The Hardy Class of Kœnigs Maps

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1. Introduction

An analytic function ϕ on $\mathbb{D}$ such that $\phi(\mathbb{D}) \subset \mathbb{D}$ induces a linear (composition) operator $C_\phi(f) = f \circ \phi$ on the functions f defined on $\mathbb{D}$. The operator C_ϕ is bounded on the Hilbert space $H^2(\mathbb{D})$ of analytic functions f on $\mathbb{D}$ such that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty \tag{1.1}$$

In this article we restrict our attention to the case when ϕ fixes a point in the disk and has nonzero derivative there. Without loss of generality we can conjugate such a fixed point to the origin with a Möbius transformation; to avoid trivial situations, we assume that ϕ is not an automorphism of $\mathbb{D}$.

Definition 1.1. We consider the family $\mathcal{A}$ of functions ϕ that are analytic on $\mathbb{D}$, with $\phi(\mathbb{D}) \subset \mathbb{D}$, $\phi(0) = 0$, and $0 < |\phi'(0)| < 1$.

For each function $\phi \in \mathcal{A}$, Kœnigs's theorem (see Proposition on p. 91 of [Sh2]) provides a function σ analytic on $\mathbb{D}$, with $\sigma(0) = 0$ and $\sigma'(0) \neq 0$, that solves Schröder's equation:

$$\sigma \circ \phi(z) = \lambda \sigma(z) \qquad \forall z \in \mathbb{D} \tag{1.2}$$

with $\lambda = \phi'(0)$.

Near the origin, $\phi \sim \lambda z$ and σ conjugates ϕ to λz (since σ is one-to-one in a neighborhood of zero). However, σ is defined in the whole unit disk and, by (1.2), it intertwines the action of ϕ on $\mathbb{D}$ with multiplication by λ on $\mathbb{C}$. Therefore, the growth of σ near $\partial\mathbb{D}$ should encapsulate the repelling properties of ϕ near $\partial\mathbb{D}$.

We first determine how we are going to measure the growth of σ. Recall that, for each $p > 0$, one defines the Hardy space $H^p(\mathbb{D})$ of analytic functions on $\mathbb{D}$ satisfying a growth condition as in (1.1) by replacing $|f(re^{i\theta})|^2$ with $|f(re^{i\theta})|^p$, and the Nevanlinna class $\mathcal{N}(\mathbb{D})$ by replacing $|f(re^{i\theta})|^2$ with $\log^+|f(re^{i\theta})|$. Then $H^{p_1}(\mathbb{D}) \subset H^{p_2}(\mathbb{D})$ for $p_1 > p_2$, and $\bigcup_{p>0} H^p(\mathbb{D})$ is strictly contained in $\mathcal{N}(\mathbb{D})$. For every analytic function f on $\mathbb{D}$ we set $h(f) = \sup\{p > 0 : f \in H^p(\mathbb{D})\} \in [0, \infty]$ and call it the *Hardy number* of f. Clearly, $f \in H^p(\mathbb{D})$ when $0 < p < h(f)$ and $f \notin H^p(\mathbb{D})$ when $h(f) < p < \infty$; however, at issue is determining

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what happens when $p = h(f) \in (0, \infty)$. In a sense, the smaller $h(f)$ is, the faster f grows. For instance, if f_θ is the Riemann map of $\mathbb{D}$ onto a straight sector of opening θ, then $h(f_\theta) = \pi/\theta$. In this case it is also well known that $f_\theta \notin H^p(\mathbb{D})$ for $p = \pi/\theta$. We will show that Kœnigs maps behave very much like the f_θ.

On the other hand, the behavior of ϕ near $\partial\mathbb{D}$ will be described by the essential spectral radius $r_e(C_\phi)$ of C_ϕ acting on $H^2(\mathbb{D})$. Recall that

$$r_e(C_\phi) = \lim_{n\to\infty} \|C_\phi^n\|_e^{1/n}, \tag{1.3}$$

where $\|\cdot\|_e$ denotes essential norm—that is, $\|\cdot\|_e$ is the distance in the operator norm to the space of compact operators on $H^2(\mathbb{D})$. Notice that $C_\phi^n = C_{\phi_n}$, where $\phi_n = \phi \circ \cdots \circ \phi$ is the nth-iterate of ϕ, and recall that Shapiro has obtained [Sh1] a formula for $\|C_\phi\|_e$ in terms of the Nevanlinna counting function N_ϕ of ϕ. Shapiro's formula states that

$$\|C_\phi\|_e^2 = \limsup_{|w|\to 1} \frac{N_\phi(w)}{\log(1/|w|)}, \tag{1.4}$$

where $N_\phi(w)$ is defined to be $\sum_j \log(1/|z_j|)$ and $\{z_j\} = \phi^{-1}(w)$, listed with multiplicity, for all $w \in \phi(\mathbb{D}) \setminus \{0\}$. Therefore, $r_e(C_\phi)$ measures the behavior near $\partial\mathbb{D}$ of the Nevanlinna counting function of an iterate ϕ_n of ϕ as n becomes large.

In this article we will show that

$$r_e(C_\phi) = |\lambda|^{h(\sigma)/2} \tag{1.5}$$

for any $\phi \in \mathcal{A}$.

