Maximal Gleason Parts for H^∞

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Introduction

It is well known that every Gleason part of the algebra H^∞, of bounded analytic functions on the unit disk, is a maximal analytic disk or a single point [7]. Furthermore, there are very different behaviors within the class of nontrivial Gleason parts. For example, it is known that some analytic disks are homeomorphic to the unit disk $\mathbb{D}$, while some others are not.

Although the Gleason parts have been studied by several authors (see e.g. [2; 5; 6]), the information at our disposal is partial and fragmented. In particular, our knowledge of the closures of Gleason parts is very limited. Far from giving the whole picture, which is probably unreachable, the present paper intends to throw some light on the behavior of the closures of Gleason parts.

First we give a criterion to check whether a point in the maximal ideal space of H^∞ is or is not in the closure of a given Gleason part. This criterion is then used to prove that if the closures of two Gleason parts have nonvoid intersection, then one of them is contained in the closure of the other. This answers a question posed by Gorkin in [5] and is the starting point of our study of maximal parts (not contained into the closure of any other part except the disk $\mathbb{D}$). We consider a class of maximal parts that contains properly the thin parts (this generalizes a result of Budde [2]) and we study the general properties of this class. Next we prove the existence of maximal parts not belonging to this class. Finally, we pose three open problems that we believe are fundamental to understanding the way in which the Gleason parts relate to each other.

Acknowledgment. I am grateful to P. Gorkin for many valuable discussions and for pointing out several misprints in the first sketch of this paper. Also, I am indebted to the referee for improving the exposition with many thoughtful suggestions.

1. Preliminaries

The maximal ideal space of H^∞ is defined by

$$M(H^\infty) = \{\, \varphi : \varphi \text{ is linear, multiplicative and } \varphi \neq 0 \,\}$$

Received December 3, 1996. Revision received March 7, 1997.
Michigan Math. J. 45 (1998).

provided with the weak $*$ topology induced by the dual space of H^∞. It is a compact Hausdorff space. We can look at a function $f \in H^\infty$ as a continuous function on $M(H^\infty)$ via the Gelfand transform $\hat{f}(\varphi) = \varphi(f)$ $(\varphi \in M(H^\infty))$. Evaluation at a point of $\mathbb{D}$ is an element of $M(H^\infty)$, so $\mathbb{D}$ is naturally imbedded into $M(H^\infty)$, and $\hat{f}$ is an extension to the whole maximal space of the function f. In what follows we avoid writing the hat for the Gelfand transform of f.

The pseudohyperbolic and hyperbolic metrics on the open unit disk are defined by

$$\rho(z, \omega) = \left| \frac{z - \omega}{1 - \bar{\omega} z} \right|$$

and

$$h(z, \omega) = \log \frac{1 + \rho(z, \omega)}{1 - \rho(z, \omega)}, \quad z, \omega \in \mathbb{D},$$

respectively. For $x, y \in M(H^\infty)$, the formula

$$\rho(x, y) = \sup\{ |f(y)| : f \in H^\infty,\ f(x) = 0,\ \|f\| \le 1\}$$

provides an extension of ρ to $M(H^\infty)$. Therefore, the Schwarz–Pick (SP) inequality takes the form $\rho(f(x), f(y)) \le \rho(x, y)$ for $x, y \in M(H^\infty)$, $f \in H^\infty$, and $\|f\| \le 1$. It will be convenient to work with the metric ρ when dealing with the SP inequality and with the metric h in calculations involving the triangular inequality. Of course, we can go from one metric to the other so long as we keep in mind that $\rho(z, \omega) \to 1$ if and only if $h(z, \omega) \to \infty$. We remark that the topology on $M(H^\infty)$ induced by ρ does not coincide with the weak $*$ topology. The Gleason part of $x \in M(H^\infty)$ is then defined as $P(x) = \{ y \in M(H^\infty) : \rho(x, y) < 1\}$. It is well known that for $x, y \in M(H^\infty)$ we have $P(x) = P(y)$ or $P(x) \cap P(y) = \emptyset$.

A first classification of Gleason parts (see [7]) shows that there are only two cases: either $P(x) = \{x\}$ $(x \in M(H^\infty))$ or $P(x)$ is an analytic disk. The former case means that there is a continuous one-to-one and onto map $L_x : \mathbb{D} \to P(x)$ such that $f \circ L_x \in H^\infty$ for every $f \in H^\infty$. Reciprocally, any analytic disk is contained in a Gleason part, and any maximal (not contained into another) analytic disk is a Gleason part. If $z \in \mathbb{D}$ then $P(z) = \mathbb{D}$ is dense in $M(H^\infty)$ by the corona theorem of Carleson [3]. This fact makes trivial all the statements of this paper for this particular part, so from now on by a Gleason part we *always* mean a Gleason part other than the open disk.

Let $S = \{z_n\} \subset \mathbb{D}$ be a sequence such that $z_n = 0$ occurs m times in S. Then the product

$$b(z) = z^m \prod_{z_n \neq 0} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}$$

converges on $\mathbb{D}$ if and only if

$$\sum (1 - |z_n|) < \infty. \tag{1.1}$$

The function b is called a *Blaschke product*. If the sequence S satisfies $\delta(S) = \delta(b) = \inf_k \prod_{n \neq k} \rho(z_n, z_k) > 0$ then (1.1) is automatically fulfilled and S is called an *interpolating sequence* (so, b is called an interpolating Blaschke product). From the work of Hoffman [7] we know that if $x \in M(H^\infty) \setminus \mathbb{D}$ then $P(x)$ is an analytic disk if and only if x belongs to the set

$$G = \{\, y \in M(H^\infty) \setminus \mathbb{D} : y \text{ is in the closure of some interpolating sequence}\,\}.$$

If $x \in M(H^\infty)$ and $f \in H^\infty$ are such that $f(x) = 0$, then the multiplicity of x as a zero of f is defined as the maximum integer N such that $f = f_1 \dots f_N$ with $f_j(x) = 0$ for $1 \le j \le N$; the multiplicity is infinite if there is no such N. It is well known that the multiplicity is infinite if and only if $f \equiv 0$ on $P(x)$. One of the features of interpolating sequences that we will use frequently is that disjoint subsequences of the same sequence have disjoint closures in $M(H^\infty)$ [7, Thm. 6.1].

2. Sequences and Parts

Let $z \in \mathbb{D}$ and $S, T \subset \mathbb{D}$. We write $\rho(z, S) = \inf_{\omega \in S} \rho(z, \omega)$ and $\rho(S, T) = \inf_{z \in T,\, \omega \in S} \rho(z, \omega)$, with similar conventions for the metric h.

DEFINITION. Let $x \in G$ and $E \subset \mathbb{D}$. We say that x *avoids* E if, for every interpolating sequence S such that $x \in \bar{S}$ and any number $0 < \rho_0 < 1$, there is a subsequence $S_1 \subset S$ such that $x \in \bar{S}_1$ and $\rho(S_1, E) \ge \rho_0$.

