\} = \mathrm{Spin}^+(7) \cap \mathrm{Spin}^-(7).$$

To see that the subgroup $\mathrm{Spin}^+(7)$ is a double cover of SO(7), notice that for a triple (A, B, C) in $\mathrm{Spin}^+(7)$, $B = C$; hence $A(1) = 1$, and we think of $A \in \mathrm{SO}(7)$. Once A is determined, B and C are also determined, up to sign. A similar argument shows $\mathrm{Spin}^-(7)$ is another double cover of SO(7).

We define the action of Spin(8) on $S^7 \times S^7$ via $(A, B, C)\colon (x, y) \mapsto (Ax, By)$. To show that this action is transitive we take any point $(x, y) \in S^7 \times S^7$ and construct a map from (x, y) to $(1, 1)$. We can first find $(A, B, C)\colon (x, y) \mapsto (1, y')$ for some y', since A can be any element of SO(8). Next, we use that $\mathrm{Spin}^+(7)$ fixes the first component of $(1, *)$ and acts transitively on S^7 in the second component to know there exists an element (A', B', B') of $\mathrm{Spin}^+(7)$ mapping $(1, y') \mapsto (1, 1)$. The composition takes (x, y) to $(1, 1)$.

Next we determine the isotropy subgroup $H \subset \mathrm{Spin}(8)$ fixing the point $(1, 1)$. Just as $\mathrm{Spin}^+(7)$ fixes the first component of $(1, *)$, $\mathrm{Spin}^-(7)$ fixes the second component of $(*, 1)$. Hence $H \subset \mathrm{G}_2 = \mathrm{Spin}^+(7) \cap \mathrm{Spin}^-(7)$, the group of automorphisms of $\mathbb{O}$. Every element of G_2 takes $(1, 1)$ to itself, so $\mathrm{G}_2 \subset H$ and thus $H = \mathrm{G}_2$. This shows that $\mathrm{Spin}(8)/\mathrm{G}_2 \cong S^7 \times S^7$.

Under the double-covering homomorphism $(A, B, C) \mapsto C$ from Spin(8) to SO(8), the subgroups $\mathrm{Spin}^+(7)$ and $\mathrm{Spin}^-(7)$ are isomorphic to their images in SO(8). We use this homomorphism to identify the Lie algebras $\mathfrak{spin}(8) \cong \mathfrak{so}(8)$. If we order our basis for the Octonians in the following way: $\{1, j, \varepsilon, j\varepsilon, i, k, i\varepsilon, -k\varepsilon\}$, then

$$\mathrm{G}_2 \subset \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(7) \end{pmatrix} \subset \mathrm{SO}(8).$$

Then $\mathfrak{g}_2$ is invariant under the triality automorphism of $\mathfrak{so}(8)$, which interchanges the Lie subalgebras $\mathfrak{so}(7)$, $\mathfrak{spin}^+(7)$, and $\mathfrak{spin}^-(7)$.

We decompose $\mathfrak{so}(8)$ into $\mathfrak{g}_2 \oplus \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{g}_2^\perp$ with respect to the inner product Q. Using any of the three following fibrations, we see that $\mathfrak{p}$ is the sum of two equivalent copies of the standard orthogonal 7-dimensional representation of G_2, denoted φ:

$$S^7 = \mathrm{Spin}^{\pm}(7)/\mathrm{G}_2 \to \mathrm{Spin}(8)/\mathrm{G}_2 \to \mathrm{Spin}(8)/\mathrm{Spin}^{\pm}(7) = S^7,$$
$$\mathbb{R}P^7 = \mathrm{SO}(7)/\mathrm{G}_2 \to \mathrm{SO}(8)/\mathrm{G}_2 \to \mathrm{SO}(8)/\mathrm{SO}(7) = S^7.$$

We have three natural ways to decompose $\mathfrak{p}$: We can choose $\mathfrak{p}_1$ so that $\mathfrak{g}_2 \oplus \mathfrak{p}_1$ is $\mathfrak{spin}^+(7)$, $\mathfrak{spin}^-(7)$, or $\mathfrak{so}(7)$, then set $\mathfrak{p}_2$ to be the Q-orthogonal complement to $\mathfrak{p}_1$ in $\mathfrak{p}$. We choose $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ so that $\mathfrak{spin}^+(7) = \mathfrak{g}_2 \oplus \mathfrak{p}_1$.

We now describe $\mathcal{M}_{\mathrm{Spin}(8)}$ for $S^7 \times S^7$. The representation φ is orthogonal; hence, by Schur's lemma, the space of intertwining maps is 1-dimensional. For an appropriately ordered Q-orthonormal basis for $\mathfrak{p}_1 \oplus \mathfrak{p}_2$, every $\mathrm{Ad}(\mathrm{G}_2)$-invariant inner product on $\mathfrak{p}$ is represented by a linear map of the form

$$g = \begin{pmatrix} x_1 \,\mathrm{Id}_7 & \lambda \,\mathrm{Id}_7 \\ \lambda \,\mathrm{Id}_7 & x_2 \,\mathrm{Id}_7 \end{pmatrix},$$

where $x_1, x_2 > 0$ and λ is any real number ($g = (g_{ij})$ is the metric). That is, $\langle \cdot, \cdot \rangle|_{\mathfrak{p}_i} = x_i Q|_{\mathfrak{p}_i}$ and $\langle \mathfrak{p}_1, \mathfrak{p}_2 \rangle = \lambda Q(J\mathfrak{p}_1, \mathfrak{p}_2)$ for J an isometry

$$J = \begin{pmatrix} 0 & \mathrm{Id}_7 \\ \mathrm{Id}_7 & 0 \end{pmatrix}.$$

Using $(*)$ and Maple, we obtain the scalar curvature functional in terms of entries g_{ij}:

$$S(g) = -\frac{7(x_1^3 - 12x_1^2x_2 - 9x_1x_2^2 + 6x_1\lambda^2 + 18x_2\lambda^2)}{2(x_1x_2 - \lambda^2)^2}.$$

We normalize to restrict to volume 1, $\tilde{S} = S - k(x_1x_2 - \lambda^2)^7$, where k is the Lagrange multiplier. Then we can solve for the critical points (again using Maple) as follows:

$$\begin{aligned}
\frac{\partial \tilde{S}}{\partial x_1} &= \frac{-7(18x_1x_2\lambda^2 - 6\lambda^4 + 9x_1x_2^3 - 27x_2^2\lambda^2 + x_1^3x_2 - 3x_1^2\lambda^2)}{2(x_1x_2 - \lambda^2)^3} \\
&\quad - 7kx_2(x_1x_2 - \lambda^2)^6, \\
\frac{\partial \tilde{S}}{\partial x_2} &= \frac{7(-6x_1^3x_2 + 9\lambda^4 + x_1^4)}{(x_1x_2 - \lambda^2)^3} - 7kx_1(x_1x_2 - \lambda^2)^6, \\
\frac{\partial \tilde{S}}{\partial \lambda} &= \frac{-14\lambda(-9x_1^2x_2 + 3x_1\lambda^2 + 9x_2\lambda^2 + x_1^3)}{(x_1x_2 - \lambda^2)^3} - 14k\lambda(x_1x_2 - \lambda^2)^6.
\end{aligned}$$

