Holomorphic Flows, Cocycles, and Coboundaries

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Dedicated to Professor Frank Forelli in fond remembrance

1. Introduction

Cocycles appear in many areas of analysis (harmonic analysis, representation theory, operator theory, ergodic theory, etc.) and, indeed, they are present whenever a group or a semigroup acts as a transformation group on some space. In a sense, cocycles are generalizations of the exponential function and provide a measure of "normality" of the underlying group action. We are primarily concerned with the semigroup action provided by a holomorphic flow on a domain in the complex plane.

The properties of semigroups of holomorphic flows may be studied by replacing these semigroups by any member of a large class of isospectral operators generated from the above semigroups by certain types of cocycles called *coboundaries*. This motivation has led us to investigate when cocycles are coboundaries, and in doing so, we are led to a complete description of all holomorphic flows on $\mathbb{C}$. Our approach and techniques are quite direct and independent of operator-theoretic considerations.

The relatively recent study of holomorphic flows was initiated by Berkson and Porta [BP], who showed the strong continuity of these flows on Hardy spaces. Cowen [C1] provided an interesting application of holomorphic flows on Hardy spaces to prove, among other things, that the Cesàro operator is subnormal. Siskakis [S1; S2] extended the results of [BP] to Bergman spaces and applied weighted holomorphic flows on Hardy spaces to the study of the Cesàro operator. König [Ko] investigated weighted holomorphic flows on the unit disc and gave a characterization of the smooth cocycles on the Hardy space. Some of our results complement those found in [BP] and [Ko], but our techniques are considerably different. Related ideas also appear in [EJ; F; H; J; JY; SM; Y]. An extensive article outlining the history of translation flows and their applications to dynamical systems is given by Latushkin and Stepin [LS].

Our notation and terminology are as follows. Let G be a domain (open, connected and nonempty) in the complex plane $\mathbb{C}$, and let $H(G)$ be the set of holomorphic functions on G. We shall use Δ to denote the open unit disc in $\mathbb{C}$. A one-parameter family $\varphi(t, z)$ of nonconstant holomorphic functions from G to G that satisfy $\varphi(0, z) = z$ and $\varphi(s + t, z) = \varphi(s, \varphi(t, z))$ for all $s, t \geq 0$ and $z \in G$

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is called a *semigroup flow*. Likewise, if $s, t \in \mathbb{R}$ then this is called a *group flow*. For technical reasons, we will require the flow to be continuous on $[0, \infty) \times G$. We say z_0 is a *fixed point* for φ if $\varphi(t, z_0) = z_0$ for all $t \geq 0$.

A continuous complex-valued function m on $[0, \infty) \times G$ is said to be a (multiplicative) *cocycle* for φ when m satisfies:

$$\begin{aligned} m(t, \cdot) &\in H(G) && \text{for all } t \geq 0, \\ m(0, z) &= 1 && \text{for all } z \in G, \\ m(t + s, z) &= m(s, z)m(t, \varphi(s, z)) && \text{for all } t, s \geq 0, z \in G. \end{aligned} \tag{1.1}$$

When φ is understood from the context, m will simply be called a *cocycle*. The third equation above is often called the cocycle identity; it implies that $m(0, z)$ is 1 or 0, so the second equation is simply a nontriviality condition. König [Ko, Lemma 2.1(b)] shows that the cocycle condition implies that m is nonvanishing.

Every function of the form $m(t, z) = \exp(it\lambda)$ for all $z \in \mathbb{C}$ and some fixed $\lambda \in \mathbb{C}$ is a cocycle for every flow; cocycles of this type are called *constant cocycles*. Conversely, *every* cocycle $m(t, z)$ that is constant in z for each fixed $t \geq 0$ is a constant cocycle.

A continuous complex-valued function m on $[0, \infty) \times G$ is said to be a *coboundary* for φ when there exists a nonvanishing function $\alpha \in H(G)$ such that

$$m(t, z) = \frac{\alpha(\varphi(t, z))}{\alpha(z)} \quad \text{for all } (t, z) \in [0, \infty) \times G. \tag{1.2}$$

It is easy to verify that every coboundary for φ is a cocycle for φ. The constant cocycle $\exp(it\lambda)$ is a coboundary for the particular flow $\varphi(t, z) = z - t$ on $\mathbb{C}$, since $\exp(it\lambda) = \exp(-i\lambda(z - t))/\exp(-i\lambda z)$. However, the constant cocycle $\exp(it\lambda)$ is not a coboundary for all flows, as shown in Section 5.

Finally, it is also convenient to introduce the additive analogs of multiplicative cocycles and coboundaries. We shall say that $a(t, z)$ is an *additive cocycle* for a semigroup flow φ if $a(0, z) = 0$ and $a(t, z)$ satisfies the cocycle identity

$$a(t + s, z) = a(t, z) + a(s, \varphi(t, z)) \quad \text{for every } s, t \geq 0,\ z \in G. \tag{1.3}$$

Similarly, a is an *additive coboundary* if there exists a function $\beta \in H(G)$ such that

$$a(t, z) = \beta(\varphi(t, z)) - \beta(z) \quad \text{for each } t \geq 0,\ z \in G. \tag{1.4}$$

It is easy to see that letting $m(t, z) = \exp(a(t, z))$ relates the two notions of cocycles. We distinguish between the additive and the multiplicative cocycles by specifying additive where appropriate. Thus, "cocycle" shall refer to the multiplicative cocycles.