LEMMA 2.1. *Let $x \in G$ and $y \in M(H^\infty) \setminus \overline{P(x)}$. If $U \subset M(H^\infty)$ is an open neighborhood of y such that $\bar{U} \cap \overline{P(x)} = \emptyset$, then x avoids $U \cap \mathbb{D}$.*

Proof. Because $y \notin \overline{P(x)}$, there exists an open neighborhood U of y such that $\bar{U} \cap \overline{P(x)} = \emptyset$. Let $S \subset \mathbb{D}$ be a interpolating sequence with $x \in \bar{S}$. For an arbitrary number $0 < \rho_0 < 1$, consider the following subsequence of S:

$$S_0 = \{\, \omega_k \in S : \rho(\omega_k, U) < \rho_0 \,\}. \tag{2.1}$$

We shall prove that $x \notin \bar{S}_0$. Clearly, this is the case if S_0 is either finite or the empty set. Let $z_k \in U \cap \mathbb{D}$ such that $\rho(\omega_k, z_k) < \rho_0$. Thus, $\overline{\{z_k\}} \subset \bar{U}$ is disjoint from $\overline{P(x)}$. Put $\varepsilon = (1 - \rho_0)/4$. Combining the definition of $P(x)$ with the compactness of $\overline{\{z_k\}}$, we obtain finitely many functions $f_1, \dots, f_n \in H^\infty$ such that $\|f_j\| \le 1$, $f_j(x) = 0$ for $1 \le j \le n$, and

$$\max_{1 \le j \le n} |f_j(\xi)| > 1 - \varepsilon \quad \text{for all } \xi \in \overline{\{z_k\}}. \tag{2.2}$$

Suppose that $x \in \bar{S}_0$. Since $f_j(x) = 0$ for all j, we also have

$$x \in \overline{\{\, \omega_k \in S_0 : |f_j(\omega_k)| < \varepsilon \; \forall j \,\}} = \overline{\{\omega_{k_s}\}}.$$

By the SP inequality, for $1 \le j \le n$ and every s,

$$\rho(f_j(z_{k_s}), f_j(\omega_{k_s})) \le \rho(z_{k_s}, \omega_{k_s}) < \rho_0.$$

An easy estimate shows that if $z, \omega \in \mathbb{D}$ and $|\omega| < \varepsilon$ then $|z| < \varepsilon + \rho(z, \omega)$. Consequently,

$$|f_j(z_{k_s})| < \varepsilon + \rho_0 \quad \text{for } 1 \le j \le n \text{ and all } z_{k_s}.$$

On the other hand, (2.2) implies that for every s there exists $1 \le j_s \le n$ such that $|f_{j_s}(z_{k_s})| > 1 - \varepsilon$. Thus, $1 - \varepsilon < \varepsilon + \rho_0$, contradicting our choice of ε. Hence, x must belong to the closure of

$$S_1 = \{\, \omega_k \in S : \rho(\omega_k, U) \ge \rho_0 \,\},$$

and the lemma follows. □

The next two lemmas are easy consequences of Lemmas 1.4 and 1.5 in [4, Ch. X], respectively.

LEMMA 2.2. *Let* $0 < \alpha < 1$. *Then there exist* $\sigma = \sigma(\alpha) > 0$ *and* $\delta = \delta(\alpha, \sigma) \in (0, 1)$ *such that if* b *is any interpolating Blaschke product with* $\delta(b) \ge \delta$, *then* $|b(z)| > \alpha$ *for every* z *in*

$$K_\sigma(b) = \{\, z \in \mathbb{D} : h(z, z_k) \ge \sigma \text{ for all zeroes } z_k \text{ of } b \,\}.$$

LEMMA 2.3. *Let* S *be an interpolating sequence and* $0 < \delta < 1$. *Suppose that* $x \in \bar{S}$. *Then there exists a subsequence* $S_1 \subset S$ *such that* $x \in \bar{S}_1$ *and* $\delta(S_1) > \delta$.

Now we are ready to prove the main theorem of this section. The idea of the proof comes from Gorkin's paper [5, Thm. 2.2]. Let $S = \{z_n\}_{n\ge1}$ be a sequence in $\mathbb{D}$. By a *tail* of S we mean a sequence of the form $\{z_n\}_{n\ge k}$ for some $k \ge 1$.

THEOREM 2.4. *Let* $S \subset \mathbb{D}$ *be an interpolating sequence and let* $x \in \bar{S} \setminus S$. *Suppose that* $T \subset \mathbb{D}$ *is a subset and let* $0 < \beta < 1$.

(I) *If* x *avoids* T *then there is a Blaschke product* b_x *such that* $b_x \equiv 0$ *on* $P(x)$ *and* $|b_x(z)| > \beta$ *for all* $z \in T$.

(II) *If, in addition to* (I), T *is also an interpolating sequence,* $y \in \bar{T}$, *and* y *avoids* S, *then there is a Blaschke product* b_y *such that* $b_y \equiv 0$ *on* $P(y)$, $|b_y(z)| > \beta$ *for all* $z \in S$, *and*

$$\max\{|b_x(z)|, |b_y(z)|\} > \beta \quad \textit{for all } z \in \mathbb{D}.$$

Proof. (I) Fix $\alpha > \beta$ and take $\sigma_1 = \sigma(\alpha^{1/2})$ as in Lemma 2.2. Since x avoids T there is a subsequence $R_1 \subset S$ such that $h(R_1, T) \ge 4\sigma_1$ and $x \in \bar{R}_1$. By Lemma 2.3 there is a subsequence $S_1 \subset R_1$ such that $x \in \bar{S}_1$ and $\delta(S_1) \ge \delta(\alpha^{1/2}, \sigma_1)$. We also can assume, by taking a tail of S_1 if necessary, that

$$\sum_{\omega\in S_1} (1 - |\omega|) \le \frac{1}{2} \sum_{\omega\in S} (1 - |\omega|).$$

By Lemma 2.2, the interpolating Blaschke product b_1 associated to S_1 satisfies $b_1(x) = 0$ and $|b_1(z)| \ge \alpha^{1/2}$ for all $z \in A_1 = \{\, z \in \mathbb{D} : h(z, S_1) \ge \sigma_1 \,\}$.

We can repeat this process with σ_1 replaced by $\sigma_2 = \sigma(\alpha^{1/4})$, S replaced by S_1, and so forth. At the Nth step we have N interpolating sequences $S_1 \supset S_2 \supset \cdots \supset S_N$ and the corresponding Blaschke products $b_1, \ldots, b_N$, so that for any $1 \le n \le N$:

(i) $b_n(x) = 0$ (i.e., $x \in \bar{S}_n$);
(ii) $h(S_n, T) \geq 4\sigma_n$, where $\sigma_n = \sigma(\alpha^{1/2^n})$;
(iii) $|b_n(z)| \geq \alpha^{1/2^n}$ for every $z \in A_n = \{ z \in \mathbb{D} : h(z, S_n) \geq \sigma_n \}$; and
(iv) $\sum_{\omega \in S_n}(1 - |\omega|) \leq (1/2^n) \sum_{\omega \in S}(1 - |\omega|)$.

The product $b = \prod_{n=1}^{\infty} b_n$ converges because, by (iv),

$$\sum_{n=1}^{\infty} \sum_{\omega \in S_n}(1 - |\omega|) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{\omega \in S}(1 - |\omega|) = \sum_{\omega \in S}(1 - |\omega|) < \infty.$$

Condition (i) implies that b has a zero of infinite multiplicity at x, so $b \equiv 0$ on $P(x)$. By (iii), for every $z \in A = \bigcap_{n \geq 1} A_n$ we have

$$|b(z)| \geq \alpha^{1/2 + 1/2^2 + \cdots} = \alpha > \beta. \tag{2.3}$$

Finally, (ii) implies that $T \subset A$ and then $|b(z)| > \beta$ for $z \in T$. Thus, $b_x = b$ satisfies the conclusion of the theorem.

(II) We keep the notation of (I). Furthermore, we repeat the previous construction with the pairs (y, T) and (x, S) interchanged, thereby obtaining a decreasing sequence of interpolating sequences $T \supset T_1 \supset \cdots$ and the corresponding Blaschke product b_y. Put $B_k = \{ z \in \mathbb{D} : h(z, T_k) \geq \sigma_k \}$. By (I) we have

(i′) $b_y \equiv 0$ on $P(y)$;
(ii′) $h(S, T_k) \geq 4\sigma_k$ for all $k \geq 1$;
(iii′) $|b_y(z)| \geq \beta$ for every $z \in B = \bigcap_{k \geq 1} B_k$; and
(iv′) $S \subset B$.

