Existence of Multiple Refinable Distributions

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1. Introduction and Main Results

Wavelets and subdivision schemes are based on refinement equations of the form

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(2 \cdot -\alpha), \tag{1.1}$$

where $\{a(\alpha)\}_{\alpha \in \mathbb{Z}^s}$ is a sequence of complex numbers. If a is finitely supported and $\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = 2^s$, then the refinement equation (1.1) has a unique solution ϕ of compactly supported distribution on $\mathbb{R}^s$ subject to the normalized condition $\hat{\phi}(0) = 1$. Here $\hat{\phi}$ is the Fourier transform of ϕ, which is defined for an integrable function f on $\mathbb{R}^s$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^s} f(x) e^{-i\xi \cdot x}\, dx, \quad \xi \in \mathbb{R}^s,$$

and has a natural extension to compactly supported distributions. This fact of existence was proved by Cavaretta, Dahmen, and Micchelli in [1]; see also [4] and [5].

In this paper we investigate the existence of multiple refinable distributions in multiwavelets. The theory of multiwavelets began with the work of Goodman, Lee, and Tang [7; 8] and the orthogonal multiwavelet basis construction of Donovan, Geronimo, Hardin, Kessler, and Massopust [6; 9]. There have been many discussions concerning different aspects (see e.g. [2; 11; 14; 15]). All these are based on multiple refinable distributions or functions. Given a positive integer r, called the *multiplicity,* and a sequence $a := \{a(\alpha)\}_{\alpha \in \mathbb{Z}^s}$ of $r \times r$ complex matrices, a vector $\phi := (\phi_1, \dots, \phi_r)^T$ of distributions on $\mathbb{R}^s$ is called a *multiple refinable distribution* associated with the *refinement mask a* if it satisfies the following *matrix refinement equation*

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(2 \cdot -\alpha). \tag{1.2}$$

Note that the scalar refinement equation (1.1) is the special form of (1.2) with $r = 1$. We denote $\hat{\phi} = (\hat{\phi}_1, \dots, \hat{\phi}_r)^T$ and always set

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$$M := 2^{-s} \sum_{\alpha \in \mathbb{Z}^s} a(\alpha). \tag{1.3}$$

Throughout the paper we assume that the multiplicity $r = 2$.

The first purpose of this paper is to consider the existence of compactly supported multiple refinable distribution solutions to (1.2) when the refinement mask a is finitely supported—that is, $a(\alpha) = 0$ except for finitely many $\alpha \in \mathbb{Z}^s$.

THEOREM 1. *Let $\{a(\alpha)\}_{\alpha \in \mathbb{Z}^s}$ be a finitely supported sequence of 2×2 complex matrices. Assume that 2^n is not an eigenvalue of M for any $n \in \mathbb{N}$. Then (1.2) has a solution $\phi = (\phi_1, \phi_2)^T$, a vector of compactly supported distributions on $\mathbb{R}^s$, subject to $\hat{\phi}(0) \neq 0$ if and only if the matrix M has an eigenvalue 1. In this case, $\hat{\phi}(0)$ is an eigenvector of M associated with the eigenvalue 1.*

In the univariate case $s = 1$, when the spectral radius of M is less than 2, the existence for arbitrary multiplicity was proved by Heil and Colella [10], Cohen, Daubechies, and Plonka [2], and Hervé [12]; see also the work of Long, Chen, and Yuan [17]. The proof for this special case is based on an "infinite matrix product" approach introduced in [10; 18; 12] as follows. For a finitely supported refinement mask a, define a sequence $\{\prod_n(\xi)\}_{n \in \mathbb{N} \cup \{0\}}$ of 2×2 matrices of functions on $\mathbb{C}^s$ by

$$\prod_n(\xi) = \prod_{n-1}(\xi)\tilde{a}\left(\frac{\xi}{2^n}\right),$$

where $\prod_0(\xi) \equiv I$ is the 2×2 identity matrix and

$$\tilde{a}(\xi) = 2^{-s} \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) e^{-i\xi \cdot \alpha} =: \begin{bmatrix} \tilde{a}_{11}(\xi) & \tilde{a}_{12}(\xi) \\ \tilde{a}_{21}(\xi) & \tilde{a}_{22}(\xi) \end{bmatrix}, \quad \xi \in \mathbb{C}^s,$$

is the symbol of the mask a. Here $\xi \cdot \alpha := \sum_{j=1}^s \xi_j \alpha_j$ for $\xi := (\xi_1, \dots, \xi_s) \in \mathbb{C}^s$ and $\alpha := (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s$. Note that $\tilde{a}(0) = M$.

Suppose that M has an eigenvector $v \neq 0$ associated with eigenvalue 1. The approach in [10; 18; 12; 2] is to define a compactly supported distribution solution ϕ to (1.2) by its Fourier transform as

$$\hat{\phi}(\xi) = \lim_{n \to \infty} \prod_n(\xi) v, \quad \xi \in \mathbb{R}^s. \tag{1.4}$$

Two questions arise with respect to this approach. One question is whether all the solutions to (1.2) can be obtained via (1.4). In fact, Heil and Colella conjecture in [10] that some matrix refinement equations have compactly supported distribution solutions, not obtainable via the infinite matrix product (1.4). The second purpose of this paper is to confirm this conjecture by presenting some examples in Section 3. The other question is when the limit in (1.4) exists; the final purpose of this paper is to answer this question.

Let us give a canonical form of the refinement mask for convenience. Let $\alpha_0 \in \mathbb{Z}^s$, and let P be an invertible 2×2 complex matrix. Then ϕ is a compactly supported multiple refinable distribution associated with the mask a subject to

$\hat{\phi}(0) = v$ if and only if $\psi := P\phi(\cdot - \alpha_0)$ is a compactly supported multiple refinable distribution associated with the new mask $Pa(\cdot - \alpha_0)P^{-1}$ subject to $\hat{\psi}(0) = Pv$. Moreover, the limit in (1.4) exists for this new mask and the new eigenvector $Pv \neq 0$ if and only if it exists for the mask a and the eigenvector v. Thus, to investigate the existence of (1.2) and the convergence of (1.4), we may replace a by $Pa(\cdot - \alpha_0)P^{-1}$ for suitable α_0 and P, and assume that the mask a is supported in $\mathbb{Z}_+^s$ and that the matrix M given by (1.3) takes the Jordan form

$$M = \begin{bmatrix} 1 & \mu \\ 0 & \lambda \end{bmatrix}. \tag{1.5}$$

Here $\mu = \lambda = 1$ when the eigenvalue 1 is degenerate; otherwise, $\mu = 0$.