We find three solutions:

$$x_1 = x_2, \qquad \lambda = 0; \tag{9}$$
$$x_1 = 3x_2, \qquad \lambda = 0; \tag{10}$$
$$x_1 = \frac{3}{5}x_2, \qquad \lambda = \pm\frac{1}{\sqrt{3}}x_1. \tag{11}$$

The first solution is the metric induced by the Killing form (recall that Q is a multiple of the Killing form). We show that the third solution is a pair of metrics that is homothetic to the product metric and in which the tangent spaces to $S^7 \times \{1\}$

and $\{1\} \times S^7$ are orthogonal. The tangent space to $S^7 \times \{1\}$ is $\mathfrak{p}_2^*$, where $\mathfrak{p}_2^*$ denotes the (Q-orthogonal) complement to $\mathfrak{g}_2$ in $\mathfrak{spin}^-(7)$; the tangent space to $\{1\} \times S^7$ is $\mathfrak{p}_1$. With respect to Q, the tangent spaces to the two spheres meet at angle $\pi/3$, so we expect that the product metric will be a nondiagonal solution. We can write $\mathfrak{so}(8) = \mathfrak{g}_2 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2^*$. The product metric is

$$\langle\langle \mathfrak{p}_1, \mathfrak{p}_1 \rangle\rangle = x_1^* Q(\mathfrak{p}_1, \mathfrak{p}_1), \qquad \langle\langle \mathfrak{p}_2^*, \mathfrak{p}_2^* \rangle\rangle = x_2^* Q(\mathfrak{p}_2^*, \mathfrak{p}_2^*),$$
$$\langle\langle \mathfrak{p}_1, \mathfrak{p}_2^* \rangle\rangle = 0, \quad \text{and} \quad x_1^* = x_2^*.$$

When we project the subspace $\mathfrak{p}_2^*$ to $\mathfrak{p}_1 \oplus \mathfrak{p}_2$, we see that the product metric indeed corresponds to the nondiagonal metric. Here is a typical Q-unit vector in $\mathfrak{p}_2^*$, decomposed with respect to $\mathfrak{p}_1 \oplus \mathfrak{p}_2$:

$$\begin{aligned}\frac{1}{2\sqrt{3}}(3E_{18} - E_{27} + E_{36} - E_{45}) = {} & \frac{1}{4\sqrt{3}}(3E_{18} + E_{27} - E_{36} + E_{45}) \\ & + \frac{\sqrt{3}}{4}(E_{18} - E_{27} + E_{36} - E_{45}).\end{aligned}$$

Whereas $x_1^* = x_1$ (since $\mathfrak{p}_1$ is projected to itself), we have $x_2^* = x_1/4 + \sqrt{3}\lambda/2 + 3x_2/4$. The equality $x_1^* = x_2^*$ induces $x_1 = x_1/4 + \sqrt{3}\lambda/2 + 3x_2/4$, which simplifies to $x_1 = x_2 + 2\lambda/\sqrt{3}$. This is our third metric exactly.

The second metric is not a new metric but rather the product metric in an unexpected form; it is conjugated by $R(\pi/3)$, the map that rotates by $\pi/3$:

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3\lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \lambda\sqrt{3} & \lambda \\ \lambda & \frac{5\lambda}{\sqrt{3}} \end{pmatrix}.$$

This rotation is the action of the triality automorphism: a homothety of our space. Notice that conjugating a second time by $R(\pi/3)$ gives us $\left(\begin{smallmatrix} \lambda\sqrt{3} & -\lambda \\ -\lambda & 5\lambda/\sqrt{3} \end{smallmatrix}\right)$, which shows that the third solution is a pair of metrics homothetic to the product metric. Thus there are exactly two distinct Spin(8)-invariant Einstein metrics on $S^7 \times S^7$.

6. The Product $S^6 \times S^7$

Our next product of symmetric spaces is $S^6 \times S^7$, the unit spheres in $\mathrm{Im}(\mathbb{O}) \times \mathbb{O}$, where $\mathrm{Im}(\mathbb{O})$ denotes the purely imaginary Octonians. We will show that $\mathrm{Spin}^+(7)$ acts transitively with isotropy subgroup SU(3). The product metric is one invariant Einstein metric; we find there are exactly two others.

We know $\mathrm{Spin}^+(7)$ from the previous example, and we describe two subgroups of $\mathrm{Spin}^+(7)$:

$$\begin{aligned} \mathrm{Spin}^+(7) &= \{\, (A, B, B) \in \mathrm{SO}(8)^3 \mid A(x)B(y) = B(xy)\ \forall x, y \in \mathbb{O} \,\}, \\ \mathrm{G}_2 &= \{\, (A, B, B) \in \mathrm{Spin}^+(7) \mid A = B \,\}, \\ \mathrm{SU}(4) &= \{\, (A, B, B) \in \mathrm{Spin}^+(7) \mid A(i) = i \,\}. \end{aligned}$$

We begin by showing how $\mathrm{Spin}^+(7)$ acts on $S^6 \times S^7$. For any point (x, y) in $S^6 \times S^7$, the action is $(A, B, B)\colon (x, y) \mapsto (Ax, By)$. To see that $\mathrm{Spin}^+(7)$ acts transitively, we map (x, y) to $(i, 1)$. We know SU(4) acts transitively on S^7, so we can find a map $(A, B, B) \in \mathrm{SU}(4)$ such that $(Ax, By) = (x', 1)$ for some $x' \in S^6$. Note that the definition of $\mathrm{Spin}^+(7)$ implies the first component of $(1, *)$ is fixed, hence $x \in \mathrm{Im}(\mathbb{O})$ implies $x' \in \mathrm{Im}(\mathbb{O})$. Next we use that G_2 acts transitively on $S^6 \subset \mathrm{Im}(\mathbb{O})$, leaving 1 fixed, to find a map $(A', A', A') \in \mathrm{G}_2$ satisfying $(A', A', A')(x', 1) = (i, 1)$. The composition takes (x, y) to $(i, 1)$.

We next determine the isotropy subgroup H fixing $(i, 1)$. For any element of H, we have $(A(i), B(1)) = (i, 1)$. We see $A(i) = i$ implies $(A, B, B) \in \mathrm{SU}(4)$, so $H \subset \mathrm{SU}(4)$. Then $B(1) = 1$ implies $A(x)B(1) = B(x)$ for all x. Hence $A = B$, and we have $H \subset \mathrm{G}_2$. One knows that any subgroup of G_2 fixing an imaginary Octonian is isomorphic to SU(3), and in our case, $H \cong \mathrm{SU}(3)$ is the subgroup of SU(4) fixing the complex line spanned by $\{1, i\}$.