In particular, (iii′) and (iv′) imply that $|b_y(x)| \geq \beta$. By (2.3) and (iii′),

$$\max\{|b_x(z)|, |b_y(z)|\} \geq \beta \quad \text{for all } z \in A \cup B.$$

Therefore (II) will follow if we show that $\mathbb{D} = A \cup B$. The complement of $A \cup B$ is

$$\begin{aligned} \mathbb{D} \setminus (A \cup B) &= \left[\bigcup_{n \geq 1} (\mathbb{D} \setminus A_n) \right] \cap \left[\bigcup_{k \geq 1} (\mathbb{D} \setminus B_k) \right] \\ &= \bigcup_{n, k \geq 1} [(\mathbb{D} \setminus A_n) \cap (\mathbb{D} \setminus B_k)]. \end{aligned}$$

Let $z \in (\mathbb{D} \setminus A_n) \cap (\mathbb{D} \setminus B_k)$ for some $n, k \geq 1$. Then $h(z, S_n) < \sigma_n$ and $h(z, T_k) < \sigma_k$. Consequently, there exist $\omega_s \in S_n$ and $\omega_t \in T_k$ such that $h(z, \omega_s) < \sigma_n$ and $h(z, \omega_t) < \sigma_k$. Hence,

$$h(S_n, T_k) \leq h(\omega_s, \omega_t) \leq h(\omega_s, z) + h(z, \omega_t) < \sigma_n + \sigma_k.$$

On the other hand, (ii) and (ii′) imply that

$$h(S_n, T_k) \geq \max\{h(S_n, T), h(S, T_k)\} \geq 4 \max\{\sigma_n, \sigma_k\}.$$

The last two inequalities imply that $4 \max\{\sigma_n, \sigma_k\} < \sigma_n + \sigma_k$, which is obviously false. Thus, $\mathbb{D} \setminus (A \cup B) = \emptyset$ and we are done. □

We are able now to give a converse of Lemma 2.1.

COROLLARY 2.5. *Let $x \in G$ and $y \in M(H^\infty)$. Then the following conditions are equivalent.*

(1) $y \notin \overline{P(x)}$.

(2) *There is an open neighborhood U of y such that x avoids $U \cap \mathbb{D}$.*

Additionally, if $y \in G$ then there are two more equivalent conditions.

(3) *For every interpolating sequence T such that $y \in \bar{T}$, there is a subsequence $T_1 \subset T$ such that $y \in \bar{T}_1$ and x avoids T_1.*

(4) *There is an interpolating sequence T_1 such that $y \in \bar{T}_1$ and x avoids T_1.*

Proof. (1) implies (2) is Lemma 2.1. Now assume that (2) holds, and let U be an open neighborhood of y such that x avoids $U \cap \mathbb{D}$. By Theorem 2.4(I), there is a Blaschke product b such that $b \equiv 0$ on $P(x)$ and $\inf_{z \in U\cap\mathbb{D}} |b(z)| > 0$. Thus, every point $\xi \in \overline{U \cap \mathbb{D}}$ (in particular y) is not in $\overline{P(x)}$. Hence (2) implies (1). If (2) holds and $y \in \bar{T}$ with T an interpolating sequence, then it is clear that $T_1 = T \cap U$ satisfies (3). Obviously (3) implies (4). Now suppose that (4) holds. As before, Theorem 2.4 says that there is a Blaschke product separating $\overline{P(x)}$ from $\bar{T}_1$, so (1) holds. □

For a set $V \subset \mathbb{D}$, write $B_h(V, \sigma) = \{ z \in \mathbb{D} : h(z, V) < \sigma \}$.

COROLLARY 2.6. *Let $y \in M(H^\infty) \setminus \mathbb{D}$ and $x \in G$. Then $y \in \overline{P(x)}$ if and only if, for every set $V \subset \mathbb{D}$ such that $y \in \bar{V}$, there exists $\sigma = \sigma(V) > 0$ such that $x \in \overline{B_h(V, \sigma)}$.*

Proof. Let $S \subset \mathbb{D}$ be an interpolating sequence such that $x \in \bar{S}$. Suppose that there is a set $V \subset \mathbb{D}$ such that $y \in \bar{V}$ and $x \notin \overline{B_h(V, \sigma)}$ for any $\sigma > 0$. Then $x \in \bar{S}_0 = \overline{\{ z_n \in S : h(z_n, V) \geq \sigma \}}$, meaning that x avoids V. By Theorem 2.4(I) there is a Blaschke product separating $\overline{P(x)}$ from $\bar{V}$.

Suppose now that $y \notin \overline{P(x)}$. By Corollary 2.5 there is an open neighborhood U of y such that x avoids $V = U \cap \mathbb{D}$. We will see that for any $\sigma > 0$, x also avoids $B_h(V, \sigma)$. Let $S \subset \mathbb{D}$ be an interpolating sequence, so that $x \in \bar{S}$. Since x avoids V, for every $M > 1$ there is a subsequence $S_0 \subset S$ such that $x \in \bar{S}_0$ and $h(S_0, V) \geq M\sigma$. For $z_n \in S_0$ and $\omega \in B_h(V, \sigma)$ we have

$$h(z_n, \omega) \geq |h(z_n, V) - h(V, \omega)| = h(z_n, V) - h(\omega, V) \geq (M - 1)\sigma.$$

That is, $h(S_0, B_h(V, \sigma)) \geq (M - 1)\sigma$. Then, by Theorem 2.4, there is a Blaschke product b such that $b(x) = 0$ and $\inf\{ |b(\omega)| : \omega \in B_h(V, \sigma) \} > 0$. Therefore, $x \notin \overline{B_h(V, \sigma)}$. □

COROLLARY 2.7. *Let P_1 and P_2 be nontrivial Gleason parts so that $\bar{P}_1 \cap \bar{P}_2 \neq \emptyset$. Then one of them is contained into the closure of the other.*

Proof. Suppose that the conclusion of the corollary does not hold. Then there are $x \in P_1 \setminus \bar{P}_2$ and $y \in P_2 \setminus \bar{P}_1$. By Corollary 2.5 there are two interpolating sequences S and T such that $x \in \bar{S}$, $y \in \bar{T}$, x avoids T, and y avoids S. Hence, Theorem 2.4 asserts that there are two Blaschke products b_x and b_y such that $b_x \equiv 0$ on $\bar{P}_1$,

$b_y \equiv 0$ on $\bar{P}_2$, and $\inf_{z\in\mathbb{D}} |b_x(z)| + |b_y(z)| > 0$. So, by the corona theorem, the pair (b_x, b_y) never takes the value $(0, 0)$ on $M(H^\infty)$. Consequently, $\bar{P}_1 \cap \bar{P}_2 = \emptyset$. $\square$

Corollary 2.7 allows us to define an equivalence relation on $M(H^\infty) \setminus \mathbb{D}$ that is weaker than Gleason's relation. For $x, y \in M(H^\infty) \setminus \mathbb{D}$ we say that y is *equivalent* to x if there is a Gleason part P such that $x, y \in \bar{P}$. The equivalence class of $x \in M(H^\infty) \setminus \mathbb{D}$ is

$$K(x) = \bigcup \{ \bar{P} : P \text{ is a Gleason part and } x \in \bar{P} \}.$$