Under this canonical form, we set $v = e_1 := (1, 0)^T$. Our main result on convergence can be stated as follows.

THEOREM 2. *Suppose that a is finitely supported in $\mathbb{Z}_+^s$, and that $M = \left[\begin{smallmatrix} 1 & \mu \\ 0 & \lambda \end{smallmatrix}\right]$ with $\mu = 0$ when $\lambda \neq 1$. If the function $\tilde{a}_{21}(\xi)$ has a zero of exact order $d \in \mathbb{N}$ at the origin, then the limit $\lim_{n\to\infty} \prod_n(\xi)e_1$ exists for every $\xi \in \mathbb{R}^s$ if and only if $|\lambda| < 2^d$. In this case, the limit is the Fourier transform of a compactly supported distribution solution ϕ of* (1.2) *with $\hat{\phi}(0) = e_1$.*

REMARK. If $\tilde{a}_{21}(\xi) \equiv 0$ then it can be directly seen that

$$\lim_{n\to\infty} \prod_n(\xi)e_1 = \left(\prod_{j=1}^{\infty} \tilde{a}_{11}\left(\frac{\xi}{2^j}\right), 0\right)^T = (\hat{\phi}_1(\xi), 0)^T \quad \text{for } \xi \in \mathbb{R}^s,$$

where ϕ_1 is a compactly supported distribution on $\mathbb{R}^s$ with $\hat{\phi}_1(0) = 1$.

THEOREM 3. *Given the conditions of Theorem 2, if $|\lambda| \geq 2^d$ then, for any nonzero vector $u \in \mathbb{C}^2$, $\lim_{n\to\infty} \prod_n(\xi)u$ does not converge.*

Let us point out that the technique for proving Theorems 2 and 3 does not generalize beyond the 2×2 case. Therefore, in this paper we always assume $r = 2$, though the technique for the proof of Theorem 1 can be extended to arbitrary multiplicity.

As a corollary of Theorem 2, it can be seen that the assumptions of Theorem 1 cannot be dropped. Let us consider the simple case $\lambda = 2$.

THEOREM 4. *Suppose that a is finitely supported in $\mathbb{Z}_+^s$ and that $M = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$. Then* (1.2) *has a solution $\phi = (\phi_1, \phi_2)^T$, a vector of compactly supported distributions on $\mathbb{R}^s$, subject to $\hat{\phi}(0) \neq 0$ if and only if the function $\tilde{a}_{21}(\xi)$ has a zero of order at least 2 at the origin.*

2. Proofs of the Main Results

In this section we prove the main results.

For $\xi := (\xi_1, \dots, \xi_s) \in \mathbb{C}^s$, set $|\xi| = \sqrt{\sum_{j=1}^s |\xi_j|^2}$, $\operatorname{Im} \xi = (\operatorname{Im} \xi_1, \dots, \operatorname{Im} \xi_s)$, $\operatorname{Re} \xi = (\operatorname{Re} \xi_1, \dots, \operatorname{Re} \xi_s)$, and $1-\xi = (1-\xi_1, \dots, 1-\xi_s)$. For two multi-indices

$\alpha = (\alpha_1, \dots, \alpha_s)$ and $\beta = (\beta_1, \dots, \beta_s) \in \mathbb{Z}^s$, $\alpha \geq \beta$ means $\alpha_j \geq \beta_j$ for $1 \leq j \leq s$. In this case,

$$\binom{\alpha}{\beta} = \prod_{j=1}^{s} \binom{\alpha_j}{\beta_j} \quad \text{if } \alpha, \beta \in \mathbb{Z}_+^s.$$

Denote $\xi^\alpha = \prod_{j=1}^s \xi_j^{\alpha_j}$.

We need the following lemma, which is proved by standard techniques; for completeness, a detailed proof is provided here.

LEMMA. *Suppose that a is finitely supported and that M has spectral radius $|\lambda|$, where $\lambda \in \mathbb{C}$ is an eigenvalue. Then, for any $\varepsilon > 0$ and an arbitrary norm $\|\cdot\|$ of $\mathbb{C}^2$, there exist positive constants A_1, B_1, C_1 such that, for any $n \in \mathbb{N}$ and $\xi \in \mathbb{C}^s$,*

$$\left\| \prod_n (\xi) \right\| \leq C_1(|\lambda| + \varepsilon)^n (1 + |\xi|)^{B_1} e^{A_1 |\mathrm{Im}\,\xi|}. \tag{2.1}$$

Moreover, if all the eigenvalues with modulus equal to $|\lambda|$ are nondegenerate and $M \neq 0$, then the term $|\lambda| + \varepsilon$ in (2.1) can be replaced by $|\lambda|$.

Proof. Let $|||\cdot|||$ be a norm of $\mathbb{C}^2$ such that, as an operator on $(\mathbb{C}^2, |||\cdot|||)$, the matrix M has norm either less than $|\lambda| + \varepsilon$ or equal to $|\lambda|$ when the eigenvalues with modulus $|\lambda|$ are nondegenerate and $M \neq 0$.