One direction of (1.5) was proved recently by Bourdon and Shapiro in [BS]. Namely, they prove that $r_e(C_\phi) \geq |\lambda|^{h(\sigma)/2}$. Moreover, they show that if $\phi \in \mathcal{A}$ is inner then $\sigma \notin \mathcal{N}(\mathbb{D})$, hence $h(\sigma) = 0$ and $r_e(C_\phi) = 1$. In [BS], the authors obtain $r_e(C_\phi) \leq |\lambda|^{h(\sigma)/2}$ when ϕ extends to be analytic in a neighborhood of $\partial\mathbb{D}$, and ask if this inequality holds for general maps $\phi \in \mathcal{A}$. At almost the same time, in [P-C1] we showed that $r_e(C_\phi) \leq |\lambda|^{h(\sigma)/2}$ under the hypothesis that ϕ is univalent, without any smoothness assumption about ϕ on $\partial\mathbb{D}$. In the present paper, we drop the univalence condition and extend our previous result to every $\phi \in \mathcal{A}$. Moreover, we show that when ϕ is not inner then $h(\sigma) > 0$, that is, $r_e(C_\phi) < 1$. The property of ϕ being inner thus provides a dichotomy for the mean growth behavior of the Kœnigs map σ.

QUESTION 1.2. Is it possible to relate $h(\sigma)$ to a quantity that describes the mapping properties of ϕ alone, without having to consider the whole sequence of iterates ϕ_n?

In the particular case when ϕ is univalent, this question was settled in [P-C2]. Namely, we studied the dynamics of ϕ near $\partial\mathbb{D}$ and computed $h(\sigma)$ in terms of the smallest product of angular derivatives evaluated along the boundary-repelling cycles of ϕ. In particular, the case where ϕ has no boundary-repelling cycles corresponded to $h(\sigma) = \infty$. However, for nonunivalent functions, angular derivatives are not the right object to look at because there are examples of infinite Blaschke products, hence with $h(\sigma) = 0$, that have no angular derivatives (see [Sh2, Sec. 10.2]).

2. Statement of Results

The fact that σ satisfies the functional equation (1.2) makes it possible to estimate the Hardy number of σ in terms of the harmonic measure of the level sets of σ, and to establish that $\sigma \notin H^p(\mathbb{D})$ when $p = h(\sigma) \in (0, \infty)$. We have gathered the properties of σ in Theorem 2.1. If z_0 is a point in a region $\Omega \subset \mathbb{C}$ and F is a Borel subset of $\partial\Omega$, we let $\omega(z_0, F, \Omega)$ denote the harmonic measure of F in Ω at the point z_0 (see Section 3 for a precise definition).

THEOREM 2.1. *Let ϕ be an analytic function on $\mathbb{D}$, with $\phi(\mathbb{D}) \subset \mathbb{D}$, $\phi(0) = 0$, and $\phi'(0) = \lambda \neq 0$, and let σ be the Kœnigs function of ϕ. Let $h(\sigma) = \sup\{\, p > 0 : \sigma \in H^p(\mathbb{D}) \,\}$ be the Hardy number of σ. For $\alpha > 0$, let Ω_α be the component of $\{\, z \in \mathbb{D} : |\sigma(z)| < \alpha \,\}$ containing the origin, let $F_\alpha = \partial\Omega_\alpha \cap \mathbb{D}$, and let $\omega_\alpha = \omega(0, F_\alpha, \Omega_\alpha)$. Then:*

(i) *the limit $\mu(\sigma) = \lim_{\alpha\to\infty}(\log(1/\omega_\alpha)/\log\alpha)$ exists in $[0, \infty]$;*
(ii) *$h(\sigma) = \mu(\sigma)$; and*
(iii) *if $\mu(\sigma) \in (0, \infty)$ then $\sigma \notin H^p(\mathbb{D})$ for $p = \mu(\sigma)$.*

Moreover, if ϕ is not an inner function then $\mu(\sigma) > 0$.

Using this description, we then obtain the following theorem.

THEOREM 2.2. *Let ϕ, λ, and $h(\sigma)$ be defined as in Theorem 2.1. Let C_ϕ be the composition operator induced by ϕ on the Hardy space $H^2(\mathbb{D})$, and let $r_e(C_\phi)$ denote the essential spectral radius of C_ϕ. Then, if ϕ is not an inner function,*

$$r_e(C_\phi) \le |\lambda|^{h(\sigma)/2} < 1.$$

The next corollary follows from Theorem 2.2 and the results of [BS].

COROLLARY 2.3. *In the notation of Theorem 2.2, $r_e(C_\phi) = |\lambda|^{h(\sigma)/2}$.*

The next corollary follows from Theorem 2.1 and Theorem 2.2 of [Du].

COROLLARY 2.4. *Let ϕ and σ be as in Theorem 2.1. If ϕ is not inner, then σ has nontangential limits almost everywhere on $\partial\mathbb{D}$.*

The working tools throughout this paper are the level sets F_α of σ as defined in Theorem 2.1. In Section 3, we prove some technical lemmas about the mapping properties of ϕ with respect to the sets F_α. In Section 4, we first prove Theorem 2.2 using Proposition 4.1, which estimates the essential norms of C_ϕ^n in terms of the harmonic measure of the sets F_α. We then relate the Nevanlinna counting function of ϕ to harmonic measure, and prove Proposition 4.1. In Section 5 we relate the Hardy class of an arbitrary analytic function on $\mathbb{D}$ to the behavior of the harmonic measure of its level sets. In Section 6 we show that, with respect to their level sets, Kœnigs maps behave like the Riemann map of $\mathbb{D}$ onto a straight sector, and finally we prove Theorem 2.1.