We identify $\mathrm{Spin}^+(7)$ with its image under the double-covering homomorphism from Spin(8) to SO(8) mapping (A, B, B) to B. Giving the Octonians our usual ordering, $\{1, j, \varepsilon, j\varepsilon, i, k, i\varepsilon, -k\varepsilon\}$, we know that G_2 is a subgroup of

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(7) \end{pmatrix} \subset \mathrm{SO}(8),$$

and that $\mathrm{SU}(4) \subset \mathrm{SO}(8)$ is embedded to respect the complex structure

$$L_i : X + iY \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

On the Lie algebra level, we can take $\mathfrak{p}$ to be the orthogonal complement to $\mathfrak{su}(3)$ in $\mathfrak{spin}^+(7)$ with respect to Q, so that $\mathfrak{spin}^+(7) = \mathfrak{su}(3) \oplus \mathfrak{p}$. We have three fibrations of our product space, which we use to decompose $\mathfrak{p}$:

$$S^6 = \mathrm{G}_2/\mathrm{SU}(3) \to \mathrm{Spin}^+(7)/\mathrm{SU}(3) \to \mathrm{Spin}^+(7)/\mathrm{G}_2 = S^7,$$

$$S^7 = \mathrm{SU}(4)/\mathrm{SU}(3) \to \mathrm{Spin}^+(7)/\mathrm{SU}(3) \to \mathrm{Spin}^+(7)/\mathrm{SU}(4) = S^6,$$

$$S^1 = \mathrm{U}(3)/\mathrm{SU}(3) \to \mathrm{Spin}^+(7)/\mathrm{SU}(3) \to \mathrm{Spin}^+(7)/\,\mathrm{U}(3).$$

In the first fibration, the isotropy representation of the fibre is $[\mu_3]_{\mathbb{R}}$, where μ_k is the standard k-dimensional complex representation of $\mathrm{SU}(k)$. The isotropy representation of the base is φ, the orthogonal representation of G_2 in SO(7); we restrict it to SU(3), $\varphi|_{\mathrm{SU}(3)} = [\mu_3]_{\mathbb{R}} \oplus \mathrm{Id}$. In the second fibration, the isotropy representation of the fibre is $[\mu_3 \oplus \mathrm{Id}]_{\mathbb{R}} = [\mu_3]_{\mathbb{R}} \oplus \mathrm{Id}$, the sum of two irreducible subrepresentations. The isotropy representation of the base space is $[\mu_4]_{\mathbb{R}}$, and $[\mu_4]_{\mathbb{R}}|_{\mathrm{SU}(3)} = [\mu_3]_{\mathbb{R}}$. In the third fibration, the isotropy representation of the fibre is trivial. The base space is the symmetric space $\mathrm{SO}(8)/\,\mathrm{U}(4)$ (see [K]), with isotropy representation $[\mu_3]_{\mathbb{R}} \oplus [\wedge^2\mu_3]_{\mathbb{R}}$. When we restrict from U(3) to SU(3), we find $\wedge^2\mu_3 \cong \mu_3$. Thus we conclude $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_0$, with two equivalent representations $\mathfrak{p}_1 \simeq \mathfrak{p}_2 \simeq [\mu_3]_{\mathbb{R}}$ and $\mathfrak{p}_0 = \mathrm{Id}$, a trivial, 1-dimensional representation. The decomposition of $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ is not unique; we will choose our decomposition so that $\mathfrak{su}(3) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_0 = \mathfrak{su}(4)$.

We would like to consider all SU(3)-invariant inner products on $\mathfrak{p}$. We know that any such inner product satisfies $\langle \cdot, \cdot \rangle = Q(g, \cdot)$ for any Ad SU(3)-equivariant symmetric positive-definite linear operator $g\colon \mathfrak{p} \to \mathfrak{p}$. We use Schur's lemma to determine the possible entries of g. Since $[\mu_3]_{\mathbb{R}}$ is a unitary representation, $[\mu_3]_{\mathbb{R}} \otimes \mathbb{C} = \mu_3 \oplus \mu_3^*$, the sum of two inequivalent irreducible complex representations. Thus the space of intertwining maps from $\mathfrak{p}_1$ to $\mathfrak{p}_2$ is 2-dimensional. We take a Q-orthonormal basis for $\mathfrak{p}$ respecting the decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_0$ and the complex structure on $\mathfrak{p}_1 \oplus \mathfrak{p}_2$, so that intertwining maps are linear combinations of

$$J_1 = \begin{pmatrix} 0 & \mathrm{Id}_6 \\ \mathrm{Id}_6 & 0 \end{pmatrix} \quad \text{and} \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id}_3 \\ 0 & 0 & -\mathrm{Id}_3 & 0 \\ 0 & -\mathrm{Id}_3 & 0 & 0 \\ \mathrm{Id}_3 & 0 & 0 & 0 \end{pmatrix}$$

on $\mathfrak{p}_1 \oplus \mathfrak{p}_2$. Any SU(3)-invariant inner product on $\mathfrak{p}$ corresponds to a g of the form

$$g = \begin{pmatrix} x_1\,\mathrm{Id}_3 & 0 & \lambda_1\,\mathrm{Id}_3 & \lambda_2\,\mathrm{Id}_3 & 0 \\ 0 & x_1\,\mathrm{Id}_3 & -\lambda_2\,\mathrm{Id}_3 & \lambda_1\,\mathrm{Id}_3 & 0 \\ \lambda_1\,\mathrm{Id}_3 & -\lambda_2\,\mathrm{Id}_3 & x_2\,\mathrm{Id}_3 & 0 & 0 \\ \lambda_2\,\mathrm{Id}_3 & \lambda_1\,\mathrm{Id}_3 & 0 & x_2\,\mathrm{Id}_3 & 0 \\ 0 & 0 & 0 & 0 & x_3 \end{pmatrix},$$
$$x_1, x_2, x_3 > 0 \ \text{ and } \ \lambda_1, \lambda_2 \in \mathbb{R}.$$

Via $(*)$, we obtain the scalar curvature equation for $S^7 \times S^6$:

$$S(g) = \frac{-6x_1^3x_3 + x_1^2(60x_2x_3 - x_3^2) + 6x_1(8x_2^2x_3 - 3(x_2+2x_3)\lambda_1^2)}{2(x_1x_2 - \lambda_1^2)^2x_3}$$
$$+ \frac{\begin{array}{l} 18(\lambda_1^2 - 4x_2x_3)\lambda_1^2 + 4(\lambda_1^2 - x_2^2)x_3^2 \\ \quad +\lambda_2^2(2x_3^2 + 9(\lambda_2^2 + 2\lambda_1^2 - x_1x_2 - 2x_1x_3 - 4x_2x_3)) \end{array}}{2(x_1x_2 - \lambda_1^2)^2x_3}.$$

Before searching for critical points, we can simplify. The normalizer of SU(3) in $\mathrm{Spin}^+(7)$ is U(3), with Lie algebra $\mathfrak{su}(3) \oplus \mathfrak{p}_0$ (since $\mathfrak{su}(3)$ and $\mathfrak{p}_0$ commute). Thus, conjugation by any element of $N(\mathrm{SU}(3))/\mathrm{SU}(3) = \mathrm{U}(1)$ is a diffeomorphism preserving $\mathfrak{p}$. This gives us a 1-parameter family of homotheties of $S^6 \times S^7$. We use these homotheties to reduce the number of parameters of g. If Z is our Q-unit vector spanning $\mathfrak{p}_0$ and if $\{X_i\}_{i=1}^{12}$ is a basis for $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ as described before, we have

$$[Z, X_i] = \begin{cases} (2/\sqrt{3})X_{i+3} & \text{for } 1 \le i \le 3, \\ -(2/\sqrt{3})X_{i-3} & \text{for } 4 \le i \le 6; \end{cases}$$
$$[Z, X_j] = \begin{cases} -(1/\sqrt{3})X_{j+3} & \text{for } 7 \le j \le 9, \\ (1/\sqrt{3})X_{j-3} & \text{for } 10 \le j \le 12. \end{cases}$$