Observe that, for $x \in \mathbb{R}$,

$$|e^x - 1| \leq e^{|x|}(1 - e^{-|x|}) \leq e^{|x|} \min\{|x|, 2\}$$

and

$$|e^{-ix} - 1| \leq \min\{|x|, 2\}.$$

Hence, for $\alpha \in \mathbb{Z}^s$ and $\xi \in \mathbb{C}^s$,

$$\begin{aligned} |e^{-i\xi\cdot\alpha} - 1| &= |e^{\mathrm{Im}\,\xi\cdot\alpha} e^{-i\,\mathrm{Re}\,\xi\cdot\alpha} - e^{-i\,\mathrm{Re}\,\xi\cdot\alpha} + e^{-i\,\mathrm{Re}\,\xi\cdot\alpha} - 1| \\ &\leq e^{|\mathrm{Im}\,\xi\cdot\alpha|} \min\{|\mathrm{Im}\,\xi\cdot\alpha|, 2\} + \min\{|\mathrm{Re}\,\xi\cdot\alpha|, 2\} \\ &\leq 2e^{|\mathrm{Im}\,\xi\cdot\alpha|} \min\{|\xi\cdot\alpha|, 2\}. \end{aligned}$$

Suppose that $a(\alpha) = 0$ for $\alpha \in \mathbb{Z}^s$ with $|\alpha| > L$. Then, for $\xi \in \mathbb{C}^s$,

$$\begin{aligned} |||\tilde{a}(\xi) - \tilde{a}(0)||| &= \left|\left|\left| 2^{-s} \sum_{|\alpha|\leq L} a(\alpha)(e^{-i\xi\cdot\alpha} - 1) \right|\right|\right| \\ &\leq \sum_{|\alpha|\leq L} 2^{-s} |||a(\alpha)||| 2e^{|\mathrm{Im}\,\xi\cdot\alpha|} \min\{|\xi\cdot\alpha|, 2\} \\ &\leq \sum_{|\alpha|\leq L} 2^{1-s} |||a(\alpha)||| e^{L|\mathrm{Im}\,\xi|} \min\{L|\xi|, 2\} \\ &\leq C_0 e^{A_0|\mathrm{Im}\,\xi|} \min\{|\xi|, 1\}, \end{aligned}$$

where $C_0 = 2^{1-s}(L+2) \sum_{|\alpha|\leq L} |||a(\alpha)|||$ and $A_0 = L$. Hence

$$|||\tilde{a}(\xi)||| \leq |||M||| + C_0 e^{A_0|\mathrm{Im}\,\xi|} \min\{|\xi|, 1\}.$$

Now we estimate (2.1). Note that $1+|x| \le e^{|x|}$ for $x \in \mathbb{R}$.

Let $n \in \mathbb{N}$ and $\xi \in \mathbb{C}^s$. We first consider the case that $M \neq 0$, that is, $|||M||| > 0$. If $|\xi| \le 2$, then

$$\begin{aligned}
\left|\left|\left|\prod_n(\xi)\right|\right|\right| &\le \prod_{j=1}^n \left|\left|\left|\tilde{a}\left(\frac{\xi}{2^j}\right)\right|\right|\right| \\
&\le \prod_{j=1}^n \left\{|||M||| + C_0 e^{A_0|\operatorname{Im}\xi/2^j|}\left|\frac{\xi}{2^j}\right|\right\} \\
&\le \prod_{j=1}^n \{(|||M|||)e^{A_0|\operatorname{Im}\xi/2^j|}\} \prod_{j=1}^n \{e^{C_0|\xi/2^j|/|||M|||}\} \\
&\le e^{2C_0/|||M|||}(|||M|||)^n e^{A_0|\operatorname{Im}\xi|}.
\end{aligned}$$

If $2^k < |\xi| \le 2^{k+1}$ for some $k \in \mathbb{N}$, then

$$\begin{aligned}
\left|\left|\left|\prod_n(\xi)\right|\right|\right| &\le \prod_{j=1}^k \left|\left|\left|\tilde{a}\left(\frac{\xi}{2^j}\right)\right|\right|\right| \prod_{j=k+1}^n \left|\left|\left|\tilde{a}\left(\frac{\xi}{2^j}\right)\right|\right|\right| \\
&\le \prod_{j=1}^k \{|||M||| + C_0 e^{A_0|\operatorname{Im}\xi/2^j|}\} \prod_{j=k+1}^n \left\{|||M||| + C_0 e^{A_0|\operatorname{Im}\xi/2^j|}\left|\frac{\xi}{2^j}\right|\right\} \\
&\le (|||M|||)^n e^{A_0|\operatorname{Im}\xi|}\left(1+\frac{C_0}{|||M|||}\right)^k \prod_{j=k+1}^n \{e^{C_0|\xi/2^j|/|||M|||}\} \\
&\le e^{2C_0/|||M|||}(|||M|||)^n |\xi|^{\log_2(1+C_0/|||M|||)} e^{A_0|\operatorname{Im}\xi|}.
\end{aligned}$$

Combining the foregoing two cases with the norm of M on $(\mathbb{C}^2, |||\cdot|||)$, we know that for some positive constants A_1, B_1, C_1, (2.1) holds when $M \neq 0$, since the two norms $\|\cdot\|$ and $|||\cdot|||$ of $\mathbb{C}^2$ are equivalent.

When $M = 0$, the same estimates for $\left|\left|\left|\prod_n(\xi)\right|\right|\right|$ hold if we replace $|||M|||$ by ε. Therefore, (2.1) is also valid in this case. This completes the proof of the Lemma. □

Now we can prove our first main result.

PROOF OF THEOREM 1. (Necessity) Suppose that $\phi = (\phi_1, \phi_2)^T$ is a compactly supported distribution solution to (1.2) with $\hat{\phi}(0) \neq 0$. Then (1.2) has an equivalent form:

$$\hat{\phi}(\xi) = \tilde{a}(\xi/2)\hat{\phi}(\xi/2), \quad \xi \in \mathbb{C}^s. \tag{2.2}$$

In particular, $\hat{\phi}(0) = M\hat{\phi}(0)$. Hence 1 is an eigenvalue of M with an eigenvector $\hat{\phi}(0)$.