3. A Different Form of Schwarz's Lemma

We first need to collect some preliminary facts about the level sets F_α and the sets Ω_α defined in Theorem 2.1. Recall that, for $\alpha > 0$, Ω_α is the component of $\{ z \in \mathbb{D} : |\sigma(z)| < \alpha \}$ that contains the origin and F_α is the relative boundary of Ω_α in $\mathbb{D}$. Thus $\{\Omega_\alpha\}$ is an increasing family of nonempty (because $\sigma(0) = 0$) simply connected regions, the sets F_α are disjoint for different α, and if $\alpha_1 < \alpha_2 < \alpha_3$ then F_{α_2} separates F_{α_1} from F_{α_3} in Ω_{α_3}. Observe that, since σ satisfies Schröder's equation (1.2), the following properties hold for all $\alpha > 0$:

$$\phi(\Omega_\alpha) \subset \Omega_{|\lambda|\alpha} \quad \text{and} \quad \phi(F_\alpha) \subset F_{|\lambda|\alpha}. \tag{3.1}$$

In fact, if $z \in \Omega_\alpha$ then there is a path $\gamma \subset \Omega_\alpha$ connecting 0 to z; by (1.2), the path $\phi(\gamma)$ connects 0 to $\phi(z)$ and is contained in the set $\{ z \in \mathbb{D} : |\sigma(z)| < |\lambda|\alpha \}$. Hence $\phi(z) \in \Omega_{|\lambda|\alpha}$. A similar argument yields the statement about F_α.

If E is a closed subset of $\mathbb{D}$ we assume that $\omega(z, E, \mathbb{D} \setminus E)$, the harmonic measure of E at z in the region $\mathbb{D} \setminus E$, is the Perron solution of the Dirichlet problem in $\mathbb{D} \setminus E$ with boundary data equal to the characteristic function χ_E of the set E. Therefore, $\omega(z, E, \mathbb{D} \setminus E)$ is the harmonic function obtained as the supremum of all the subharmonic functions $v(z)$ in $\mathbb{D} \setminus E$ that satisfy $\limsup_{z\to\zeta} v(z) \le \chi_E(\zeta)$ for every $\zeta \in \partial(\mathbb{D} \setminus E)$ (such v are often called "candidates" for the Perron solution). We refer the reader to [Co, p. 266] for more information on Perron's method. Also, recall that the hyperbolic distance between two points $a, b \in \mathbb{D}$ is defined by:

$$\rho_{\mathbb{D}}(a, b) = \log \frac{1 + \left|\dfrac{b-a}{1-\bar{b}a}\right|}{1 - \left|\dfrac{b-a}{1-\bar{b}a}\right|}.$$

Suppose E is a closed set in $\mathbb{D}\setminus\Omega_\alpha$ for some $\alpha > 0$, so that F_α separates E from 0 in $\mathbb{D}$. Then, by (3.1), $F_{\alpha/|\lambda|}$ separates $\phi^{-1}(E)$ from 0 in $\mathbb{D}$ and, by the invariant form of Schwarz's lemma (Theorem I.4.1 of [CG]), $\rho_{\mathbb{D}}(F_\alpha, E) \le \rho_{\mathbb{D}}(F_{\alpha/|\lambda|}, \phi^{-1}(E))$. We claim that a similar inequality holds if we use harmonic measure instead of hyperbolic distance. Hence, we think of the following lemma as a different form of Schwarz's lemma. For simplicity we restrict ourselves to nice sets E.

LEMMA 3.1. *Let E be a closed set in $\mathbb{D}$. Let $u(z) = \omega(z, E, \mathbb{D} \setminus E)$ and $\tilde{u}(z) = \omega(z, \phi^{-1}(E), \mathbb{D} \setminus \phi^{-1}(E))$. Suppose also that $\partial(\mathbb{D} \setminus E) \cap E$ is locally an analytic arc. If $E \cap \Omega_\alpha = \emptyset$ for some $\alpha > 0$, then $\phi^{-1}(E) \cap \Omega_{\alpha/|\lambda|} = \emptyset$ and*

$$\sup_{\zeta \in F_\alpha} u(\zeta) \ge \sup_{\xi \in F_{\alpha/|\lambda|}} \tilde{u}(\xi).$$

Proof. Let $\tilde{v}$ be a candidate for $\tilde{u}$ on $\mathbb{D} \setminus \phi^{-1}(E)$. Note that $\phi(\mathbb{D} \setminus \phi^{-1}(E)) \subset \mathbb{D} \setminus E$, so $\tilde{v} - u \circ \phi$ is subharmonic on $\mathbb{D} \setminus \phi^{-1}(E)$. For $z \in \mathbb{D} \setminus \phi^{-1}(E)$ and $\zeta \in \partial(\mathbb{D} \setminus \phi^{-1}(E))$,

$$\limsup_{z\to\zeta} \tilde{v}(z) - u \circ \phi(z) \le 0. \tag{3.2}$$

In fact, when $\zeta \in \partial\mathbb{D}$, (3.2) holds because $\limsup_{z\to\zeta} \tilde{v}(z) \le 0$ and $u \circ \phi(z) \ge 0$. For $\zeta \in \phi^{-1}(E)$, we have $\limsup_{z\to\zeta} \tilde{v}(z) \le 1$ by definition, but on the other hand $\lim_{z\to\zeta} u \circ \phi(z) = 1$, because $\phi(\zeta) \in E$ and—since $\partial(\mathbb{D} \setminus E) \cap E$ is locally an analytic arc—there is a barrier for $\mathbb{D} \setminus E$ at $\phi(\zeta)$ (see [Co, pp. 269–271]).

Thus, by the maximum principle for subharmonic functions, (3.2) implies that $\tilde{v} \le u \circ \phi$ and, since $\tilde{v}$ is arbitrary, $\tilde{u} \le u \circ \phi$ on $\mathbb{D} \setminus \phi^{-1}(E)$. Then, by (3.1), $\phi^{-1}(E) \cap \Omega_{\alpha/|\lambda|} = \emptyset$. Thus

$$\begin{aligned}\sup_{\xi\in F_{\alpha/|\lambda|}} \tilde{u}(\xi) &\le \sup_{\xi\in F_{\alpha/|\lambda|}} u(\phi(\xi)) \\ &\le \sup_{\zeta\in\phi(F_{\alpha/|\lambda|})} u(\zeta) \\ &\le \sup_{\zeta\in F_\alpha} u(\zeta),\end{aligned}$$

where the last inequality follows from (3.1). □

Lemma 3.1 will be used in conjunction with another maximum principle trick known as "the conditional probability estimate", which we now state and prove for convenience.