Setting $\tilde{g}(t) = \mathrm{Ad}(\exp tZ) \cdot g$, we see that g and $\tilde{g}(t)$ are homothetic; choose t so that $\tilde{\lambda}_2 = 0$. Next, consider the normalized scalar curvature functional (i.e., restricted to those metrics of volume 1): $\tilde{S} = S - k(x_1x_2 - \lambda_1^2 - \lambda_2^2)^6x_3$, where k is the Lagrange multiplier. Notice that $\tilde{S}$ is a function of λ_2^2, so in setting $\lambda_2 = 0$ we do not miss any critical points. Using Maple, we obtain the following results:

$$\begin{aligned}
x_2 &= \frac{13}{6}x_1, & x_2 &= \frac{1}{2}x_1, & x_2 &= \zeta x_1,\\
x_3 &= \frac{3}{2}x_1, & x_3 &= \frac{9}{7}x_1, & x_3 &= \left(\frac{-72\zeta^2+120\zeta-33}{7}\right)x_1,\\
\lambda_1 &= \frac{1}{\sqrt{2}}x_1, & \lambda_1 &= 0, & \lambda_1 &= 0,
\end{aligned}$$

where ζ is a real, positive solution to $24\zeta^3 - 28\zeta^2 + 5\zeta - 5 = 0$. (This gives $x_2 \sim 1.144x_1$ and $x_3 \sim 1.437x_1$.)

We find that the first metric is the product metric: we show that it is the only metric of the three with the symmetric metric on S^7. Since $S^7 = \mathrm{SU}(4)/\mathrm{SU}(3)$ is not a symmetric pair, we must project to determine what the symmetric metric is. Recall we chose $\mathfrak{p}_1$ so that $\mathfrak{su}(4) = \mathfrak{su}(3) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_0$; we will project a typical element in $\mathfrak{p}_1$ and a typical element in $\mathfrak{p}_0$ to $\tilde{\mathfrak{p}}$, the Q-orthogonal complement to $\mathfrak{so}(7)$ in $\mathfrak{so}(8)$. In $\mathfrak{p}_1$, $(1/\sqrt{2})(E_{12} + E_{56}) \mapsto (1/\sqrt{2})E_{12}$; in $\mathfrak{p}_0$, $(1/2\sqrt{3})(3E_{15} - E_{26} - E_{37} - E_{48}) \mapsto (3/2\sqrt{3})E_{15}$. Thus $x_1 \mapsto \frac{1}{2}$ and $x_3 \mapsto \frac{3}{4}$, so the symmetric metric satisfies $\frac{3}{2}x_1 = x_3$; this is the first metric exactly.

We show the second metric is a fibration metric, coming from the fibration

$$S^1 = \mathrm{U}(3)/\mathrm{SU}(3) \to \mathrm{Spin}^+(7)/\mathrm{SU}(3) \to \mathrm{Spin}^+(7)/\,\mathrm{U}(3).$$

Consider the 1-parameter family of metrics $g_t = g_B + tg_F$ (for $t > 0$) obtained by scaling in the direction of the fibre and keeping the metric fixed in the horizontal directions. Recall that $B = \mathrm{Spin}^+(7)/\,\mathrm{U}(3) \cong \mathrm{SO}(8)/\,\mathrm{U}(4)$. There are two $\mathrm{Spin}^+(7)$-invariant Einstein metrics on the base space; we will show that the symmetric metric induces an Einstein metric on $S^6 \times S^7$ and the other does not.

The symmetric metric satisfies $x_2 = \frac{1}{2}x_1$; for X a unit vector in $\mathfrak{p}_1$, for Y a unit vector in $\mathfrak{p}_2$, and for A the O'Neill tensor (of our Riemannian submersion),

$$|A_X|^2 = \frac{1}{3x_1^2} \quad \text{and} \quad |A_Y|^2 = \frac{1}{12x_2^2} = \frac{4}{12x_1^2} = \frac{1}{3x_1^2}.$$

Thus the O'Neill tensor is a constant multiple of Q. A constant O'Neill tensor implies there is an Einstein metric in the 1-parameter family, g_t. (Since the fibre is flat, the Einstein metric is unique.) Proposition 9.70 in [Bes, p. 253] implies it occurs when $t = \frac{9}{7}x_1$, which is exactly our second metric.

It is reasonable to ask why the other $\mathrm{Spin}^+(7)$-invariant Einstein metric on $\mathrm{Spin}^+(7)/\,\mathrm{U}(3)$ does not induce an Einstein metric on $S^6 \times S^7$. The answer lies in the O'Neill tensor. This time we substitute $x_2 = \frac{3}{4}x_1$:

$$|A_X|^2 = \frac{1}{3x_1^2} \quad \text{and} \quad |A_Y|^2 = \frac{1}{12x_2^2} = \frac{4}{27x_1^2};$$

we do not obtain a constant multiple of Q. Hence, by [Bes, Prop. 9.70], there can be no Einstein metrics arising from this fibration.

Our three metrics are not isometric. We compare a scale-invariant constant and show that the constants, and thus the metrics, are distinct. Our constant is

$(S)^{13/2}(V)^{1/2}$, where S is the scalar curvature and V is the volume for each metric. The first metric yields $(S)^{13/2}(V)^{1/2} \cong 1.245982 \times 10^{11}$. The second metric yields $(S)^{13/2}(V)^{1/2} \cong 1.0350064 \times 10^{11}$; the third metric yields $(S)^{13/2}(V)^{1/2} \cong 9.308607 \times 10^{10}$.

7. The Product $S^7 \times G_2^+(\mathbf{R}^8)$

The next product of symmetric spaces we consider is $S^7 \times G_2^+(\mathbb{R}^8)$, the product of the 7-sphere with the Grassmannian of oriented 2-planes in $\mathbb{R}^8$. We can write this space homogeneously as $\mathrm{Spin}(8)/\mathrm{U}(3)$. We know the product metric is one homogeneous Einstein metric. We will show that there is exactly one other $\mathrm{Spin}(8)$-invariant Einstein metric on $S^7 \times G_2^+(\mathbb{R}^8)$. We again identify $\mathbb{R}^8 \cong \mathbb{O}$.