As is the case for Gleason parts, the class $K(x)$ can be very big or a single point. Let us illustrate this situation with two extreme cases. If $x \in S(H^\infty)$, the Shilov boundary of H^∞ (see [4, p. 188]), then it is known that x does not belong to the closure of any nontrivial Gleason part P. The reason is that there exists a Blaschke product b such that $b \equiv 0$ on P while $|b| \equiv 1$ on $S(H^\infty)$, where the last condition holds for every inner function (see [4, p. 194]). Hence, $K(x) = \{x\}$ for every $x \in S(H^\infty)$. In the other extreme we have the closure of a thin part. An interpolating sequence $S = \{z_k\}$ is called *thin* if

$$\prod_{n:n\neq k} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| \to 1 \quad \text{when } k \to \infty.$$

If $x \in M(H^\infty) \setminus \mathbb{D}$ is in the closure of some thin sequence then so is every point in $P(x)$. This makes it consistent to say that $P(x)$ is a thin part. Thin sequences and thin parts have many special features. In particular, I learned from P. Gorkin that in Budde's dissertation [2] it is proved that no thin part is contained in the closure of another Gleason part. A part with this characteristic is called *maximal*. If P is a Gleason part, we say that $\bar{P}$ is maximal if for every part Q such that $\bar{P} \subset \bar{Q}$ we have $\bar{P} = \bar{Q}$. So, if P is a maximal part then $\bar{P}$ is a maximal closure of part. The converse would hold if two different parts have different closures, which seems to be unknown. My guess is that this is not true. It is clear that if $x \in M(H^\infty)$ then $\overline{P(x)}$ is maximal if and only if $K(x) = \overline{P(x)}$. In particular, if $P(x)$ is a thin part then $K(x) = \overline{P(x)}$ is homeomorphic to $M(H^\infty)$ and thus, very big (see [7, p. 107]).

Our next section is essentially devoted to study a particular class of maximal parts that properly contains the class of thin parts.

3. Weakly Thin Sequences

Definition. An interpolating sequence $\{z_k\} \subset \mathbb{D}$ satisfying

$$\lim_{n\to\infty} \rho(z_n, \{z_k\}_{k\neq n}) = 1 \tag{3.1}$$

will be called a *weakly thin* (w-thin) sequence. We say that a Gleason part P is w-thin if it contains a point x in the closure of some w-thin sequence. As was the case for thin parts, this definition does not depend on the particular choice of $x \in P$.

LEMMA 3.1. *Let S be a w-thin sequence. If S is a finite union of thin sequences then S is a thin sequence.*

Proof. By induction it is enough to assume that $S = S_1 \cup S_2$, where each S_j is a thin sequence and they are disjoint. Let b_j be the Blaschke product with zero sequence S_j $(j = 1, 2)$. For an arbitrary $0 < \alpha < 1$ consider the constants $\sigma = \sigma(\alpha)$ and $\delta = \delta(\alpha, \sigma)$ of Lemma 2.2. Since S is w-thin there is a tail T_1 of S_1 such that the hyperbolic distance between T_1 and S_2 is bigger than σ. Let a_1 be a Blaschke product with zero sequence T_1. Since S_1 is a thin sequence we can also assume that $\delta(a_1) \geq \delta$. Thus, Lemma 2.2 implies that $|a_1(z_n)| \geq \alpha$ for all z_n in S_2. Write $b_1 = a_1 c_1$, where c_1 is a finite Blaschke product with zeroes $S_1 \setminus T_1$. Then, for $z_n \in S_2$,

$$|b_1(z_n)| = |a_1(z_n)||c_1(z_n)| \geq \alpha |c_1(z_n)| \to \alpha \quad \text{when } n \to \infty.$$

Because $0 < \alpha < 1$ is arbitrary, $|b_1(z_n)|$ tends to 1 when $n \to \infty$. Analogously, $|b_2(z_n)| \to 1$ when $n \to \infty$ for $z_n \in S_1$. Then S is a thin sequence. □

We now turn to H^∞ of the complex half plane $\mathbb{C}_+ = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$. In this case,

$$\rho(z, \omega) = \left| \frac{z - \omega}{z - \bar{\omega}} \right|, \quad z, \omega \in \mathbb{C}_+.$$

Let Q be a closed square of $\mathbb{C}_+$ (closed in the topology of $\mathbb{C}_+$) with base on the real line $\mathbb{R}$. We write $l(Q)$ for the side length of Q. A (positive) measure μ on $\mathbb{C}_+$ is called a *Carleson measure* if there exists a constant $C > 0$ such that $\mu(Q) \leq Cl(Q)$ for every square Q as before.

Suppose that $\{z_n\} \subset \mathbb{C}_+$ is a *bounded* sequence and consider the measure

$$\mu = \sum_n y_n \delta_{z_n},$$

where δ_z denotes the probability measure with mass concentrated at the point z and where y_n is the imaginary part of z_n. The points z_n will be called the *localizations* of μ. It is well known [4, Ch. VII] that $\{z_n\}$ is a Blaschke sequence if and only if $\mu(\mathbb{C}_+) < \infty$, and that it is an interpolating sequence if and only if μ is a Carleson measure and $\rho(z_n, z_k) > \alpha > 0$ for all $n \neq k$ (i.e., $\{z_n\}$ is separated).

THEOREM 3.2. *There exist*:

(a) *a non-Blaschke sequence satisfying* (3.1);
(b) *a separated Blaschke sequence satisfying* (3.1) *that is not interpolating*; *and*
(c) *a w-thin sequence that is not a finite union of thin sequences.*

Proof of (a). We construct a family of closed intervals as follows: $I_{1,1} = [0, 1]$ and for each integer $n \geq 2$, we divide $I_{1,1}$ into $n!$ intervals of length $1/n!$, $I_{n,1}, \ldots, I_{n,n!}$. Denote by $Q_{n,j}$ the closed square in $\mathbb{C}_+$ with base $I_{n,j}$. The family $\mathcal{F} = \{ Q_{n,j} : n \geq 1, 1 \leq j \leq n! \}$ is a decomposition of $Q_{1,1} = [0, 1] \times (0, 1]$. For a square of the form $Q_{n,nk+1}$, with $0 \leq k \leq (n-1)! - 1$, let $z_{n,nk+1}$ denote the midpoint of the upper side. We will say that $Q_{n,nk+1}$ is a *marked* square (see Figure 1). A straightforward calculation shows that the sequence $S = \{ z_{n,nk+1} : n \geq 1, 0 \leq k \leq (n-1)! - 1 \}$ satisfies (3.1).