(Sufficiency) As observed in Section 1, we may assume that a is supported in $\mathbb{Z}_+^s$ and $M = \begin{bmatrix} 1 & \mu \\ 0 & \lambda \end{bmatrix}$, where $\mu = 0$ if $\lambda \neq 1$. Recall that, for a finitely supported sequence $b := \{b(\alpha)\}_{\alpha\in\mathbb{Z}^s}$ of complex numbers, $\tilde{b}$ is defined by

$$\tilde{b}(\xi) := 2^{-s}\sum_{\alpha\in\mathbb{Z}} b(\alpha)e^{-i\xi\cdot\alpha}, \quad \xi \in \mathbb{C}^s.$$

Let h be a sequence of complex numbers, finitely supported in $\mathbb{Z}_+^s$ with $\tilde{h}(0) = 0$, that will be determined later for different purposes. Denote $e_2 := (0, 1)^T$. Define a sequence of functions $\{\hat{\Phi}_n\}_{n\in\mathbb{N}}$ on $\mathbb{C}^s$ by

$$\hat{\Phi}_n(\xi) = \prod_n(\xi)\begin{bmatrix} 1 \\ \tilde{h}(\xi/2^n) \end{bmatrix}, \quad \xi \in \mathbb{C}^s. \tag{2.3}$$

Then, for $n \in \mathbb{N}$,

$$\hat{\Phi}_n(0) = e_1,$$

and for $n \geq 2$,

$$\hat{\Phi}_n(\xi) = \tilde{a}(\xi/2)\hat{\Phi}_{n-1}(\xi/2), \quad \xi \in \mathbb{C}^s. \tag{2.4}$$

Moreover, for $2 \leq n \in \mathbb{N}$ and $\xi \in \mathbb{C}^s$,

$$\begin{aligned}
\hat{\Phi}_n(\xi) &= \prod_{n-1}(\xi)\begin{bmatrix} \tilde{a}_{11}(\xi/2^n) + \tilde{a}_{12}(\xi/2^n)\tilde{h}(\xi/2^n) \\ \tilde{a}_{21}(\xi/2^n) + \tilde{a}_{22}(\xi/2^n)\tilde{h}(\xi/2^n) \end{bmatrix} \\
&= [\tilde{a}_{11}(\xi/2^n) + \tilde{a}_{12}(\xi/2^n)\tilde{h}(\xi/2^n)]\hat{\Phi}_{n-1}(\xi) \\
&\quad + [\tilde{a}_{21}(\xi/2^n) + \tilde{a}_{22}(\xi/2^n)\tilde{h}(\xi/2^n) - \tilde{h}(\xi/2^{n-1}) \\
&\qquad \times (\tilde{a}_{11}(\xi/2^n) + \tilde{a}_{12}(\xi/2^n)\tilde{h}(\xi/2^n))]\prod_{n-1}(\xi)e_2 \\
&:= \tilde{f}(\xi/2^n)\hat{\Phi}_{n-1}(\xi) + \tilde{g}(\xi/2^n)\prod_{n-1}(\xi)e_2.
\end{aligned}$$

Then, by induction, for $2 \leq n \in \mathbb{N}$ and $\xi \in \mathbb{C}^s$ we have

$$\hat{\Phi}_n(\xi) = \sum_{j=2}^{n}\left[\prod_{k=j+1}^{n}\tilde{f}\left(\frac{\xi}{2^k}\right)\right]\tilde{g}\left(\frac{\xi}{2^j}\right)\prod_{j-1}(\xi)e_2 + \left[\prod_{k=2}^{n}\tilde{f}\left(\frac{\xi}{2^k}\right)\right]\hat{\Phi}_1(\xi). \tag{2.5}$$

The expression (2.5) is valid for any trigonometric polynomial $\tilde{h}$ with $\tilde{h}(0) = 0$. We apply (2.5) with suitable sequences h to the estimate (2.1). Since $\tilde{f}(0) = 1$, there are positive constants A_2, B_2, C_2 such that, for $n \in \mathbb{N}$ and $\xi \in \mathbb{C}^s$,

$$\left|\prod_{j=1}^{n}\tilde{f}\left(\frac{\xi}{2^j}\right)\right| \leq C_2(1 + |\xi|)^{B_2}e^{A_2|\operatorname{Im}\xi|}. \tag{2.6}$$

If $|\lambda| < 2$, we choose $h \equiv 0$. Then

$$\tilde{g}(\xi) = \tilde{a}_{21}(\xi), \quad \xi \in \mathbb{C}^s. \tag{2.7}$$

If $2^m \leq |\lambda| < 2^{m+1}$ for some $m \in \mathbb{N}$ then, by the assumption, $\lambda \neq 2^m$. Observe that

$$\tilde{g}(\xi) = \tilde{a}_{21}(\xi) + \tilde{a}_{22}(\xi)\tilde{h}(\xi) - \tilde{h}(2\xi)\tilde{a}_{11}(\xi) - \tilde{h}(\xi)\tilde{h}(2\xi)\tilde{a}_{12}(\xi).$$

We want to choose h such that $\tilde{g}$ has a zero of order at least $m + 1$ at the origin; that is, for $\alpha \in \mathbb{Z}_+^s$ with $|\alpha|_1 := |\alpha_1| + \cdots + |\alpha_s| \leq m$,

$$D^\alpha\tilde{g}(0) := \frac{\partial^{\alpha_1+\cdots+\alpha_s}}{\partial\xi_1^{\alpha_1}\cdots\partial\xi_s^{\alpha_s}}\tilde{g}(0) = 0.$$

Toward this end, define a sequence $\{t_\alpha\}_{|\alpha|_1 \le m}$ by $t_0 := 0$ and (inductively) for $\alpha \ge 0$, $\alpha \ne 0$, and $|\alpha|_1 \le m$,

$$t_\alpha := \frac{1}{2^{|\alpha|_1} - \lambda}\Bigg\{D^\alpha \tilde{a}_{21}(0) + \sum_{0\le\beta\le\alpha,\beta\ne\alpha} \binom{\alpha}{\beta} t_\beta \Bigg[D^{\alpha-\beta}\tilde{a}_{22}(0) - 2^{|\beta|_1} D^{\alpha-\beta}\tilde{a}_{11}(0) - \sum_{0\le\gamma\le\alpha-\beta}\binom{\alpha-\beta}{\gamma} 2^{|\gamma|_1} t_\gamma D^{\alpha-\beta-\gamma}\tilde{a}_{12}(0)\Bigg]\Bigg\}.$$