Recall that

$$\mathrm{Spin}(8) = \{ (A, B, C) \in \mathrm{SO}(8)^3 \mid A(x)B(y) = C(xy)\ \forall x, y \in \mathbb{O} \}.$$

For $(x, P) \in S^7(1) \times G_2^+(\mathbb{R}^8)$ the action is $(A, B, C)\colon (x, P) \mapsto (A(x), B(P))$. We show that this action is transitive and find the isotropy subgroup; then we can describe the space of homogeneous metrics on $S^7 \times G_2^+(\mathbb{R}^8)$. To see that the action is transitive, we take any element (x, P) in $S^7(1) \times G_2^+(\mathbb{R}^8)$ and then construct a map in $\mathrm{Spin}(8)$ taking (x, P) to $(1, P_0)$, where P_0 is the oriented 2-plane $\mathrm{span}\{1, i\}$. We will use our knowledge of the subgroups of $\mathrm{Spin}(8)$. Since $\mathrm{Spin}^-(7)$ acts transitively on $S^7 \times \{1\}$, there is an element (A, B, A) in $\mathrm{Spin}^-(7)$ mapping (x, P) to $(1, P')$. Similarly since $\mathrm{Spin}^+(7)$ acts transitively on $\{1\} \times G_3^+(\mathbb{R}^8)$, $\mathrm{Spin}^+(7)$ acts transitively on $\{1\} \times G_2^+(\mathbb{R}^8)$. Hence there is a map in $\mathrm{Spin}^+(7)$ taking $(1, P')$ to $(1, P_0)$. Their composition yields $(x, P) \mapsto (1, P') \mapsto (1, P_0)$.

Next we show that the isotropy subgroup H of $\mathrm{Spin}(8)$ fixing $(1, P_0)$ is $\mathrm{U}(3) \cong S(\mathrm{U}(1)\mathrm{U}(3)) \subset \mathrm{SU}(4) \subset \mathrm{Spin}^+(7)$. For $(A, B, C) \in \mathrm{Spin}(8)$, $A(1) = 1$ implies that $B = C$ and hence $H \subset \mathrm{Spin}^+(7)$. Then $B(P_0) = P_0$ means that, for some angle θ, $B(1) = e^{i\theta}$ and $B(i) = ie^{i\theta}$; hence $A(i) = i$ and $H \subset \mathrm{SU}(4)$. In fact, we have shown $H \subset S(\mathrm{U}(1)\mathrm{U}(3))$. By a dimension count, $H = S(\mathrm{U}(1)\mathrm{U}(3))$.

As in the previous examples we identify $\mathfrak{spin}(8)$ with $\mathfrak{so}(8)$ via the differential of the map taking (A, B, C) to C. On the Lie algebra level we have $\mathfrak{so}(8) = \mathfrak{u}(3) \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{u}(3)$ with respect to our usual comparison metric $Q(X, Y) = -\frac{1}{2}\,\mathrm{tr}(XY)$. We have three fibrations of our space; using them, we decompose $\mathfrak{p}$ into its irreducible representations of $\mathfrak{u}(3)$:

$$\mathbb{C}P^3 \cong \mathrm{SU}(4)/\mathrm{U}(3) \to \mathrm{Spin}(8)/\mathrm{U}(3) \to \mathrm{Spin}(8)/\mathrm{SU}(4) \cong V_2(\mathbb{R}^8),$$

$$G_2^+(\mathbb{R}^8) \cong \mathrm{Spin}^+(7)/\mathrm{U}(3) \to \mathrm{Spin}(8)/\mathrm{U}(3) \to \mathrm{Spin}(8)/\mathrm{Spin}^+(7) \cong S^7,$$

$$S^7 \cong \mathrm{U}(4)/\mathrm{U}(3) \to \mathrm{Spin}(8)/\mathrm{U}(3) \to \mathrm{Spin}(8)/\mathrm{U}(4) \cong G_2^+(\mathbb{R}^8).$$

Let ρ_k denote the standard representation of $\mathrm{SO}(k)$, and let μ_k denote the standard representation of $\mathrm{U}(k)$. In the first fibration, the fibre is an irreducible symmetric space with isotropy representation $\mathfrak{p}_1 = [\mu_1 \hat{\otimes} \mu_3]_{\mathbb{R}}$. The base space is isomorphic to $\mathrm{SO}(8)/\mathrm{SO}(6)$; we know that the isotropy representation of the base is $\rho_6 \oplus \rho_6 \oplus \mathrm{Id}$. When we restrict this to $\mathrm{U}(3)$, we obtain $\mathfrak{p}_2 \oplus \mathfrak{p}_3 \oplus \mathfrak{p}_0 = [\mu_3]_{\mathbb{R}} \oplus [\mu_3]_{\mathbb{R}} \oplus \mathrm{Id}$.

In the second fibration, although the fibre is an irreducible symmetric space, $\mathrm{Spin}^+(7)$ is not the full isometry group. Hence the isotropy representation is reducible: via $\mathrm{U}(3) \subset \mathrm{SU}(4) \subset \mathrm{Spin}^+(7)$ the isotropy representation is $\mathfrak{p}_1 \oplus \mathfrak{p}_2 = [\mu_1 \hat{\otimes} \mu_3]_{\mathbb{R}} \oplus [\mu_3]_{\mathbb{R}}$. The base space, $\mathrm{SO}(8)/\mathrm{SO}(7)$, is also a symmetric space; its isotropy representation is ρ_7. When we restrict ρ_7 to $\mathrm{U}(3) \subset \mathrm{SO}(6) \subset \mathrm{SO}(7)$, we get $\mathfrak{p}_3 \oplus \mathfrak{p}_0 = [\mu_3]_{\mathbb{R}} \oplus \mathrm{Id}$.

In the third fibration, the fibre is symmetric but not a symmetric pair; the isotropy representation is $\mathfrak{p}_1 \oplus \mathfrak{p}_0 = [\mu_1 \hat{\otimes} \mu_3]_{\mathbb{R}} \oplus \mathrm{Id}$. The base space is isomorphic to $\mathrm{SO}(8)/\mathrm{SO}(2)\mathrm{SO}(6)$, an irreducible symmetric space with isotropy representation $\rho_2 \otimes \rho_6$; when we restrict the action to $\mathrm{U}(3)$ this gives $\mathfrak{p}_2 \oplus \mathfrak{p}_3 = [\mu_3]_{\mathbb{R}} \oplus [\mu_3]_{\mathbb{R}}$. We conclude that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \oplus \mathfrak{p}_0$. This decomposition is not unique, since $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are equivalent representations of $\mathfrak{u}(3)$. We choose the decomposition to be Q-orthogonal so that $\mathfrak{su}(4) = \mathfrak{u}(3) \oplus \mathfrak{p}_1$, $\mathfrak{u}(4) = \mathfrak{u}(3) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_0$, and $\mathfrak{spin}^+(7) = \mathfrak{u}(3) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