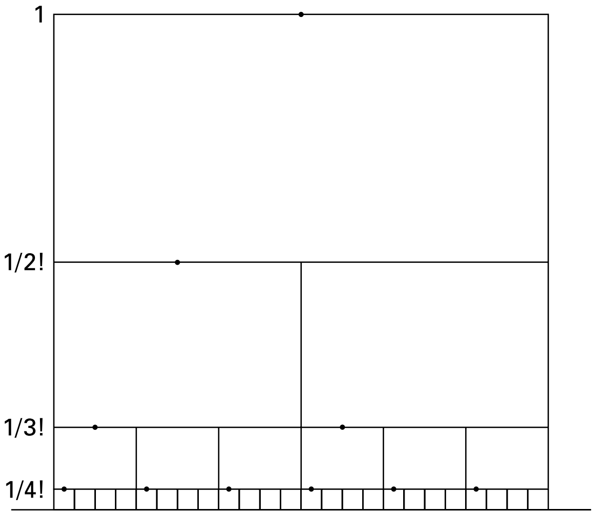

Figure 1

Let $y_{n,\,nk+1}$ be the imaginary part of $z_{n,\,nk+1}$. Then

$$\sum_{n\geq 1}\sum_{k=0}^{(n-1)!-1} y_{n,\,nk+1} = \sum_{n\geq 1}\sum_{k=0}^{(n-1)!-1}\frac{1}{n!} = \sum_{n\geq 1}\frac{1}{n} = \infty.$$

Hence, S is not a Blaschke sequence. □

We will construct the examples for (b) and (c) as suitable subsequences of S. An auxiliary result is needed. For two integers $1 \leq p \leq q$, let $\nu_{p,\,q}$ be the measure

$$\nu_{p,\,q} = \sum_{n=p}^{q}\sum_{k=0}^{(n-1)!-1} y_{n,\,nk+1}\delta_{z_{n,\,nk+1}}.$$

LEMMA 3.3. *Let R be an arbitrary square of the decomposition $\mathcal{F}$ with $l(R) = 1/n!$.*

(i) *If $n > q$ then $\nu_{p,\,q}(R) = 0$.*
(ii) *If $p \leq n \leq q$ then*

$$\nu_{p,\,q}(R) = \frac{1}{n!}\left[1 + \sum_{j=n+1}^{q}\frac{1}{j}\right]$$

when R is a marked square, and

$$\nu_{p,\,q}(R) = \frac{1}{n!}\sum_{j=n+1}^{q}\frac{1}{j}$$

when R is not a marked square. The sum $\sum_{j=n+1}^{q} 1/j$ reduces to zero if $n + 1 > q$.

(iii) *If* $n < p$ *then*

$$\nu_{p,q}(R) = \frac{1}{n!} \sum_{j=p}^{q} \frac{1}{j}.$$

Proof. If $n > q$ then R lies below the localizations of $\nu_{p,q}$, so (i) follows.

If $p \le n \le q$ and R is a marked square, then the middle point of its upper side is a localization of the measure $\nu_{p,q}$. The contribution of this point to $\nu_{p,q}(R)$ is $1/n! = l(R)$. Further, R contains $n+1$ squares of $\mathcal{F}$ of length $1/(n+1)!$, where only one of them is marked. Each one of these squares contains $n+2$ squares of $\mathcal{F}$ of length $1/(n+2)!$, where only one is marked, and so forth. Moreover, the points that contribute to $\nu_{p,q}(R)$ are only those corresponding to squares in $\mathcal{F}$ of length at least $1/q!$. Therefore

$$\begin{aligned}\nu_{p,q}(R) &= \frac{1}{n!} + \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} \\ &\quad + \frac{(n+1)(n+2)}{(n+3)!} + \cdots + \frac{(n+1)\ldots(q-1)}{q!} \\ &= \frac{1}{n!}\left[1 + \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{q}\right)\right],\end{aligned}$$

as claimed. When R is not a marked square, the summand $1/n!$ does not appear in the preceding expression.

Finally, if $n < p$ then (ii) of the lemma says that

$$\begin{aligned}\nu_{p,q}(R) &= \nu_{n,q}(R) - \nu_{n,p-1}(R) \\ &= \frac{1}{n!}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{q}\right) \\ &\quad - \frac{1}{n!}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{p-1}\right) \\ &= \frac{1}{n!}\left(\frac{1}{p} + \frac{1}{p+1} + \cdots + \frac{1}{q}\right),\end{aligned}$$

whether R is marked or not (when R is marked we must add $1/n!$ to both quantities $\nu_{n,q}(R)$ and $\nu_{n,p-1}(R)$, not affecting the difference). □

The Constructions for (b) *and* (c). Let $\{a_j\} \subset \mathbb{R}$ be an arbitrary sequence, where $a_j \ge 1$ for all j. We define inductively two sequences of positive numbers as follows,

$$p_1 = 1 \quad \text{and} \quad q_1 = \min\left\{ q : \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{q} > a_1 \right\}.$$

Supposing that the first numbers $p_1, \ldots, p_{j-1}, q_1, \ldots, q_{j-1}$ are given, we choose

$$p_j = q_{j-1} + 1 \quad \text{and} \quad q_j = \min\left\{ q : \frac{1}{p_j + 1} + \frac{1}{p_j + 2} + \cdots + \frac{1}{q} > a_j \right\}.$$

Hence,

$$1 = p_1 < q_1 < \cdots < q_{j-1} + 1 = p_j < q_j < q_j + 1 = \cdots . \tag{3.2}$$

Notice that since $(p_j + 1)^{-1} \leq 1/2$ for all $j \geq 1$, by the choice of q_j we also have

$$a_j + 1 > \frac{1}{p_j + 1} + \frac{1}{p_j + 2} + \cdots + \frac{1}{q_j} > a_j \quad \forall j. \tag{3.3}$$

For $j \geq 1$ let R_j be the first (at the left) square in $\mathcal{F}$ such that $l(R_j) = 1/p_j!$. We consider the measures $\mu_j = \chi_{R_j} \nu_{p_j, q_j}$ for $j \geq 1$ and $\mu = \sum_{j \geq 1} \mu_j$, where χ_R denotes the characteristic function of the set R. Because R_j is a marked square, by Lemma 3.3 and (3.2) we have

$$\begin{aligned} \mu(R_j) &= \mu_j(R_j) + \mu_{j+1}(R_j) + \cdots \\ &= \nu_{p_j, q_j}(R_j) + \nu_{p_{j+1}, q_{j+1}}(R_{j+1}) + \cdots \\ &= \sum_{k \geq j} \nu_{p_k, q_k}(R_k) = \sum_{k \geq j} \frac{1}{p_k!} \left[1 + \frac{1}{p_k + 1} + \frac{1}{p_k + 2} + \cdots + \frac{1}{q_k} \right]. \end{aligned}$$

Thus, by (3.3),

$$\sum_{k \geq j} \frac{1}{p_k!}(1 + a_k) < \mu(R_j) < \sum_{k \geq j} \frac{1}{p_k!}(2 + a_k). \tag{3.4}$$

Moreover, since $p_{k+1} = q_k + 1 \geq p_k + 1$, it follows that $p_{k+1}! \geq (p_k + 1)! = (p_k + 1) p_k! \geq 2 p_k!$. So, for $k > j$,

$$\frac{p_j!}{p_k!} = \frac{p_j!}{p_{j+1}!} \frac{p_{j+1}!}{p_{j+2}!} \cdots \frac{p_{k-1}!}{p_k!} \leq \frac{1}{2^{k-j}}.$$

Obviously, this also holds for $k = j$. Hence, by (3.4),

$$\frac{1}{p_j!}(1 + a_j) < \mu(R_j) < \frac{1}{p_j!} \sum_{k \geq j} \frac{1}{2^{k-j}}(2 + a_k). \tag{3.5}$$

The examples for (b) and (c) will be constructed by choosing different sequences $\{a_j\}$, and taking the localizations of the respective measures μ.