Then we choose the sequence h such that

$$\tilde{h}(\xi) = \sum_{0\le\alpha,0<|\alpha|_1\le m} H(\alpha)(1-e^{-i\xi})^\alpha$$

satisfies

$$D^\alpha \tilde{h}(0) = t_\alpha \quad \text{for } \alpha \in \mathbb{Z}_+^s,\ |\alpha|_1 \le m.$$

This is fulfilled by setting inductively for $\alpha \ge 0$, $\alpha \ne 0$, and $|\alpha|_1 \le m$,

$$H(\alpha) = \frac{1}{i^{|\alpha|_1}\prod_{j=1}^s \alpha_j!}\Bigg\{t_\alpha - D^\alpha\Bigg(\sum_{0\le\beta,0<|\beta|_1<|\alpha|_1} H(\beta)(1-e^{-i\xi})^\beta\Bigg)(0)\Bigg\}.$$

Under this choice of h, we know that $\tilde{g}$ has a zero of order at least $m+1$ at the origin. Hence, for some finitely supported sequence G on $\mathbb{Z}_+^s$,

$$\tilde{g}(\xi) = \sum_{|\alpha|_1\ge m+1} G(\alpha)(1-e^{-i\xi})^\alpha. \tag{2.8}$$

Combining (2.7) and (2.8), one can see from the proof of the Lemma that, for $\xi \in \mathbb{C}^s$,

$$|\tilde{g}(\xi)| \le C_3|\xi|^{m+1}e^{A_3|\mathrm{Im}\,\xi|}, \tag{2.9}$$

where A_3 and C_3 are positive constants and $m = 0$ if $|\lambda| < 2$.

Using (2.5), (2.6), (2.9), and the Lemma, for some ε with $\max\{0, 1-|\lambda|\} < \varepsilon < 2^{m+1} - |\lambda|$ we conclude that, for $2 \le n \in \mathbb{N}$ and $\xi \in \mathbb{C}^s$,

$$\begin{aligned}|\hat{\Phi}_n(\xi)| \le{}& \sum_{j=2}^n C_2(1+|\xi/2^j|)^{B_2}e^{A_2|\mathrm{Im}\,\xi/2^j|}C_3|\xi/2^j|^{m+1}e^{A_3|\mathrm{Im}\,\xi/2^j|}\\
&\times C_1(|\lambda|+\varepsilon)^{j-1}(1+|\xi|)^{B_1}e^{A_1|\mathrm{Im}\,\xi|} + C_2(1+|\xi/2|)^{B_2}e^{A_2|\mathrm{Im}\,\xi/2|}\\
&\times C_1(|\lambda|+\varepsilon)(1+|\xi|)^{B_1}e^{A_1|\mathrm{Im}\,\xi|}(1+|\tilde{h}(\xi/2)|)\\
\le{}& C_1C_2C_3(1+|\xi|)^{B_1+B_2+m+1}e^{(A_1+A_2+A_3)|\mathrm{Im}\,\xi|}2^{m+1}/(2^{m+1}-|\lambda|-\varepsilon)\\
&+ C_1C_2(|\lambda|+\varepsilon)\Bigg(1+\sum_{\alpha\in\mathbb{Z}}|h(\alpha)|\Bigg)(1+|\xi|)^{B_1+B_2}e^{(A_1+A_2+m)|\mathrm{Im}\,\xi|}.\end{aligned}$$

Hence

$$|\hat{\Phi}_n(\xi)| \le C_4(1+|\xi|)^{B_4}e^{A_4|\mathrm{Im}\,\xi|}, \tag{2.10}$$

where A_4, B_4, C_4 are positive constants independent of n and ξ.

The next step is to show the uniform convergence of the sequence on bounded sets. Let K be a bounded subset of $\mathbb{C}^s$. Then (2.10) and $\tilde{f}(0) = 1$ imply that there is a positive constant C_K such that, for $n \in \mathbb{N}$ and $\xi \in K$,

$$|\hat{\Phi}_n(\xi)| + |\xi|^{m+1}e^{A_3|\mathrm{Im}\,\xi|} + 2^n|\tilde{f}(\xi/2^n) - 1| \le C_K.$$

Thus, for $2 \le n \in \mathbb{N}$,

$$\begin{aligned}
&|\hat{\Phi}_n(\xi) - \hat{\Phi}_{n-1}(\xi)| \\
&\le \left|\tilde{f}\left(\frac{\xi}{2^n}\right) - 1\right| |\hat{\Phi}_{n-1}(\xi)| + \left|\tilde{g}\left(\frac{\xi}{2^n}\right)\right| \left|\prod_{n-1}(\xi)e_2\right| \\
&\le 2^{-n}C_K^2 + 2^{-n(m+1)}C_3C_KC_1(|\lambda| + \varepsilon)^{n-1}(1 + |\xi|)^{B_1}e^{A_1|\mathrm{Im}\,\xi|} \\
&\le \mathrm{const}\left\{2^{-n} + \left(\frac{|\lambda| + \varepsilon}{2^{m+1}}\right)^n\right\}.
\end{aligned}$$

Therefore, the sequence $\{\hat{\Phi}_n(\xi)\}_{n\in\mathbb{N}}$ converges uniformly on K. This implies that the function $\lim_{n\to\infty} \hat{\Phi}_n(\xi)$ is analytic on $\mathbb{C}^s$.

By the Paley–Wiener theorem, we conclude from the estimate (2.10) that there is a compactly supported distribution vector $\phi = (\phi_1, \phi_2)^T$ on $\mathbb{R}^s$ such that

$$\hat{\phi}(\xi) = \lim_{n\to\infty} \hat{\Phi}_n(\xi), \quad \xi \in \mathbb{C}^s.$$

By (2.4), $\hat{\phi}(0) = e_1$ and $\hat{\phi}$ satisfies (2.2), and so (1.2) holds for ϕ.