Any $\mathrm{U}(3)$-invariant inner product $\langle \cdot, \cdot \rangle$ must satisfy $\langle \cdot, \cdot \rangle = Q(g \cdot, \cdot)$, where g is an $\mathrm{Ad}\,\mathrm{U}(3)$-equivariant positive-definite symmetric linear operator. Since $\mathfrak{p}_2 \cong \mathfrak{p}_3 \cong [\mu_3]_{\mathbb{R}}$ is a unitary representation, we have two dimensions of intertwining maps. Take a Q-orthonormal basis, ordered to respect the complex structure L_i on $\mathfrak{p}_2 \oplus \mathfrak{p}_3$. Any intertwining map is a linear combination of

$$I = \begin{pmatrix} 0 & \mathrm{Id}_6 \\ \mathrm{Id}_6 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 0 & 0 & \mathrm{Id}_3 \\ 0 & 0 & -\mathrm{Id}_3 & 0 \\ 0 & -\mathrm{Id}_3 & 0 & 0 \\ \mathrm{Id}_3 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$g = \begin{pmatrix} x_1 \mathrm{Id}_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 \mathrm{Id}_3 & 0 & \lambda_1 \mathrm{Id}_3 & \lambda_2 \mathrm{Id}_3 & 0 \\ 0 & 0 & x_2 \mathrm{Id}_3 & -\lambda_2 \mathrm{Id}_3 & \lambda_1 \mathrm{Id}_3 & 0 \\ 0 & \lambda_1 \mathrm{Id}_3 & -\lambda_2 \mathrm{Id}_3 & x_3 \mathrm{Id}_3 & 0 & 0 \\ 0 & \lambda_2 \mathrm{Id}_3 & \lambda_1 \mathrm{Id}_3 & 0 & x_3 \mathrm{Id}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_4 \end{pmatrix}$$

with scaling factors $x_1, x_2, x_3, x_4 > 0$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. This parameterizes the space of $\mathrm{Spin}(8)$-invariant metrics, but without loss of generality we can simplify as in the previous sections. The normalizer of $\mathrm{U}(3)$ in $\mathrm{Spin}(8)$ is $\mathrm{U}(1) \cdot \mathrm{U}(3) \subset \mathrm{U}(4)$; its corresponding Lie algebra is $\mathfrak{u}(3) \oplus \mathfrak{p}_0$ (since $\mathfrak{u}(3)$ and $\mathfrak{p}_0$ commute). As in the previous example, conjugation by any element of $\mathrm{U}(1) = N(\mathrm{U}(3))/\mathrm{U}(3)$ is a diffeomorphism fixing $\mathfrak{p}$; this gives a 1-parameter group of isometries of homogeneous metrics. With any basis $\{X_i\}_{i=1}^6$ for $\mathfrak{p}_2$ and $\{Y_i\}_{i=1}^6$ for $\mathfrak{p}_3$ that satisfies the preceding description, these Lie bracket relations hold (for Z a Q-unit vector spanning $\mathfrak{p}_0$):

$$[Z, X_i] = Y_i \quad \text{and} \quad [Z, Y_j] = -X_j.$$

Hence, via $\mathrm{Ad}(\exp tZ) \cdot g$, any homogeneous metric is homothetic to one with $\lambda_1 = 0$. From $(*)$ we derive the following scalar curvature functional:

$$S = 3\Bigg(\frac{x_1^2 x_4(2\lambda_2^2 - (x_2^2 + x_3^2)) + 8x_4(2\lambda_2^2 - x_2x_3)(\lambda_2^2 - x_2x_3) + 4(x_1 + 2x_4)\lambda_1^4}{x_1x_4(x_2x_3 - \lambda_1^2 - \lambda_2^2)^2}$$

$$- \frac{\begin{array}{r} x_1(12x_4(x_2 + x_3)(\lambda_2^2 - x_2x_3) + \lambda_2^2(x_4^2 - (x_2 - x_3)^2) \\ +x_2x_3(x_4^2 + (x_2 - x_3)^2)) \end{array}}{x_1x_4(x_2x_3 - \lambda_1^2 - \lambda_2^2)^2}$$

$$+ \frac{\begin{array}{r} (x_1(x_2^2 + x_3^2 + x_4^2 - 3(x_2x_3 + 4x_2x_4 + 4x_3x_4)) \\ -2x_4(x_1^2 + 8x_2x_3 + 12\lambda_2^2))\lambda_1^2 \end{array}}{x_1x_4(x_2x_3 - \lambda_1^2 - \lambda_2^2)^2} \Bigg).$$

We normalize for volume 1 to get $\tilde{S} = S - k(x_1^6(x_2x_3 - \lambda_1^2 - \lambda_2^2)^6 x_4)$. Since $\tilde{S}$ is a function of λ_1^2, we can set $\lambda_1 = 0$ and know that we will not omit any critical points of the scalar curvature. We find, using Maple, that there are two real solutions such that each $x_i > 0$:

$$x_1 = \frac{2x_3}{3}, \qquad x_2 = \frac{x_3}{3}, \quad x_4 = \frac{2x_3}{3}, \qquad \lambda_2 = \frac{x_3}{3},$$

$$x_1 = \left(\frac{91\zeta^4 + 54\zeta^2 - 1}{8}\right)x_3, \quad x_2 = x_3, \quad x_4 = \left(\frac{91\zeta^4 - 2\zeta^2 + 7}{7}\right)x_3, \quad \lambda_2 = \zeta x_3,$$

where ζ is a real solution to $91\zeta^6 + 131\zeta^4 + 41\zeta^2 - 7 = 0$. (Approximately, $x_1 \simeq 0.853x_3$, $x_4 \simeq 1.154x_3$, and $\lambda_2 \simeq 0.347x_3$.)

We check that these are not isometric by comparing a scale-invariant constant $(S)^{19/2}(V)^{1/2}$. For the first metric that constant is (approx.) 4.3401636×10^{18}; the second metric yields (approx.) 2.8743704×10^{18}. We show that the first metric is the product metric. With that metric the Grassmannian is a symmetric space. In our fibration, we do not have the symmetric pair presentation; we project the tangent space to $G_2^+(\mathbb{R}^8)$, which is $\mathfrak{p}_1 \oplus \mathfrak{p}_2$, onto $\tilde{\mathfrak{p}}$, the Q-orthogonal complement to $\mathfrak{so}(2) \oplus \mathfrak{so}(6)$ in $\mathfrak{so}(8)$. We see that $\mathfrak{p}_1 \subset \tilde{\mathfrak{p}}$, so that a unit element in $\mathfrak{p}_1$ projects to a unit element in $\tilde{\mathfrak{p}}$. On the other hand, writing a basic element of $\mathfrak{p}_2$ to respect the decomposition $\mathfrak{so}(8) = \tilde{\mathfrak{p}} \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(6)$, we have

$$\tfrac{1}{2}(E_{16} - E_{25} + E_{38} - E_{47}) = \tfrac{1}{2}(E_{16} - E_{25}) + \tfrac{1}{2}(E_{38} - E_{47}).$$

A unit element in $\mathfrak{p}_2$ projects to an element of length $\frac{1}{2}$. Hence the symmetric metric is the metric in which $2x_2 = x_1$; this is the first metric.

Appendix

We describe of the geometry of the G-homogeneous metrics on $G_2^+(\mathbb{R}^7) \times S^7$ and $S^7 \times G_3^+(\mathbb{R}^8)$, and we give their moduli spaces of G-invariant metrics. In each case, the critical points of the scalar curvature functional will be Einstein metrics. The differential equation was unmanageable, however, even with the help of Maple.