Proof of (b). Take $a_j = j$ for every $j \geq 1$, and let μ be the associated measure. Then, by (3.5),

$$\mu(\mathbb{C}_+) = \mu(R_1) \leq \sum_{k \geq 1} \frac{1}{2^{k-1}}(2 + k) < \infty.$$

Therefore, the localizations of μ form a Blaschke sequence. By (3.5),

$$\frac{\mu(R_j)}{l(R_j)} = p_j! \mu(R_j) > 1 + j \to \infty \quad \text{when} \quad j \to \infty.$$

Hence μ is not a Carleson measure, and consequently its localizations do not form an interpolating sequence. $\square$

Proof of (c). Now take $a_j = 1$ for all $j \geq 1$. We claim that the associated measure μ is Carleson. Let $Q \subset \mathbb{C}_+$ be an arbitrary square with base on $\mathbb{R}$. We can assume without loss of generality that $Q \subset [0,1] \times (0,1]$ and that $l(Q) < 1/q_1!$. Let j be the integer that satisfies $1/q_{j-1}! > l(Q) \geq 1/q_j!$. Then $\mu_1(Q) = \mu_2(Q) = \cdots = \mu_{j-1}(Q) = 0$, because the localizations of all these measures lie above Q. Therefore $\mu(Q) = \mu_j(Q) + \sum_{k \geq j+1} \mu_k(Q)$ where, by (3.5) and (3.2),

$$\begin{aligned}\sum_{k\geq j+1} \mu_k(Q) &\leq \sum_{k \geq j+1} \mu_k(\mathbb{C}_+) = \mu(R_{j+1}) \\ &< \frac{1}{p_{j+1}!} \sum_{k\geq j+1} \frac{3}{2^{k-j-1}} = \frac{6}{p_{j+1}!} = \frac{6}{(q_j+1)!} < \frac{6}{q_j!} \leq 6l(Q).\end{aligned}$$

Hence

$$\mu(Q) \leq \mu_j(Q) + 6l(Q). \tag{3.6}$$

Let n be smallest integer such that $l(Q) \geq 1/n!$. Thus,

$$q_{j-1} < n \leq q_j. \tag{3.7}$$

Clearly, no localization of ν_{p_j,q_j} corresponding to a square of length bigger than $1/n!$ can lie in Q. Suppose that Q meets s squares of $\mathcal{F}$ of length $1/n!$, say $Q_1, \ldots, Q_s$. Then, either $s \leq 2$ or $l(Q) \geq (s-2)/n!$. By Lemma 3.3(ii), (3.7), (3.2), and (3.3),

$$\begin{aligned}\mu_j(Q) &\leq \nu_{p_j,q_j}(Q) \leq \nu_{p_j,q_j}(Q_1) + \cdots + \nu_{p_j,q_j}(Q_s) \\ &\leq \frac{s}{n!}\left[1 + \sum_{k=n+1}^{q_j} \frac{1}{k}\right] \leq \frac{s}{n!}\left[1 + \sum_{k=q_{j-1}+2}^{q_j} \frac{1}{k}\right] \\ &= \frac{s}{n!}\left[1 + \sum_{k=p_j+1}^{q_j} \frac{1}{k}\right] \leq \frac{s}{n!}(1+2) = \frac{3s}{n!}.\end{aligned}$$

Now, if $s \leq 2$ then $\mu_j(Q) \leq 6/n! \leq 6l(Q)$, and if $s \geq 3$ then

$$\mu_j(Q) \leq \frac{3(s-2)}{n!} + \frac{6}{n!} \leq 3l(Q) + 6l(Q) = 9l(Q).$$

These estimates together with (3.6) show that $\mu(Q) \leq 15l(Q)$ for every square Q such that $l(Q) < 1/q_1!$. This proves our claim.

Finally, we will see that the sequence $\{z_n\}$ of localizations of μ is not a thin sequence, which together with Lemma 3.1 proves (c). By (3.5),

$$2l(R_j) = \frac{2}{p_j!} < \mu(R_j) \quad \text{for all} \quad j. \tag{3.8}$$

Let z_{n_j} be the middle point in the upper side of R_j. By construction this point belongs to the sequence $\{z_n\}$ of localizations of μ. Let b be the Blaschke product with zeroes $\{z_n\}$, and let $b^{(j)}$ be the same product with the point z_{n_j} deleted. As before, y_n denotes the imaginary part of z_n. Then, by [4, p. 288] and (3.8),

$$\begin{aligned}\log|b^{(j)}(z_{n_j})|^{-2} &\geq 4 \sum_{n:n\neq n_j} \frac{y_{n_j} y_n}{|z_{n_j} - \bar{z}_n|^2} \geq 4 \sum_{\substack{n\neq n_j \\ z_n \in R_j}} \frac{y_{n_j} y_n}{|z_{n_j} - \bar{z}_n|^2} \\ &\geq 4 \sum_{\substack{n\neq n_j \\ z_n \in R_j}} \frac{y_{n_j} y_n}{5y_{n_j}^2} = \frac{4}{5} \sum_{\substack{n\neq n_j \\ z_n \in R_j}} \frac{y_n}{y_{n_j}} = \frac{4}{5}\frac{1}{l(R_j)}[\mu(R_j) - l(R_j)] \\ &> \frac{4}{5}\frac{1}{l(R_j)}[2l(R_j) - l(R_j)] = \frac{4}{5}\end{aligned}$$

for all j. That is, $|b^{(j)}(z_{n_j})| \leq \exp(-2/5)$ for every $j \geq 1$, and then $\{z_n\}$ is not a thin sequence. □

PROPOSITION 3.4. *Let T be an interpolating sequence and let $y \in \bar{T} \setminus T$. If P is a Gleason part so that $y \in \bar{P} \setminus P$, then y is in the closure of $\bar{T} \cap P$. In particular, $\bar{T} \cap P$ is an infinite set.*

Proof. First we show that $\bar{T} \cap P \neq \emptyset$. Let $x \in G$ so that $P = P(x)$, and let b be an interpolating Blaschke product whose zero sequence is T. Consider the map $L_x : \mathbb{D} \to P(x)$ mentioned in Section 1. By [6, Lemma 1.8], $b \circ L_x(z) = B(z)g(z)$, where B is an interpolating Blaschke product (including constants of modulus 1) and g is an invertible function in H^∞. Thus, if $\bar{T} \cap P = \emptyset$ then B is constant. Therefore

$$\inf\{\, |b(\xi)| : \xi \in P \,\} = \inf\{\, |g(z)| : z \in \mathbb{D} \,\} > 0,$$

and hence $|b| > 0$ on $\bar{P}$. This is not possible because $y \in \bar{P}$ and $b(y) = 0$.

Now suppose that $y \in \bar{P} \setminus P$ is not in the closure E of $\bar{T} \cap P$. Then there is an open neighborhood V_y of y such that $\overline{V_y} \cap E = \emptyset$. Therefore, the sequence $T_1 = T \cap V_y$ contains the point y in its closure but $\bar{T}_1 \cap P = \emptyset$, contradicting the fact just proved. □

My original proof of Proposition 3.4 was more complicated than the one given here. I learned independently and almost at the same time of two different easier proofs from P. Gorkin and R. Mortini. The proof given here is a combination of their arguments.

COROLLARY 3.5. *Let S be an interpolating sequence. Then the following statements are equivalent.*

(1) *S is a w-thin sequence.*
(2) *For every Gleason part P, $\bar{P} \cap \bar{S}$ has at most one point.*
(3) *For every Gleason part P, $P \cap \bar{S}$ has at most one point.*

Proof. Suppose that S is w-thin and let $x, y \in \bar{S} \setminus S$, $x \neq y$. Then there are disjoint subsequences $S_x, S_y \subset S$ such that $x \in \bar{S}_x$ and $y \in \bar{S}_y$. Since S is w-thin, for any $0 < \rho_0 < 1$ there is a tail of S_x whose pseudohyperbolic distance to S_y is bigger than ρ_0. That is, x avoids S_y and, by Corollary 2.5, $y \notin \overline{P(x)}$. Trivially, (2) implies (3). If S is not w-thin, then $\underline{\lim}\, \rho(z_n, S \setminus \{z_n\}) = \rho_0 < 1$. Hence there are two disjoint subsequences $S_1 = \{\omega_n\}$ and $S_2 = \{\xi_n\}$ of S such that $\rho(\omega_n, \xi_n) < \rho_0^{1/2}$ for all n. If (ω_α) is a subnet of $\{\omega_n\}$ that converges to a point $x \in M(H^\infty)$, then there is a corresponding subnet (ξ_α) of $\{\xi_n\}$ and we can assume that $\xi_\alpha \to y \in M(H^\infty)$. Thus $x \neq y$ because $x \in \bar{S}_1$ and $y \in \bar{S}_2$. Since ρ is lower semicontinuous (see [7, Thm. 6.2]), $\rho(x, y) \leq \lim_\alpha \rho(\omega_\alpha, \xi_\alpha) \leq \rho_0^{1/2}$. Thus (3) does not hold. □