LEMMA 3.2. *Suppose E is a closed set in $\mathbb{D} \setminus \Omega_\alpha$ for some $\alpha > 0$, and let $\omega_\alpha = \omega(0, F_\alpha, \Omega_\alpha)$. Then*

$$\omega(0, E, \mathbb{D} \setminus E) \le \omega_\alpha \sup_{\zeta\in F_\alpha} \omega(\zeta, E, \mathbb{D} \setminus E).$$

Proof. Let $u(z) = \omega(z, F_\alpha, \Omega_\alpha)$ for $z \in \Omega_\alpha$. Let v be a candidate for $\omega(z, E, \mathbb{D} \setminus E)$ in $\mathbb{D} \setminus E$, and let $S = \sup_{\zeta\in F_\alpha} \omega(\zeta, E, \mathbb{D} \setminus E)$. Then, for $z \in \Omega_\alpha$ and $\zeta \in \partial\Omega_\alpha$,

$$\limsup_{z\to\zeta} v(z) - Su(z) \le 0.$$

In fact, if $\zeta \in \partial\mathbb{D}$, then $\limsup_{z\to\zeta} v(z) \le 0$ and $Su(z) \ge 0$. On the other hand, if $\zeta \in F_\alpha$, then $\limsup_{z\to\zeta} v(z) \le S$ and $\lim_{z\to\zeta} Su(z) = S$ because F_α is locally an analytic arc. Therefore, by the maximum principle for subharmonic functions, $v \le Su$ on Ω_α. Since v is arbitrary, Lemma 3.2 follows by evaluating at zero. □

4. The Nevanlinna Counting Function and Harmonic Measure

The following key estimate relates the essential norm of C_ϕ^n to harmonic measure.

PROPOSITION 4.1. *For $\alpha > 0$, let ω_α be defined as in Theorem 2.1. Consider the sequence $\alpha_n = |\lambda|^{-n}$, $n = 1, 2, \dots$, which tends to infinity. Then there exists a constant $C_0 > 0$ depending only on ϕ such that, for all $n \ge 1$,*

$$\|C_{\phi_n}\|_e^2 \le C_0 \omega_{\alpha_n}.$$

We will prove Proposition 4.1 at the end of this section, but first we show how, together with Theorem 2.1, it implies Theorem 2.2.

Proof of Theorem 2.2. By Proposition 4.1,

$$\|C_{\phi_n}\|_e^{2/n} \leq C_0^{1/n} \exp\{-\log(1/\omega_{\alpha_n})/n\}.$$

Letting n tend to infinity and observing that $\log \alpha_n = n \log(1/|\lambda|)$, by Theorem 2.1(i) and (ii) we obtain that

$$r_e(C_\phi) \leq \exp\{-\log(1/|\lambda|)\mu(\sigma)/2\} = |\lambda|^{h(\sigma)/2}. \qquad \square$$

In order to prove Proposition 4.1 we will use (1.4) and estimate the Nevanlinna counting function in terms of harmonic measure.

Because $\sigma'(0) \neq 0$, it follows that σ is one-to-one on some disk $s\mathbb{D}$ with $0 < s < 1$. To simplify the notation, in the following we will always assume, without loss of generality, that $2\mathbb{D} \subset \sigma(s\mathbb{D})$. In fact, we can just multiply σ by a large enough constant; then (1.2) still holds and $h(\sigma)$ is unchanged. Hence,

$$\overline{\Omega_1} \subset s\mathbb{D}. \tag{4.1}$$

Let $\delta > 0$ be chosen so that the hyperbolic disk of radius δ centered at 0 is $s\bar{\mathbb{D}}$, that is, $\delta = \log(1+s)/(1-s)$. For every $w \in \mathbb{D} \setminus s\bar{\mathbb{D}}$, we let Δ_w denote the closed hyperbolic disk of radius δ centered at w. Then

$$\omega(0, \Delta_w, \mathbb{D} \setminus \Delta_w) = \omega(w, s\bar{\mathbb{D}}, \mathbb{D} \setminus s\bar{\mathbb{D}}) = \frac{\log(1/|w|)}{\log(1/s)}. \tag{4.2}$$

The next lemma estimates the Nevanlinna counting function $N_\phi(w)$ in terms of the harmonic measure of $\phi^{-1}(\Delta_w)$ at 0.

LEMMA 4.2. *Let $\phi \in \mathcal{A}$. For every $w \in \phi(\mathbb{D}) \setminus s\bar{\mathbb{D}}$, let $\{z_j\}_1^\infty = \phi^{-1}(w)$ and let $N_\phi(w) = \sum_{j=1}^\infty \log(1/|z_j|)$. Then*

$$N_\phi(w) \leq \log(1/s)\omega\big(0, \phi^{-1}(\Delta_w), \mathbb{D} \setminus \phi^{-1}(\Delta_w)\big)$$

Proof. Let M be a Möbius transformation of $\mathbb{D}$ that sends w to 0, and set $\psi = M \circ \phi$. Then $\{z_j\}_1^\infty = \psi^{-1}(0)$ and $\psi(\mathbb{D}) \subset \mathbb{D}$. Fix an integer $n \geq 1$. By the maximum modulus principle, for all $z \in \mathbb{D}$ we have

$$|\psi(z)| < \prod_{j=1}^{n} \left| \frac{z_j - z}{1 - \overline{z_j} z} \right|. \tag{4.3}$$