The Product $G_2^+(\mathbb{R}^7) \times S^7$

We identify $\mathbb{R}^8 \cong \mathbb{O}$, the Octonians, and $\mathbb{R}^7 \cong \mathrm{Im}(\mathbb{O})$, the imaginary Octonians. Our Grassmannian $G_2^+(\mathbb{R}^7)$ is the space of oriented 2-planes through the origin in $\mathrm{Im}(\mathbb{O})$, and S^7 is the unit sphere in $\mathbb{O}$. Recall that

$$\mathrm{Spin}^+(7) = \{\, (A, B, B) \in \mathrm{SO}(8)^3 \mid A(x)B(y) = B(xy)\ \forall x, y \in \mathbb{O}\,\}.$$

The action of $\mathrm{Spin}^+(7)$ is $(A, B, B)\colon (P, x) \mapsto (AP, Bx)$. To see that this action is transitive, we will send any (P, x) to $(P_0, 1)$, where P_0 is the oriented 2-plane $\mathrm{span}\{j, k\}$. We will use the subgroups G_2 and $\mathrm{SU}(4)$ (G_2 is the group of automorphisms of $\mathbb{O}$ and $\mathrm{SU}(4)$ is unitary with respect to left multiplication by i) of $\mathrm{Spin}^+(7)$. We know that $\mathrm{SU}(4)$ acts transitively on S^7, so there exists an (A, B, B) taking (P, x) to $(P', 1)$, for some P' in $G_2^+(\mathbb{R}^7)$. Then, since G_2 acts transitively on $G_2^+(\mathbb{R}^7)$ (and fixes 1 in S^7), we can find some (A', A', A') taking $(P', 1)$ to $(P_0, 1)$.

The isotropy subgroup H fixing $(P_0, 1)$ satisfies $H \subset \mathrm{G}_2$. This is because elements of H take 1 to itself, so that $A(x)B(1) = B(x)$ for all $x \in \mathbb{O}$. Elements of H also take P_0 to itself and $i = jk$, so i is fixed by the isotropy subgroup. This shows (cf. [K]) that the isotropy subgroup of G_2 fixing P_0 is $\mathrm{U}(2) \cong S(\mathrm{U}(1)\mathrm{U}(2)) \subset \mathrm{SU}(3) \subset \mathrm{G}_2$, where $\mathrm{SU}(3)$ is the subgroup fixing i.

As in the previous examples, we consider the Lie algebra of $\mathfrak{spin}^+(7)$ as a Lie subalgebra of $\mathfrak{so}(8)$. We use the $\mathrm{Ad}\,\mathrm{U}(2)$-invariant inner product Q to choose an invariant complement $\mathfrak{p} = \mathfrak{u}(2)^\perp$ to $\mathfrak{u}(2)$ in $\mathfrak{spin}^+(7)$. There are many fibrations of our space; we give three here (and use one to determine the isotropy representation of $\mathrm{U}(2)$ on $\mathfrak{p}$):

$$\begin{aligned} S^5 \cong \mathrm{U}(3)/\mathrm{U}(2) &\to \mathrm{Spin}^+(7)/\mathrm{U}(2) \to \mathrm{Spin}^+(7)/\mathrm{U}(3),\\ G_2^+(\mathbb{R}^7) \cong \mathrm{G}_2/\mathrm{U}(2) &\to \mathrm{Spin}^+(7)/\mathrm{U}(2) \to \mathrm{Spin}^+(7)/\mathrm{G}_2 \cong S^7,\\ \mathrm{SU}(4)/\mathrm{U}(2) &\to \mathrm{Spin}^+(7)/\mathrm{U}(2) \to \mathrm{Spin}^+(7)/\mathrm{SU}(4) \cong S^6. \end{aligned}$$

In the first fibration, the isotropy representation of the fibre is $[\mu_1 \hat{\otimes} \mu_2]_{\mathbb{R}} \oplus \mathrm{Id}$, where μ_k is the standard k-dimensional representation of $\mathrm{SU}(k)$. The base space is a symmetric space but not a symmetric pair; its isotropy representaton is $[\mu_3]_{\mathbb{R}} \oplus [\wedge^2\mu_3]_{\mathbb{R}}$ (cf. [K]). We must restrict the representations of the base to $\mathrm{U}(2)$: notice that $[\mu_3]_{\mathbb{R}}$ and $[\wedge^2\mu_3]_{\mathbb{R}}$ are equivalent representations of $\mathrm{SU}(3)$, so their restrictions both give $[\mu_1 \hat{\otimes} \mathrm{Id}]_{\mathbb{R}} \oplus [\mathrm{Id} \hat{\otimes} \mu_2]_{\mathbb{R}}$. Thus our total isotropy representation decomposition is $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \oplus \mathfrak{p}_4 \oplus \mathfrak{p}_5 \oplus \mathfrak{p}_0$, the sum of six irreducible real represenatations of $\mathrm{U}(2)$, where: $\mathfrak{p}_1 \simeq \mathfrak{p}_2 \simeq [\mu_1 \hat{\otimes} \mathrm{Id}]_{\mathbb{R}}$; $\mathfrak{p}_3 \simeq [\mu_1 \hat{\otimes} \mu_2]_{\mathbb{R}}$; $\mathfrak{p}_4 \simeq \mathfrak{p}_5 \simeq [\mathrm{Id} \hat{\otimes} \mu_2]_{\mathbb{R}}$; and, finally, $\mathfrak{p}_0$ is a 1-dimensional trivial representation. The decomposition of $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ and $\mathfrak{p}_4 \oplus \mathfrak{p}_5$ into irreducible representations is not unique. We have $\mathfrak{su}(3) = \mathfrak{u}(2) \oplus \mathfrak{p}_3$ and $\mathfrak{u}(3) = \mathfrak{su}(3) \oplus \mathfrak{p}_0$. If we require that $\mathfrak{su}(4) = \mathfrak{u}(3) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_4$, this determines a choice of $\mathfrak{p}_1$ and $\mathfrak{p}_4$. Choose $\mathfrak{p}_2$ and $\mathfrak{p}_5$ to be the Q-orthogonal complement to $\mathfrak{su}(4)$.

Each of $[\mu_1 \hat{\otimes} \mathrm{Id}]_{\mathbb{R}}$ and $[\mathrm{Id} \hat{\otimes} \mu_2]_{\mathbb{R}}$ is a unitary representation; there are two dimensions of intertwining maps for each pair. Hence the space of $\mathrm{Ad}\,\mathrm{U}(2)$-equivariant symmetric positive definite maps g is 10-dimensional. Let $x_i > 0$ for

$i = 0, \ldots, 5$ and $\lambda_j \in \mathbb{R}$ for $j = 1, \ldots, 4$. Then $g = \operatorname{diag}(A_1, x_3 \operatorname{Id}_4, A_2, x_0)$, where

$$A_1 = \begin{pmatrix} x_1 & 0 & \lambda_1 & \lambda_2 \\ 0 & x_1 & -\lambda_2 & \lambda_1 \\ \lambda_1 & -\lambda_2 & x_2 & 0 \\ \lambda_2 & \lambda_1 & 0 & x_2 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} x_4 \operatorname{Id}_2 & 0 & \lambda_3 \operatorname{Id}_2 & \lambda_4 \operatorname{Id}_2 \\ 0 & x_4 \operatorname{Id}_2 & -\lambda_4 \operatorname{Id}_2 & \lambda_3 \operatorname{Id}_2 \\ \lambda_3 \operatorname{Id}_2 & -\lambda_4 \operatorname{Id}_2 & x_5 \operatorname{Id}_2 & 0 \\ \lambda_4 \operatorname{Id}_2 & \lambda_3 \operatorname{Id}_2 & 0 & x_5 \operatorname{Id}_2 \end{pmatrix}.$$