COROLLARY 3.6. *No w-thin part is in the closure of another Gleason part.*

Proof. Let $x \in M(H^\infty)$ be such that $P(x)$ is a w-thin part, and let $S \subset \mathbb{D}$ be a w-thin sequence such that $x \in \bar{S}$. If $P(x)$ is contained properly in the closure of another Gleason part Q, then $x \in \bar{Q} \setminus Q$. So, by Proposition 3.4, $\bar{S} \cap Q$ is an infinite set, contradicting Corollary 3.5. □

Let $S, T \subset \mathbb{D}$ be two sequences. Suppose that there are $\sigma > 0$ and an integer N such that $S \cap B_h(z_n, \sigma)$ has no more than N points for every $z_n \in T$, and such that

$$S \subset \bigcup_{z_n \in T} B_h(z_n, \sigma).$$

Then a routine argument (see [4, p. 310]) shows that whenever T is a finite union of interpolating, thin, or w-thin sequences, then so is S, respectively.

PROPOSITION 3.7. *Let S be an interpolating sequence and $K \subset \bar{S} \setminus S$ a compact set. If every point of K lies in a thin (w-thin) part, then there is an open neighborhood V of K such that $S \cap V$ is a finite union of thin (w-thin) sequences, respectively.*

Proof. By compactness we can assume that $K = \{x\}$. Let T be a thin (w-thin) sequence such that $x \in \bar{T}$, and let b be the corresponding Blaschke product. For $0 < \alpha < 1$, let $\sigma = \sigma(\alpha)$ as in Lemma 2.2. By Lemmas 2.2 and 2.3 we can also assume that $\delta(T)$ is so close to 1 so that the balls $B_h(z_n, \sigma)$, $z_n \in T$, are pairwise disjoint, and $|b(z)| > \alpha$ for all $z \notin B = \bigcup_{z_n \in T} B_h(z_n, \sigma)$. Since S is interpolating, there exists some positive number N such that S contains no more than N points in each of the balls $B_h(z_n, \sigma)$. By the comment preceding the proposition, $S \cap B$ is a finite union of thin (w-thin) sequences, respectively. Thus, $V = \{\, y \in M(H^\infty) : |b(y)| < \alpha \,\}$ is an open neighborhood of x and $S \cap V \subset S \cap B$ satisfies the proposition. □

COROLLARY 3.8. *Let $S \subset \mathbb{D}$ be an interpolating sequence. If $P(x)$ is a thin (w-thin) part for every $x \in \bar{S} \setminus S$, then S is a finite union of thin (w-thin) sequences, respectively.*

Proof. The same argument works for thin and w-thin parts, so let us assume that every point in $\bar{S} \setminus S$ lies in a thin part. By Proposition 3.7, there is an open neighborhood V of $\bar{S} \setminus S$ such that $S_1 = V \cap S$ is a finite union of thin sequences. The corollary follows because the set $S \setminus S_1$ has only finitely many points. □

COROLLARY 3.9. *There are w-thin parts* (*and hence maximal parts*) *that are not thin parts.*

Proof. Every w-thin part is maximal by Corollary 3.6. The other assertion is an immediate consequence of Theorem 3.2(c) and Corollary 3.8. □

4. Yet Another Kind of Maximal Parts

Although the next proposition is well known, I was unable to find it expressly stated in the literature, so we give here a proof. The particular case $\rho_0 = 0$ already appeared in [7, Thm. 6.1].

PROPOSITION 4.1. *Let $x, y \in G$. Then $\rho(x, y) \le \rho_0 \in [0, 1)$ if and only if, for every pair of interpolating sequences S, T such that $x \in \bar{S}$ and $y \in \bar{T}$, we have*

$$\underline{\lim}_{z_n \in S} \rho(z_n, T) \le \rho_0. \tag{4.1}$$

Notice that condition (4.1) *is symmetric in S and T.*

Proof. If (4.1) does not hold there are interpolating sequences S and T such that $x \in \bar{S}$, $y \in \bar{T}$, and $\underline{\lim}_{z_n \in S} \rho(z_n, T) > \rho_0$. So, by taking tails of S and T we can assume that $\rho(z_n, \omega_k) \ge \rho_1 > \rho_0$ for all $z_n \in S$ and $\omega_k \in T$. Let $\alpha \in (0, 1)$ such that $\alpha\rho_1 > \rho_0$, and take $\sigma(\alpha)$ and $\delta_0 = \delta(\alpha, \sigma)$ as in Lemma 2.2. By Lemma 2.3 there is a subsequence $S_1 = \{z_j\} \subset S$ such that $x \in \bar{S}_1$ and $\delta(S_1) > \delta_0$ is close enough to 1 so that the hyperbolic balls $B_h(z_j, \sigma)$ are pairwise disjoint. Hence, if $\omega_k \in T$ then there is at most one point $z_{j_k} \in S_1$ such that $\omega_k \in B_h(z_{j_k}, \sigma)$. Let b be the Blaschke product with zero sequence S_1. Suppose first that there exists a point z_{j_k} as before and write b_{j_k} for the Blaschke product b with the zero z_{j_k} deleted. By Lemma 2.2, we have

$$|b(\omega_k)| = |b_{j_k}(\omega_k)|\rho(z_{j_k}, \omega_k) > \alpha\rho_1.$$

If there is no point z_{j_k} then $|b(\omega_k)| > \alpha > \alpha\rho_1$. Since $\omega_k \in T$ is arbitrary, it follows that $\inf_{\omega \in T} |b(\omega)| \ge \alpha\rho_1 > \rho_0$. Thus $b(x) = 0$ and $|b(y)| > \rho_0$, implying that $\rho(x, y) > \rho_0$.

Now suppose that $\rho(x, y) = \rho_1 > \rho_0$ and take $0 < \varepsilon < \rho_1$. Then there is $f \in H^\infty$, $\|f\| \le 1$, such that $f(x) = 0$ and $|f(y)| > \rho_1 - \varepsilon/4$. Thus, if S and T are interpolating sequences whose closures contain the points x and y, respectively, then x is in the closure of $S(\varepsilon) = \{ z_n \in S : |f(z_n)| < \varepsilon/2 \}$ and y is in the closure of $T(\varepsilon) = \{ \omega_k \in T : |f(\omega_k)| > \rho_1 - \varepsilon/2 \}$. The SP inequality then gives

$$\rho(z_n, \omega_k) \ge \rho(f(z_n), f(\omega_k)) \ge \frac{|f(\omega_k)| - |f(z_n)|}{1 - |f(\omega_k)||f(z_n)|} > \rho_1 - \varepsilon$$

for all $z_n \in S(\varepsilon)$ and $\omega_k \in T(\varepsilon)$ (see [4, p. 4] for the second inequality). Hence, (4.1) does not hold for $S(\varepsilon)$ and $T(\varepsilon)$ if ε is small enough. □

It seems natural to conjecture that every maximal part is a w-thin part. Our next example in H^∞ of $\mathbb{C}_+$ shows the existence of maximal parts that are not w-thin. For $n \geq 1$ let

$$S_n = \left\{ z_{n,k} = \frac{1}{2^n} + \frac{i}{2^{nk}} : k \geq 2 \right\},$$

$T_N = \bigcup_{n \geq N} S_n$, and $S = T_1$. The following list of properties can be verified by direct (though tedious) computation.