The proof of Theorem 1 is now complete. □

From the proof of Theorem 1, it can be easily seen that if $D^\alpha \tilde{a}_{21}(0) = 0$ for any $\alpha \ge 0$ with $|\alpha|_1 \le m$, then $t_\alpha = 0$ and hence $H(\alpha) = 0$ for any $\alpha \ge 0$ with $|\alpha|_1 \le m$. This fact gives a simple proof of Theorem 2.

PROOF OF THEOREM 2. We follow the notation and discussion in the proof of Theorem 1.

(Sufficiency) Suppose that the function $\tilde{a}_{21}(\xi)$ has a zero of exact order $d \in \mathbb{N}$ at the origin, and that $|\lambda| < 2^d$. Then either $|\lambda| < 2$ or $2^m \le |\lambda| < 2^{m+1}$ with $m < d$ for some $m \in \mathbb{N}$, which implies that $D^\alpha \tilde{a}_{21}(0) = 0$ for any $\alpha \in \mathbb{Z}_+^s$ with $|\alpha|_1 \le m$. As observed previously, $h \equiv 0$. Hence, for $n \in \mathbb{N}$ and $\xi \in \mathbb{C}^s$, $\hat{\Phi}_n(\xi) = \prod_n(\xi)e_1$. The proof of Theorem 1 implies that $\lim_{n\to\infty} \prod_n(\xi)e_1$ converges for every $\xi \in \mathbb{R}^s$ and that $\lim_{n\to\infty} \prod_n(\xi)e_1 = \hat{\phi}(\xi)$ for a compactly supported distribution solution ϕ of (1.2) with $\hat{\phi}(0) = e_1$. This proves the sufficiency.

(Necessity) Suppose that $\lim_{n\to\infty} \prod_n(\xi)e_1$ converges for every $\xi \in \mathbb{R}^s$ and that $|\lambda| \ge 2^d$ for some $d \in \mathbb{N}$.

Consider a new refinement mask c given by $c(\alpha) = a(\alpha)/\lambda$ for $\alpha \in \mathbb{Z}^s$. By the proof of the sufficiency, there exists a compactly supported distribution vector ψ on $\mathbb{R}^s$ such that $\hat{\psi}(0) = e_2$ and

$$\hat{\psi}(\xi) = \lim_{n\to\infty} \lambda^{-n} \prod_n(\xi)e_2, \quad \xi \in \mathbb{R}^s.$$

If $\tilde{a}_{21}(\xi)$ has a zero of exact order d at the origin, then there exists a nonzero homogeneous polynomial p of exact degree d such that

$$|\tilde{a}_{21}(\xi) - p(\xi)| = O(|\xi|^{d+1}), \quad |\xi| \to 0.$$

By the definition of $\prod_n$, for $\xi \in \mathbb{R}^s$ we have

$$\prod_n(\xi)e_1 = \tilde{a}_{11}(\xi/2^n)\prod_{n-1}(\xi)e_1 + \tilde{a}_{21}(\xi/2^n)\prod_{n-1}(\xi)e_2.$$

This implies that, for any fixed $\xi \in \mathbb{R}^s$,

$$\begin{aligned} p(\xi)\hat{\psi}(\xi) &= \lim_{n\to\infty}\left\{2^{nd}p\left(\frac{\xi}{2^n}\right)\lambda^{1-n}\prod_{n-1}(\xi)e_2\right\} \\ &= \lim_{n\to\infty}\left\{2^{nd}\tilde{a}_{21}\left(\frac{\xi}{2^n}\right)\lambda^{1-n}\prod_{n-1}(\xi)e_2\right\} \\ &= \lim_{n\to\infty}\left\{\left(\frac{2^d}{\lambda}\right)^n\lambda\left[\prod_n(\xi)e_1 - \tilde{a}_{11}\left(\frac{\xi}{2^n}\right)\prod_{n-1}(\xi)e_1\right]\right\} = 0. \end{aligned}$$

However, $p(\xi_0) \neq 0$ for some $\xi_0 \in \mathbb{R}^s$. Hence

$$p(\xi_0/2^n)\hat{\psi}(\xi_0/2^n) = 2^{-nd}p(\xi_0)\hat{\psi}(\xi_0/2^n) \neq 0$$

for sufficiently large $n \in \mathbb{N}$, which is a contradiction. Thus, we must have $|\lambda| < 2^d$, thereby completing the proof of Theorem 2. □

Proof of Theorem 3. We use the notation in the proof of Theorems 1 and 2. Let $u = (u_1, u_2)^T$ be a nonzero vector in $\mathbb{C}^2$. Suppose that $\tilde{a}_{21}(\xi)$ has a zero of exact order $d \in \mathbb{N}$ at the origin, and that $|\lambda| \geq 2^d > 1$.

If $u_2 = 0$, then Theorem 2 implies that $\lim_{n\to\infty}\prod_n(\xi)u$ does not converge. If $u_2 \neq 0$, then we first need to estimate $|\prod_n(\xi)e_1|$.

By the condition on $\tilde{a}_{21}$, we know that (2.9) holds for $g = a_{21}$ and $m = d - 1$. As in the proof of Theorem 1, this in connection with (2.5) and (2.6) for $h \equiv 0$ and the Lemma implies that, for $2 \leq n \in \mathbb{N}$ and $\xi \in \mathbb{C}^s$,

$$\begin{aligned} \left|\prod_n(\xi)e_1\right| &\leq \sum_{j=2}^{n} C_2(1+|\xi/2^j|)^{B_2}e^{A_2|\operatorname{Im}\xi/2^j|} \\ &\quad\times C_3|\xi/2^j|^d e^{A_3|\operatorname{Im}\xi/2^j|}C_1|\lambda|^{j-1}(1+|\xi|)^{B_1}e^{A_1|\operatorname{Im}\xi|} \\ &\quad + C_2(1+|\xi/2|)^{B_2}e^{A_2|\operatorname{Im}\xi/2^j|}C_1|\lambda|(1+|\xi|)^{B_1}e^{A_1|\operatorname{Im}\xi|} \\ &\leq \left\{\frac{C_1C_2C_3}{|\lambda|-2^d} + C_1C_2|\lambda|\right\}\left(\frac{|\lambda|}{2^d}\right)^n \\ &\quad\times(1+|\xi|)^{B_1+B_2+d}e^{(A_1+A_2+A_3)|\operatorname{Im}\xi|}. \end{aligned}$$