Consider the finite Blaschke product

$$B_n(z) = \prod_{j=1}^{n} \frac{z_j - z}{1 - \overline{z_j} z}.$$

Then, by (4.3) and since $M(\Delta_w) = s\bar{\mathbb{D}}$,

$$B_n^{-1}(s\bar{\mathbb{D}}) \subset \psi^{-1}(s\bar{\mathbb{D}}) = \phi^{-1}(\Delta_w).$$

Now $\omega\big(z, B_n^{-1}(s\bar{\mathbb{D}}), \mathbb{D}\setminus B_n^{-1}(s\bar{\mathbb{D}})\big) = \log(1/|B_n(z)|)/\log(1/s)$. Thus, for $z\in\mathbb{D}\setminus \phi^{-1}(\Delta_w)$,

$$\frac{\log(1/|B_n(z)|)}{\log(1/s)} \le \omega\big(z, \phi^{-1}(\Delta_w), \mathbb{D}\setminus\phi^{-1}(\Delta_w)\big).$$

Evaluating at the origin and multiplying both sides by $\log(1/s)$, we obtain

$$\sum_{j=1}^{n}\log\frac{1}{|z_j|} \le \log\Big(\frac{1}{s}\Big)\omega\big(0, \phi^{-1}(\Delta_w), \mathbb{D}\setminus\phi^{-1}(\Delta_w)\big).$$

Lemma 4.2 follows by letting n tend to infinity. □

Proof of Proposition 4.1. Note that, if $\|C_{\phi_n}\|_e = 0$ for some $n \ge 1$, then the inequality we need to prove is trivially satisfied. So, fix an integer $n \ge 1$ and assume that $\|C_{\phi_n}\|_e > 0$. By (1.4) we can choose a sequence $\{w_m\}_{m=1}^{\infty} \subset \phi_n(\mathbb{D})$ such that $|w_m|$ tends to unity and $N_{\phi_n}(w_m)/\log(1/|w_m|)$ tends to $\|C_{\phi_n}\|_e^2$ as m tends to infinity. Since $|w_m| \to 1$, we can assume that $\Delta_{w_m}\cap s\bar{\mathbb{D}} = \emptyset$ for all $m \ge 1$, where s is the same as in (4.1). By Lemma 4.2, $N_{\phi_n}(w_m)/\log(1/s)$ is less than

$$\omega\big(0, \phi_n^{-1}(\Delta_{w_m}), \mathbb{D}\setminus\phi_n^{-1}(\Delta_{w_m})\big). \tag{4.4}$$

Because $\Delta_{w_m}\cap\Omega_1 = \emptyset$, by (3.1) we have $\phi_n^{-1}(\Delta_{w_m})\cap\Omega_{\alpha_n} = \emptyset$ for all $m \ge 1$. Fix an integer $m \ge 1$. By Lemma 3.2 with E replaced by $\phi_n^{-1}(\Delta_{w_m})$ and α by α_n, (4.4) is less than

$$\omega_{\alpha_n}\sup_{\zeta\in F_{\alpha_n}}\omega\big(\zeta, \phi_n^{-1}(\Delta_{w_m}), \mathbb{D}\setminus\phi_n^{-1}(\Delta_{w_m})\big). \tag{4.5}$$

Therefore, by Lemma 3.1 with E replaced by Δ_{w_m}, α by 1, and ϕ by ϕ_n, (4.5) is less than

$$\omega_{\alpha_n}\sup_{\zeta\in F_1}\omega(\zeta, \Delta_{w_m}, \mathbb{D}\setminus\Delta_{w_m}). \tag{4.6}$$

Since $F_1 = \partial\Omega_1 \subset s\mathbb{D}$, by Harnack's inequality there is a constant $C_0 \ge 1$ depending only on $\overline{\Omega_1}$ and s such that

$$\sup_{\zeta\in F_1}\omega(\zeta, \Delta_{w_m}, \mathbb{D}\setminus\Delta_{w_m}) \le C_0\omega(0, \Delta_{w_m}, \mathbb{D}\setminus\Delta_{w_m}).$$

Finally, by (4.2), $\omega(0, \Delta_{w_m}, \mathbb{D}\setminus\Delta_{w_m}) = \log(1/|w_m|)/\log(1/s)$. Thus, the term $\log(1/s)$ cancels out and, dividing both sides by $\log(1/|w_m|)$, we obtain

$$\frac{N_{\phi_n}(w_m)}{\log(1/|w_m|)} \le C_0\omega_{\alpha_n}.$$

Letting m tend to infinity, we obtain Proposition 4.1. □

Now it remains to prove Theorem 2.1, which we do in the next two sections.

5. The Hardy Class of Analytic Functions and Harmonic Measure

The next proposition provides a characterization of the Hardy spaces $H^p(\mathbb{D})$ in terms of the harmonic measure of the level sets. In [ESS, Thm. 7] this characterization is obtained with the extra hypothesis that the complement of the image region has positive capacity. This condition is used in [ESS] to prove the sufficiency of the integral condition, but is in fact not needed. We include here, for completeness, a proof of Proposition 5.1.