We can simplify by using the extra isometries from the normalizer of U(2) in $\operatorname{Spin}^+(7)$. We find $N(\mathrm{U}(2)) = \mathrm{U}(3)$, and so $N(\mathrm{U}(2))/\mathrm{U}(2) = \mathrm{U}(3)/\mathrm{U}(2) = \mathrm{U}(1)$. The tangent space to the quotient is $\mathfrak{p}_0$, and if X_0 is a Q-unit vector spanning $\mathfrak{p}_0$ then $\operatorname{ad}(X_0)$ takes $\mathfrak{p}$ to itself. Thus, for all real t, $\operatorname{Ad}(\exp tX_0)(g)$ is isometric to g. Because the action of $\operatorname{ad}(X_0)$ in fact rotates $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ and also $\mathfrak{p}_4 \oplus \mathfrak{p}_5$, we can find a t such that one of our off-diagonal terms is zero. That is, any $\operatorname{Spin}^+(7)$-invariant metric is isometric to one given by nine (instead of ten) parameters. Nevertheless, the scalar curvature formula is cumbersome; we were not able to find its critical points.

The Product $S^7 \times G_3^+(\mathbb{R}^8)$

Identify $\mathbb{R}^8 \cong \mathbb{O}$, the Octonians, so that S^7 is the unit sphere in $\mathbb{O}$ and $G_3^+(\mathbb{R}^8)$ is the set of 3-planes in $\mathbb{O}$ through the origin. Recall that

$$\operatorname{Spin}(8) = \{\, (A, B, C) \in \mathrm{SO}(8)^3 \mid A(x)B(y) = C(xy)\ \forall x, y \in \mathbb{O} \,\}.$$

The action of Spin(8) is $(A, B, C)\colon (x, P) \mapsto (Ax, BP)$ for any $x \in S^7$ and any $P \in G_3^+(\mathbb{R}^8)$. To see that this action is transitive, we will send any (x, P) to $(1, P_0)$, where $P_0 = \operatorname{span}\{i, j, k\}$, using the subgroups of Spin(8). Since $\operatorname{Spin}^-(7)$ acts transitively on S^7, there exists a triple (A, B, A) taking (x, P) to $(1, P')$ for some P' in $G_3^+(\mathbb{R}^8)$. Then, since $\operatorname{Spin}^+(7)$ acts transitively on $G_3^+(\mathbb{R}^8)$ (cf. [K]), there exits an (A', B', B') sending $(1, P')$ to $(1, P_0)$. The composition is the desired map.

The isotropy subgroup H fixing $(1, P_0)$ satisfies $H \subset \operatorname{Spin}^+(7)$: since $A(1) = 1$, we have $B(x) = C(x)$ for all $x \in \mathbb{O}$. Then the subgroup of $\operatorname{Spin}^+(7)$ fixing P_0 is SO(4), the subgroup of G_2 fixing the associative subalgebra of $\mathbb{O}$ generated by i and j (cf. [K]).

As in our previous examples, we identify the Lie algebra $\mathfrak{spin}(8)$ with $\mathfrak{so}(8)$ via the double-covering homomorphism $(A, B, C) \mapsto C$. We use the Ad-invariant inner product Q to choose an invariant complement $\mathfrak{p} = \mathfrak{so}(4)^\perp$ to $\mathfrak{so}(4)$ in $\mathfrak{so}(8)$. There are several fibrations of our space, we give two here, using their geometry to decompose $\mathfrak{p}$ into a sum of irreducible real representations of SO(4):

$$G_3^+(\mathbb{R}^8) \cong \operatorname{Spin}^\pm(7)/\mathrm{SO}(4) \to \operatorname{Spin}(8)/\mathrm{SO}(4) \to \operatorname{Spin}(8)/\operatorname{Spin}^\pm(7) \cong S^7,$$
$$\mathrm{G}_2/\mathrm{SO}(4) \to \operatorname{Spin}(8)/\mathrm{SO}(4) \to \operatorname{Spin}(8)/\mathrm{G}_2 \cong S^7 \times S^7.$$

Before we describe the isotropy representation, we note that $\mathfrak{so}(4)$ is not simple: $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Denote by ρ_3 the standard orthogonal representation of SO(3) in $\mathbb{R}^3$, and denote by θ_n the unique irreducible unitary representation of SU(2) of dimension n. In the second fibration, the fibre is an irreducible symmetric space with isotropy representation $[\theta_2 \hat{\otimes} \theta_4]_{\mathbb{R}}$. The base space has isotropy representation $\varphi \oplus \varphi$, where φ is the standard orthogonal representation of G_2 in SO(7). When we restrict each φ to SO(4), we get $[\mathrm{Id} \hat{\otimes} \rho_3]_{\mathbb{R}} \oplus [\theta_2 \hat{\otimes} \theta_2]_{\mathbb{R}}$. The isotropy representation of the total space decomposes into $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \oplus \mathfrak{p}_4 \oplus \mathfrak{p}_5$, with $\mathfrak{p}_1 \simeq \mathfrak{p}_2 \simeq [\mathrm{Id} \hat{\otimes} \rho_3]_{\mathbb{R}}$, $\mathfrak{p}_3 \simeq [\theta_2 \hat{\otimes} \theta_4]_{\mathbb{R}}$, and $\mathfrak{p}_4 \simeq \mathfrak{p}_5 \simeq [\theta_2 \hat{\otimes} \theta_2]_{\mathbb{R}}$. As before, $\mathfrak{p}$ does not compose uniquely. If we choose $\mathfrak{spin}^+(7) = \mathfrak{so}(4) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_3 \oplus \mathfrak{p}_4$, this fixes a choice of $\mathfrak{p}_1$ and $\mathfrak{p}_4$. Then we can choose $\mathfrak{p}_2$ and $\mathfrak{p}_5$ to be the corresponding Q-orthogonal complements.

Since both $\mathfrak{p}_1 \simeq \mathfrak{p}_2 \simeq [\mathrm{Id} \hat{\otimes} \rho_3]_{\mathbb{R}}$ and $\mathfrak{p}_4 \simeq \mathfrak{p}_5 \simeq [\theta_2 \hat{\otimes} \theta_2]_{\mathbb{R}}$ are orthogonal, each pair has a 1-dimensional space of intertwining maps. The space of Ad SO(4)-invariant symmetric positive-definite maps g is 7-dimensional. For $x_i > 0$ ($i = 1, \ldots, 5$) and $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$g = \begin{pmatrix} x_1 \mathrm{Id}_3 & \lambda_1 \mathrm{Id}_3 & & & \\ \lambda_1 \mathrm{Id}_3 & x_2 \mathrm{Id}_3 & & & \\ & & x_3 \mathrm{Id}_8 & & \\ & & & x_4 \mathrm{Id}_4 & \lambda_2 \mathrm{Id}_4 \\ & & & \lambda_2 \mathrm{Id}_4 & x_5 \mathrm{Id}_4 \end{pmatrix}.$$

The scalar curvature is a function of these variables; it can be obtained from equation $(*)$ using any Q-orthonormal basis satisfying the decomposition just described.

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