Thus, for $\xi \in \mathbb{C}^s$,

$$\begin{aligned} |\lambda|^{-n}\left|\prod_n(\xi)u\right| &= \left|u_2\lambda^{-n}\prod_n(\xi)e_2 + u_1\lambda^{-n}\prod_n(\xi)e_1\right| \\ &\to |u_2||\hat{\psi}(\xi)| \quad (n\to\infty). \end{aligned}$$

Therefore, for $\xi \in \mathbb{R}^s$ with $\hat{\psi}(\xi) \neq 0$, $\{\prod_n(\xi)u\}_{n\in\mathbb{N}}$ is unbounded. Hence $\lim_{n\to\infty}\prod_n(\xi)u$ does not converge.

The proof of Theorem 3 is complete. □

PROOF OF THEOREM 4. The sufficiency is an easy consequence of Theorem 2 with $\lambda = 2$.

Let us prove the necessity. Suppose that (1.2) has a solution $\phi = (\phi_1, \phi_2)^T$, a vector of compactly supported distributions on $\mathbb{R}^s$ with $\hat{\phi}(0) \neq 0$. Then, by (1.2),

$$\hat{\phi}(2\xi) = \tilde{a}(\xi)\hat{\phi}(\xi), \quad \xi \in \mathbb{R}^s.$$

Let $\xi = 0$. It follows that $\hat{\phi}(0) = ce_1$ for some $c \neq 0$. Taking derivatives at the origin on both sides, for $1 \leq j \leq s$ we obtain

$$2\frac{\partial \hat{\phi}}{\partial \xi_j}(0) = \frac{\partial \tilde{a}}{\partial \xi_j}(0)\hat{\phi}(0) + \tilde{a}(0)\frac{\partial \hat{\phi}}{\partial \xi_j}(0).$$

This implies that for $1 \leq j \leq s$,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{\partial \hat{\phi}}{\partial \xi_j}(0) = c \begin{bmatrix} \frac{\partial \tilde{a}_{11}}{\partial \xi_j}(0) \\ \frac{\partial \tilde{a}_{21}}{\partial \xi_j}(0) \end{bmatrix}.$$

Hence

$$\frac{\partial \tilde{a}_{21}}{\partial \xi_j}(0) = 0 \quad \forall j = 1, \ldots, s.$$

Thus, $\tilde{a}_{21}(\xi)$ has a zero of order at least 2 at the origin. This completes the proof of Theorem 4. □

3. Examples and Proof of a Conjecture

In this section we apply our theorems to some examples and prove a conjecture raised in [10].

EXAMPLE 1. Let $s = 1$. Let the refinement mask a be supported on [0, 3] and be given by

$$a(0) = \begin{bmatrix} -3 & -4\sqrt{2} \\ \frac{\sqrt{2}}{4} & \frac{3}{2} \end{bmatrix}, \qquad a(1) = \begin{bmatrix} -3 & 0 \\ -\frac{9\sqrt{2}}{4} & -5 \end{bmatrix},$$

$$a(2) = \begin{bmatrix} 0 & 0 \\ -\frac{9\sqrt{2}}{4} & \frac{3}{2} \end{bmatrix}, \qquad a(3) = \begin{bmatrix} 0 & 0 \\ \frac{\sqrt{2}}{4} & 0 \end{bmatrix}.$$

We choose an eigenvector $v = (\sqrt{2}, -2)^T$ of the matrix $M = \begin{bmatrix} -3 & -2\sqrt{2} \\ -2\sqrt{2} & -1 \end{bmatrix}$ with eigenvalue 1, and apply Theorems 2 and 3 to the investigation of the convergence of (1.4). Toward this end, set $P = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -2 & 1 \end{bmatrix}$ and define a new mask b by $b(\alpha) = P^{-1}a(\alpha)P$ for $\alpha \in \mathbb{Z}$. Then $\tilde{b}(0) = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}$.

We apply our theorems to b. Observe that $\lambda = -5$ and $\tilde{b}_{21}(\xi) = e_2^T P^{-1}\tilde{a}(\xi)Pe_1$ has a zero of exact order $d = 1$ at the origin. By Theorem 1, the matrix refinement equation (1.2) with the mask a does have a compactly supported distribution solution ϕ with $\hat{\phi}(0) = v$. In fact, it was shown in [10] that $\phi = (\sqrt{2}\chi_{[0,1)}, -\chi_{[0,2)})^T$. However, by Theorem 2, the limit in (1.4) does not exist. This proves a conjecture of Heil and Colella in [10]. Moreover, it is conjectured in [10] that, for any nonzero vector $u \in \mathbb{C}^2$, the sequence $\{\prod_n(\xi)u\}_{n\in\mathbb{N}}$ does not converge. Theorem 3 tells us that this conjecture is also true.

The next refinement mask was given in [6]; see also [2].