PROPOSITION 5.1. *Assume that ψ is an analytic function on $\mathbb{D}$. For every $\alpha > 0$, let Ω_α be the component of $\{ z \in \mathbb{D} : |\psi(z)| < \alpha \}$ containing the origin, $F_\alpha = \partial\Omega_\alpha \cap \mathbb{D}$, and $\omega_\alpha = \omega(0, F_\alpha, \Omega_\alpha)$. Then, for $0 < p < \infty$,*

$$\psi \in H^p(\mathbb{D}) \iff \int_0^\infty \alpha^{p-1} \omega_\alpha \, d\alpha < \infty.$$

Proof. Suppose first that $\int_0^\infty \alpha^{p-1}\omega_\alpha \, d\alpha < \infty$ for some $p > 0$. Fix $0 < r < 1$. Then, by Fubini's theorem,

$$\int_0^{2\pi} |\psi(re^{i\theta})|^p \, d\theta = \int_0^\infty p\alpha^{p-1} |\{ \theta : |\psi(re^{i\theta})| > \alpha \}| \, d\alpha,$$

where $|\cdot|$ denotes Lebesgue measure of a set of real numbers. Fix $\alpha > 0$ and let K_α be the set of points on the circle $|z| = r$ such that $|\psi(z)| > \alpha$. Then

$$|\{ \theta : |\psi(re^{i\theta})| > \alpha \}| = 2\pi\omega(0, K_\alpha, r\mathbb{D}).$$

Also, $K_\alpha \cap \Omega_\alpha = \emptyset$, so F_α separates K_α from 0 in $\mathbb{D}$. Thus, by the maximum principle,

$$\omega(0, K_\alpha, r\mathbb{D}) \le \omega(0, K_\alpha, \mathbb{D} \setminus K_\alpha) \le \omega_\alpha$$

and

$$\int_0^{2\pi} |\psi(re^{i\theta})|^p \, d\theta \le 2\pi p \int_0^\infty \alpha^{p-1}\omega_\alpha \, d\alpha < \infty.$$

Since $0 < r < 1$ was arbitrary, we obtain that $\psi \in H^p(\mathbb{D})$.

Before proving the converse, we recall the following definition.

DEFINITION 5.2. For $\zeta \in \partial\mathbb{D}$, consider the nontangential region

$$\Gamma(\zeta) = \{ z \in \mathbb{D} : |\zeta - z| < 2(1 - |z|) \}.$$

Then a *non-tangential maximal function* $\tilde{\psi}$ of ψ is defined at every point $\zeta \in \partial\mathbb{D}$ by putting

$$\tilde{\psi}(\zeta) = \sup_{z \in \Gamma(\zeta)} |\psi(z)|.$$

Now suppose that $\psi \in H^p(\mathbb{D})$. Then ψ has nontangential limits almost everywhere on $\partial\mathbb{D}$, and by [Du, Thm. 2.6] it follows that

$$\int_0^{2\pi} |\psi(e^{i\theta})|^p \, d\theta < \infty.$$

Also, by a theorem of Hardy and Littlewood (see [Ga, p. 57]),

$$\int_0^{2\pi} |\tilde{\psi}(e^{i\theta})|^p \, d\theta \le B \int_0^{2\pi} |\psi(e^{i\theta})|^p \, d\theta < \infty,$$

where $B > 0$ is a constant depending only on p. Hence, for all $\alpha > 0$, we let $E_\alpha = \{ e^{i\theta} : |\tilde{\psi}(e^{i\theta})| \ge \alpha \}$. Then, by Fubini's theorem,

$$\int_0^{\infty} \alpha^{p-1} \omega(0, E_\alpha, \mathbb{D}) \, d\alpha < \infty. \tag{5.1}$$

Fix $w \in F_\alpha$, and let $I = \{ e^{i\theta} : |w - e^{i\theta}| < 2(1 - |w|) \}$. We have $I \subset E_\alpha$ and $|I| \ge C_0(1 - |w|)$ for some constant $0 < C_0 < 1$ independent of w. Then

$$\omega(w, E_\alpha, \mathbb{D}) \ge \int_I \frac{1 - |w|^2}{|e^{i\theta} - w|^2} \frac{d\theta}{2\pi} \ge \frac{C_0}{8\pi} = C_1, \tag{5.2}$$

where $0 < C_1 < 1$. Thus, if $C_2 = C_1/2$ and U_α is the component containing the origin of the set of $z \in \mathbb{D}$ where $\omega(z, E_\alpha, \mathbb{D}) < C_2$, then $F_\alpha \cap \overline{U_\alpha} = \emptyset$. Hence ∂U_α separates F_α from 0 in $\mathbb{D}$, since U_α is eventually a nonempty neighborhood of 0 because, by (5.1), $\omega(0, E_\alpha, \mathbb{D})$ tends to zero as α tends to infinity. Thus

$$\omega_\alpha \le \omega(0, \partial U_\alpha \cap \mathbb{D}, U_\alpha).$$

Moreover, as we shall show,

$$\omega(0, \partial U_\alpha \cap \mathbb{D}, U_\alpha) \le \omega(0, E_\alpha, \mathbb{D})/C_2. \tag{5.3}$$

Therefore, by (5.1), $\int_0^\infty \alpha^{p-1} \omega_\alpha \, d\alpha < \infty$ and the converse is proved.

To see why (5.3) holds, notice that, for every fixed $\zeta \in \partial U_\alpha \cap \mathbb{D}$, $\omega(z, E_\alpha, \mathbb{D})$ tends to C_2 as z tends to ζ, and otherwise $\omega(z, E_\alpha, \mathbb{D}) \ge 0$. Thus, if v is an arbitrary candidate for $\omega(0, \partial U_\alpha \cap \mathbb{D}, U_\alpha)$ then we always have $\omega(z, E_\alpha, \mathbb{D})/C_2 \ge v(z)$, and the inequality follows by taking the supremum over all the v. □

We now obtain some immediate consequences of Proposition 5.1. The next definition is inspired by the behavior of the Riemann maps of $\mathbb{D}$ onto straight sectors.