(1) S is separated.
(2) $\mu = \sum_{n,k} y_{n,k} \delta_{z_{n,k}}$ is a Carleson measure.
(3) Given $0 < \alpha < 1$, there exists an $N = N(\alpha)$ such that

$$\inf_{z \in T_N} \rho(z, T_N \setminus \{z\}) > \alpha.$$

(4) Given $0 < \beta < 1$, for every $N \geq 2$ there is a $k_0 = k_0(\beta, N) \geq 2$ such that $\rho(z_{n,k}, T_N) > \beta$ for all $n \leq N - 1$ and $k \geq k_0$.
(5) $\rho(z_{n,k}, z_{n,k+1}) = (2^n - 1)/(2^n + 1)$ for all $k \geq 2$ and all $n \geq 1$.

By (1) and (2), S is an interpolating sequence. Put $K_N = \bar{T}_N$ and $K = \bigcap_{N \geq 1} K_N$. The property of finite intersection then implies that K is a nonempty compact set in $M(H^\infty(\mathbb{C}_+)) \setminus \mathbb{C}_+$.

Claim 1: If $y \in K$ then $P(y)$ is maximal. Otherwise there is some part Q such that $y \in \bar{Q} \setminus Q$. By Proposition 3.4, the set $\bar{S} \cap Q$ has infinitely many points and there are $x_1, x_2 \in \bar{S} \cap Q$ such that $\rho(x_1, x_2) = \alpha$ for some $0 < \alpha < 1$. If $L \subset S$ is an arbitrary subsequence whose closure contains x_1 and x_2, then Proposition 4.1 implies that $\underline{\lim}_{z \in L} \rho(z, L \setminus \{z\}) \leq \alpha$. Thus, taking $N = N(\alpha)$ as in (3), we have that at least one of the points x_1 or x_2, say x_1, is not in $\bar{T}_N$.

On the other hand, by (4) every point in $(\overline{S \setminus T_N}) \setminus (S \setminus T_N)$ avoids T_N (it is enough to take tails of $S \setminus T_N$). Since $y \in \bar{T}_N$ and $x_1 \in \overline{S \setminus T_N}$, Corollary 2.5 says that $y \notin \overline{P(x_1)} = \bar{Q}$.

Claim 2: There exists a point $y \in K$ that is not in a w-thin part. If every point of K belongs to a w-thin part, by Proposition 3.7 there exists some open neighborhood V of K such that $V \cap S$ is a finite union of w-thin sequences. Because $K = \bigcap_{N \geq 1} K_N$, where $\{K_N\}$ is a decreasing sequence of compact sets, there is some N_0 such that $K_{N_0} \subset V$. Therefore, $T_{N_0} \subset V \cap S$ must be a finite union of w-thin sequences. But $S_{N_0} \subset T_{N_0}$ is not a finite union of w-thin sequences, by property (5). □

5. Open Problems

PROBLEM 1. We already saw that $K(x) = \{x\}$ for every x in the Shilov boundary $S(H^\infty)$. Is $K(x) \neq \{x\}$ for every $x \in M(H^\infty) \setminus (\mathbb{D} \cup S(H^\infty))$? This question is equivalent to a problem posed by Tolokonnikov [9, p. 139].

PROBLEM 2. Is there a reasonable characterization of maximal Gleason parts? What about maximal closures of parts?

Problem 3. If $x \in G$, is $K(x)$ the closure of a Gleason part? An affirmative answer means that every $x \in G$ is in some maximal closure of Gleason part.

In [1], Alling conjectured that every nontrivial, nonmaximal closed prime ideal of H^∞ is formed by the functions that vanish identically on a given nontrivial Gleason part (see [8] for an English exposition and further information). This conjecture is related to our problem in the following way. Suppose that b is an interpolating Blaschke product with zero sequence S. We can take tails of S, $S \supset S_1 \supset S_2 \supset \cdots$ such that $\sum_{1\leq j}\sum_{z_k\in S_j}(1-|z_k|) < \infty$. If b_j denotes the Blaschke product with zero sequence S_j, then $b_0 = \prod_{j\geq 1} b_j$ converges and $b_0 \equiv 0$ on every Gleason part P such that $\bar{S} \cap P \neq \emptyset$. If $x \in \bar{S} \setminus S$ then Proposition 3.4 says that $\bar{S} \cap P \neq \emptyset$ for every part P such that $x \in \bar{P}$. This means that $b_0 \equiv 0$ on the whole class $K(x)$. Let I be the ideal of H^∞ defined by

$$I = \{\, f \in H^\infty : f \equiv 0 \text{ on } K(x)\,\}.$$

Since $b_0 \in I$, the ideal is not trivial. Clearly, it is closed and nonmaximal. In addition, I is prime. In fact, if $fg \in I$ and P is a Gleason part such that $x \in \bar{P}$, then f or g vanishes identically on P (because functions in H^∞ behave as analytic functions on P). So, say that $f \equiv 0$ and $g \not\equiv 0$ on P. Then, for every part Q such that $P \subset \bar{Q}$, we have $g \not\equiv 0$ on Q and consequently $f \equiv 0$ on Q; that is, $f \in I$. Suppose that Alling's conjecture is true. Then the set

$$\operatorname{hull} I = \{\, y \in M(H^\infty) : f(y) = 0 \text{ for all } f \in I\,\}$$

has the form $\bar{P}$ for some Gleason part P. Since $x \in K(x) \subset \bar{P}$, we have $\bar{P} \subset K(x)$. Hence $\bar{P} = K(x)$ and so $\bar{P}$ is a maximal closure of part. Thus, an affirmative answer to Alling's conjecture implies an affirmative answer to Problem 3. Although we have no valid argument to support the converse of this implication, it is very likely that Problem 3 (or a variant of it) is one of the main obstacles in proving Alling's conjecture. In addition, we see that if the conjecture holds then I is the intersection of all the closed prime nonmaximal and nontrivial ideals J such that $x \in \operatorname{hull} J$. That is, I is the minimum of such ideals.

Added in proof. After I finished writing this paper I received the preprint *Trivial points in the maximal ideal space of H^∞* by T. Ishii and K. Izuchi, where they show that Problem 1 has a negative answer.

References

[1] N. Alling, Aufgabe 2.3, Jahresbericht DMV 73 (1971/72), no. 2, 2.

[2] P. Budde, *Support sets and Gleason parts of $M(H^\infty)$*, Thesis, Univ. of California, Berkeley, 1982.

[3] L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. 76 (1962), 547–559.

[4] J. B. Garnett, *Bounded analytic functions,* Academic Press, New York, 1981.

[5] P. Gorkin, *Gleason parts and COP,* J. Funct. Anal. 83 (1989), 44–49.

[6] P. Gorkin, H. M. Lingenberg, and R. Mortini, *Homeomorphic disks in the spectrum of* H^∞, Indiana Univ. Math. J. 39 (1990), 961–983.
[7] K. Hoffman, *Bounded analytic functions and Gleason parts,* Ann. of Math. (2) 86 (1967), 74–111.
[8] R. Mortini, *Gleason parts and prime ideals in* H^∞, Lecture Notes in Math., 1573, 136–138, Springer, Berlin, 1994.
[9] V. Tolokonnikov, *Banach algebras of analytic functions,* Lecture Notes in Math., 1573, 139–144, Springer, Berlin, 1994.

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