EXAMPLE 2. Let $s = 1$ and $t \in \mathbb{R}\setminus\{-2\}$. Let the refinement mask a be supported on [0, 3] and be given by

$$a(0) = \begin{bmatrix} -\frac{t^2-4t-3}{2(t+2)} & 1 \\ -\frac{3(t-1)(t+1)(t^2-3t-1)}{4(t+2)^2} & \frac{3t^2+t-1}{2(t+2)} \end{bmatrix}, \quad a(1) = \begin{bmatrix} -\frac{t^2-4t-3}{2(t+2)} & 0 \\ -\frac{3(t-1)(t+1)(t^2-t+3)}{4(t+2)^2} & 1 \end{bmatrix},$$

$$a(2) = \begin{bmatrix} 0 & 0 \\ -\frac{3(t-1)(t+1)(t^2-t+3)}{4(t+2)^2} & \frac{3t^2+t-1}{2(t+2)} \end{bmatrix}, \quad a(3) = \begin{bmatrix} 0 & 0 \\ -\frac{3(t-1)(t+1)(t^2-3t-1)}{4(t+2)^2} & 0 \end{bmatrix}.$$

Then

$$M = \begin{bmatrix} -\frac{t^2-4t-3}{2(t+2)} & \frac{1}{2} \\ -\frac{3(t^2-1)(t-1)^2}{2(t+2)^2} & \frac{1}{2} + \frac{3t^2+t-1}{2(t+2)} \end{bmatrix}.$$

This matrix has two eigenvalues, 1 and $\lambda := t$, with corresponding eigenvectors $v := (1, (t-1)^2/(t+2))^T$ and $(1, 3(t^2-1)/(t+2))^T$, respectively. Theorem 1 tells us that for every $t \in \mathbb{R}\setminus\{-2\}\setminus\{2^n : n \in \mathbb{N}\}$, there always exists a compactly supported distribution solution ϕ with $\hat{\phi}(0) \neq 0$.

Set

$$P = \begin{bmatrix} 1 & 1 \\ \frac{(t-1)^2}{t+2} & \frac{3(t^2-1)}{t+2} \end{bmatrix}.$$

Then $P^{-1}MP = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}$. Let $b(\alpha) = P^{-1}a(\alpha)P$ for $\alpha \in \mathbb{Z}$. Note that

$$\tilde{b}_{21}(\xi) = \frac{\begin{array}{r}(-3t^3+6t^2+12t+3)e^{-i3\xi} + (3t^3-4t^2-10t-7)e^{-i2\xi} \\ + (-t^3-6t^2-11)e^{-i\xi} + (t^3+4t^2-2t+15)\end{array}}{16(t+2)^2}.$$

We see that $\tilde{b}_{21}$ has a zero of exact order d at the origin, where $d = 1$ for $|t| > 2$ while $d = 2$ for $t = 2$. Thus the discussion about the canonical form and Theorem 2 tell us that $\lim_{n\to\infty}\prod_n(\xi)v$ converges if and only if $-2 < t \leq 2$.

Let us mention that for $t \in \mathbb{R}\setminus\{-2, 1\}\setminus\{2^n : n \in \mathbb{N}\}$, the integer translates of ϕ are linearly independent. This can be proved using methods similar to those in [13; 16; 19], for which we omit the details here. This fact is interesting, since Dahmen and Micchelli [3] show that if the integer translates of an integrable multiple refinable function are stable, then M has an eigenvalue 1 and all its other eigenvalues are less than 1 in modulus. Therefore, we know that for $t \in \mathbb{R}\setminus\{-2, 1\}\setminus\{2^n : n \in \mathbb{N}\}$ with $|t| \geq 1$, ϕ is not integrable. The case $t = 1$ corresponds to $\phi = (\chi_{[0,1)}, 0)^T$.

Our final refinement mask was given by Jia, Riemenschneider, and Zhou in [15].

EXAMPLE 3. Let $s = 1$. Let the refinement mask a be supported on [0, 2] and be given by

$$a(0) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \beta & \beta \end{bmatrix}, \quad a(1) = \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix}, \quad a(2) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\beta & \beta \end{bmatrix},$$

where $\beta, y \in \mathbb{R}$ are two parameters.

It is easily seen that $M = \begin{bmatrix} 1 & 0 \\ 0 & \beta+\frac{y}{2} \end{bmatrix}$ has two eigenvalues, 1 and $\beta + y/2$, with corresponding eigenvectors e_1 and e_2, respectively. Theorem 1 tells us that when $\beta + y/2 \notin \{2^n : n \in \mathbb{N}\}$, there always exists a compactly supported distribution solution ϕ of (1.2) with this mask satisfying $\hat{\phi}(0) = e_1$.

Note that $\tilde{a}_{21}(\xi) = \beta(1 - e^{-i2\xi})$. Hence $\tilde{a}_{21}$ has a zero of exact order 1 at the origin if $\beta \neq 0$, while $\tilde{a}_{21} \equiv 0$ if $\beta = 0$. When $\beta \neq 0$, Theorem 2 tells that $\lim_{n\to\infty} \prod_n(\xi)e_1$ converges for every $\xi \in \mathbb{R}$ if and only if $|\beta + y/2| < 2$. When $\beta = 0$, $\lim_{n\to\infty} \prod_n(\xi)e_1$ always converges for every $\xi \in \mathbb{R}$; in fact,

$$\lim_{n\to\infty} \prod_n(\xi)e_1 = \left(\prod_{j=1}^{\infty}\left\{\frac{(1 + e^{-i\xi/2^j})^2}{4}\right\}, 0\right)^T \quad \text{for } \xi \in \mathbb{R},$$

and $\phi = (\phi_1, 0)^T$ with ϕ_1 the hat function supported on [0, 2].

Let $\beta \neq 0$ and $\beta + y/2 = 2$. Then Theorem 4 tells that (1.2) with this mask does not have a compactly supported distribution solution ϕ with $\hat{\phi}(0) \neq 0$.

The interest of this example arises from the case where $-\sqrt{2}/2 \leq \beta < -\frac{1}{2}$ and $y = \sqrt{2 - 4\beta^2}$; see [15]. In this case, with the refinement mask supported on [0, 2], ϕ is a solution of (1.2) and is a continuous real-valued orthogonal multiple refinable function supported on [0, 2]. Also, ϕ_1 is symmetric about 1 while ϕ_2 is antisymmetric about 1.

Note added in the revised version. After submitting this paper, I received a preprint of Q. Jiang and Z. Shen entitled "On the existence and weak stability of matrix refinable functions", in which Theorem 1 is proved for arbitrary multiplicity under the same assumption that 2^n is not an eigenvalue of M for any $n \in \mathbb{N}$. Their technique for the proof of this result is the same as that in the proof of Theorem 1.

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