DEFINITION 5.3. Let ψ be an analytic function on $\mathbb{D}$. Let F_α be defined as in Proposition 5.1. We say that ψ is *sectorlike* if:

(a) the limit

$$\mu(\psi) = \lim_{\alpha \to \infty} \frac{\log(1/\omega_\alpha)}{\log \alpha}$$

exists in $[0, \infty]$; and

(b) when $\mu(\psi) \in (0, \infty)$ there is a $C > 0$ such that, for $\alpha > 1$,

$$\mu(\psi) - \frac{\log(1/\omega_\alpha)}{\log \alpha} \ge -\frac{C}{\log \alpha}.$$

PROPOSITION 5.4. *Suppose that ψ is sectorlike. Then, for $0 < p < \infty$,*

$$\psi \in H^p(\mathbb{D}) \iff 0 < p < \mu(\psi).$$

Proof. By Proposition 5.1, $\psi \in H^p(\mathbb{D})$ if and only if

$$\int_0^\infty \alpha^{-1}\alpha^{p-\log(1/\omega_\alpha)/\log\alpha}\,d\alpha \tag{5.4}$$

converges. Now, if $0 < p < \mu(\psi)$ then (5.4) converges, so $\psi \in H^p(\mathbb{D})$. Conversely, assume that $\psi \in H^p(\mathbb{D})$ for some $0 < p < \infty$. If $\mu(\psi) < p$ then (5.4) diverges, and if $\mu(\psi) = p$ then, by Definition 5.3(b), (5.4) is bounded below by

$$\int_1^\infty e^{-C}\alpha^{-1}\,d\alpha,$$

which also diverges. We have thus reached a contradiction, so $0 < p < \mu(\psi)$. □

6. Kœnigs Maps Are Sectorlike

We are now ready to prove Theorem 2.1. Toward this end, we will show that if σ is the Kœnigs map associated to a function $\phi \in \mathcal{A}$ then σ is sectorlike, so that (by Proposition 5.4) the Hardy number $h(\sigma) = \sup\{\, p > 0 : \sigma \in H^p(\mathbb{D}) \,\}$ is equal to the limit $\mu(\sigma)$ in Definition 5.3, and $\sigma \in H^p(\mathbb{D})$ if and only if $0 < p < \mu(\sigma)$. Moreover, we will also show that, when ϕ is not an inner function, $\mu(\sigma) > 0$.

Proof of Theorem 2.1. For $\alpha > 0$, let Ω_α, F_α, and ω_α be as defined in Theorem 2.1. We assume that σ is normalized so that (4.1) holds, that is, $\overline{\Omega_1} \subset s\mathbb{D}$ for some $0 < s < 1$. Also, we assume that σ is unbounded, so that $\omega_\alpha > 0$ for all $\alpha > 0$, because otherwise Theorem 2.1 is trivially proved.

Fix $\beta > 1$ and find an integer $N \ge 1$ such that

$$(1/|\lambda|)^{N-1} \le \beta < (1/|\lambda|)^N. \tag{6.1}$$

For all $\alpha > \beta$, there is an integer $H \ge 1$ such that

$$\beta(1/|\lambda|)^{(H-1)N} \le \alpha < \beta(1/|\lambda|)^{HN}. \tag{6.2}$$

Having set the scales in which we are measuring the sizes of α and β, we let $T_h = F_{\beta|\lambda|^{-hN}}$ and $W_h = \Omega_{\beta|\lambda|^{-hN}}$ for $h \in \mathbb{Z}$. Iterating the conditional probability estimate of Lemma 3.2, we obtain

$$\omega_\alpha \le \prod_{h=0}^{H-1} \sup_{\zeta\in T_{h-1}} \omega(\zeta, T_h, W_h). \tag{6.3}$$

Notice that, by (3.1), $\phi_{hN}^{-1}(F_\beta) \cap W_h = \emptyset$ and $\phi_{hN}^{-1}(F_\beta) \supset T_h$. Therefore,

$$\sup_{\zeta\in T_{h-1}} \omega(\zeta, T_h, W_h) = \sup_{\zeta\in T_{h-1}} \omega\big(\zeta, \phi_{hN}^{-1}(F_\beta), \mathbb{D}\setminus\phi_{hN}^{-1}(F_\beta)\big).$$

We now apply Lemma 3.1 with E replaced by F_β, α by $|\lambda|^N\beta$, and ϕ by ϕ_{hN}. Then, for $h = 0, \dots, H-1$,

$$\sup_{\zeta\in T_{h-1}} \omega\big(\zeta, \phi_{hN}^{-1}(F_\beta), \mathbb{D}\setminus\phi_{hN}^{-1}(F_\beta)\big) \le \sup_{\zeta\in T_{-1}} \omega(\zeta, F_\beta, \Omega_\beta).$$

Finally, using the fact that, by (6.1), $\beta|\lambda|^N < 1$ and thus $T_{-1} \subset \overline{\Omega_1} \subset s\mathbb{D}$, (6.3) becomes

$$\omega_\alpha \leq \left(\sup_{\zeta \in \overline{\Omega_1}} \omega(\zeta, F_\beta, \Omega_\beta) \right)^H .$$

Write P_β for $\sup_{\zeta \in \overline{\Omega_1}} \omega(\zeta, F_\beta, \Omega_\beta)$. Taking logarithms and using (6.2), we obtain

$$\frac{\log(1/\omega_\alpha)}{\log \alpha} \geq \frac{H \log(1/P_\beta)}{HN \log(1/|\lambda|) + \log \beta}.$$

Letting α tend to infinity, H also tends to infinity. Hence,

$$\liminf_{\alpha \to \infty} \frac{\log(1/\omega_\alpha)}{\log \alpha} \geq \frac{\log(1/P_\beta)}{N \log(1/|\lambda|)}.$$

By (6.1), $N \log(1/|\lambda|) \leq \log \beta + \log(1/|\lambda|)$, so

$$\liminf_{\alpha \to \infty} \frac{\log(1/\omega_\alpha)}{\log \alpha} \geq \frac{\log(1/P_\beta)}{\log \beta + \log(1/|\lambda|)}. \tag{6.4}$$

This estimate will be used in the sequel.

Because σ is bounded on $s\bar{\mathbb{D}}$, there exists a $\beta_0 > 1$ such that $s\bar{\mathbb{D}} \subset \Omega_\beta$ for all $\beta > \beta_0$. Harnack's inequality yields a constant $C_0 \geq 1$ depending only on $\overline{\Omega_1}$ and s (i.e., independent of β) such that

$$P_\beta \leq C_0 \omega_\beta \quad \text{for } \beta > \beta_0.$$

Thus, (6.4) becomes

$$\liminf_{\alpha \to \infty} \frac{\log(1/\omega_\alpha)}{\log \alpha} \geq \frac{\log(1/\omega_\beta) + \log(1/C_0)}{\log \beta + \log(1/|\lambda|)}. \tag{6.5}$$

Letting β tend to infinity in (6.5), we obtain

$$\liminf_{\alpha \to \infty} \frac{\log(1/\omega_\alpha)}{\log \alpha} \geq \limsup_{\beta \to \infty} \frac{\log(1/\omega_\beta)}{\log \beta}.$$

Therefore, the limit $\mu(\sigma)$ in Definition 5.3(a) exists. Moreover, if $\mu(\sigma)$ is finite then we have from equation (6.5) that, for all $\beta > \beta_0$,

$$\mu(\sigma) \log \beta - \log(1/\omega_\beta) \geq -\mu(\sigma) \log(1/|\lambda|) - \log C_0.$$

As a result,

$$\mu(\sigma) - \frac{\log(1/\omega_\beta)}{\log \beta} \geq -\frac{C}{\log \beta}$$

for some constant $C > 0$. We have thus proved that σ is sectorlike.

We now want to show that, when ϕ is not inner, $\mu(\sigma) > 0$. By (6.4) it is enough to find a $\beta > 1$ for which

$$\sup_{\zeta \in \overline{\Omega_1}} \omega(\zeta, F_\beta, \Omega_\beta) < 1,$$

so suppose that $\phi \in \mathcal{A}$ and ϕ is not an inner function. Then there is an $A \subset \partial\mathbb{D}$ of positive Lebesgue measure such that $\phi(A) \subset \mathbb{D}$, meaning that ϕ has nontangential limit in $\mathbb{D}$ at each point of A. For all $\zeta \in A$, let $\Gamma(\zeta)$ be the cone region described in Definition 5.2. Also, write $\Gamma_r(\zeta)$ for the truncated cone $\Gamma(\zeta) \setminus r\mathbb{D}$, with $0 < r < 1$.

Without loss of generality, $\overline{\phi(A)} \subset \mathbb{D}$ (replace A by $\{ \zeta \in A : |\phi(\zeta)| \leq 1 - 1/m \}$ for some $m \in \mathbb{N}$ sufficiently large). Hence, by Schwarz's lemma, there exists an integer $N \in \mathbb{N}$ such that $\overline{\phi_N(A)} \subset \Omega_1$. Thus, to each $\zeta \in A$ we associate a truncated cone $\Gamma_r(\zeta)$, where $1 - r$ is chosen small enough so that $\phi_N(\Gamma_r(\zeta)) \subset \Omega_1$. Then, by (1.2),

$$\Gamma_r(\zeta) \subset \{ z \in \mathbb{D} : |\sigma(z)| < |\lambda|^{-N} \}. \tag{6.6}$$

Again without loss of generality, all the truncated cones $\Gamma_r(\zeta)$ can be taken to have the same height (replace A by the set of ζ in A for which the associated truncated cone has height greater than $1 - 1/m$ for some $m \in \mathbb{N}$ large enough). For $n = 1, 2, \ldots$, let $W_n = \Omega_{|\lambda|^{-n}}$ and $T_n = F_{|\lambda|^{-n}}$. Again, there is an $n_0 = n_0(r)$ such that, for all $n \geq n_0$, $r\bar{\mathbb{D}} \subset W_n$, because σ is bounded on $r\bar{\mathbb{D}}$. So, for all $\zeta \in A$ and all $n > n_1 = \max\{N, n_0\}$, $W_n \cap \Gamma_r(\zeta) \neq \emptyset$. By (6.6), we must have $\Gamma_r(\zeta) \subset W_n$. Therefore, for $n > n_1$, the set

$$V = r\mathbb{D} \cup \left(\bigcup_{\zeta \in A} \Gamma(\zeta) \right)$$

is a simply connected set contained in W_n such that $\partial V \cap \partial \mathbb{D} = A$. We want to show that $\sup_{\zeta \in \overline{\Omega_1}} \omega(\zeta, T_n, W_n) < 1$. By Harnack's inequality, it is enough to show that $\omega(0, T_n, W_n)$ is strictly less than unity, or equivalently that $\omega(0, \partial W_n \cap \partial \mathbb{D}, W_n)$ is strictly positive. By the maximum principle,

$$\omega(0, \partial W_n \cap \partial \mathbb{D}, W_n) \geq \omega(0, A, V),$$

so it is enough to show that

$$\omega(0, A, V) > 0. \tag{6.7}$$

To see why (6.7) holds, let ψ be a Riemann map of $\mathbb{D}$ onto V such that $\psi(0) = 0$. By Carathéodory's theorem, ψ extends to be continuous and one-to-one on $\bar{\mathbb{D}}$ (V satisfies Theorem 2.6(iii) of [Pom, p. 24]). We can therefore let $G = \psi^{-1}(A) \subset \partial\mathbb{D}$. Then

$$\omega(0, A, V) = \omega(0, G, \mathbb{D}) = 2\pi |G|.$$

By the McMillan sector theorem (see [Pom, Thm. 6.24, p. 146]),

$$|G| > 0 \iff |A| > 0.$$

Hence, (6.7) follows. □

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