Cocycles of the type we are discussing arise naturally in the theory of semigroups of weighted composition operators. The family $(T_t)_{t \geq 0}$ of *composition operators* on $H(G)$ is given by

$$(T_t f)(z) = f(\varphi(t, z)) \quad \text{for every } t \geq 0,\ f \in H(G).$$

The semigroup property $T_{t+s} = T_t T_s$ is satisfied because $\varphi(t, z)$ is a flow. If $(m(t, z))_{t \geq 0}$ is a family of holomorphic functions on G, then we can define a family $(S_t)_{t \geq 0}$ of *weighted composition operators* on $H(G)$ by

$$(S_t f)(z) = m(t, z)(T_t f)(z) \quad \text{for each } t \geq 0,\ f \in H(G).$$

It is easy to show that $(S_t)_{t \geq 0}$ is a semigroup if and only if m is a cocycle for φ [EJ, Thm. 2.1]. If m is a coboundary then S has the simpler representation $(S_t f)(z) = \alpha^{-1}(z)(T_t \alpha f)(z)$, which can be abbreviated as

$$S_t = \alpha^{-1} T_t \alpha.$$

This relation establishes a similarity between the operators S and T that allows one to determine properties of S from known properties of T. This reduction is familiar in the contexts of matrix and operator theory. Consequently, it is important to determine which cocycles are coboundaries.

Our terminology is adapted from group cohomology (see [E], [EM], or [Ku]). The group $\mathbb{R}$ or the semigroup $[0, \infty)$ is regarded as acting on the (multiplicative) coefficient group $H^{-1}(G)$ of *nonvanishing* holomorphic functions on G by means of the flow. Thus $t \in [0, \infty)$ acts on $h \in H^{-1}(G)$ by $(t : h)(z) = h(\varphi(t, z))$. From this viewpoint the "cocycles are coboundaries" results are simply statements that the first cohomology group is trivial. Most of the results and arguments in this paper are readily adapted to the coefficient group $M(G)$ of meromorphic functions on a domain G in the Riemann sphere S^2. However, replacing $H^{-1}(G)$ by a more restricted coefficient group, say the *bounded* invertible holomorphic functions on G, leads to considerations beyond the scope of this paper.

This paper is organized as follows. In Section 2 we provide a complete characterization of group and semigroup flows on $\mathbb{C}$. We show that every semigroup flow on $\mathbb{C}$ automatically extends to a group flow, and that these flows take one of two forms based on the number of their fixed points. We combine this result and the known result showing that flows on simply connected proper domains in $\mathbb{C}$ (e.g., the disc) are generated, under conjugation with conformal maps, by restriction of exponential and translation flows to invariant subsets of $\mathbb{C}$ to calculate the form of flows on the disc. In Section 3 we consider the question of which cocycles are coboundaries for group flows on $\mathbb{C}$. We show that, under an integrability hypothesis, cocycles are always coboundaries. Since group flows are semigroup flows, we treat the more general case of semigroup flows in Section 4. Section 4 provides a complete answer to the question of which cocycles for semigroup flows are coboundaries. The answer here is given in terms of existence of fixed points of the flow. Finally, motivated by an example, in Section 5 we find a simple condition that enables one to construct cocycles by modifying *projective cocycles*. This construction exploits the geometry of the level sets of harmonic functions. The triviality of the second cohomology group of the reals acting in the nonzero complex numbers makes a cameo appearance.

2. Holomorphic Flows

In this section we will describe the holomorphic flows on $\mathbb{C}$ and on simply connected proper domains in $\mathbb{C}$. To do so we need several elementary lemmas.

LEMMA 2.1. *Let $G \subseteq \mathbb{C}$ be a domain, and let $\varphi\colon [0,\infty) \times G \to \mathbb{C}$ be a jointly continuous family of holomorphic mappings of G into $\mathbb{C}$. Then $\varphi'(t,z) = \frac{\partial\varphi}{\partial z}(t,z)$ is also a jointly continuous family of holomorphic mappings of G into $\mathbb{C}$.*

Proof. Let $s, t \geq 0$ and let z and w be contained in a small open disk $\mathbb{D}_\varepsilon \subseteq \mathbb{D}_{2\varepsilon} \subseteq G$ of radius ε. From Cauchy's integral formula for derivatives, we obtain the estimate

$$|\varphi'(t,z) - \varphi'(s,w)| \leq \frac{B}{2\pi}\int_\gamma |(\xi - w)^2\varphi(t,\xi) - (\xi - z)^2\varphi(s,\xi)||d\xi|,$$

where $\gamma = \partial\mathbb{D}_{2\varepsilon}$ and $B = \max\{\,|1/(\xi - w)^2(\xi - z)^2| : \xi \in \gamma\,\}$. Fix $(s,w) \in I \times \mathbb{D}_\varepsilon$, where $I \subseteq \mathbb{R}$ is a closed interval. The integrand is continuous and hence bounded as a function of (t,z,ξ) on the compact set $I \times \bar{\mathbb{D}}_\varepsilon \times \gamma$. The result follows from the bounded convergence theorem. □

We remark that, under the hypotheses of Lemma 2.1, the same conclusion holds for $\frac{\partial^n\varphi}{\partial z^n}(t,z)$ for every $n \in \mathbb{N}$.

LEMMA 2.2 (Univalence of holomorphic semigroup flows). *Let $G \subseteq \mathbb{C}$ be a convex domain, and let $\varphi\colon [0,\infty) \times G \to G$ be a holomorphic semigroup flow. Then $\varphi(t,\cdot)$ is univalent in G for every $t \in [0,\infty)$.*

Proof. Since $\varphi(0,z) = z$, we have $\varphi'(0,z) = 1$ and $\Re\frac{\partial\varphi}{\partial z}(0,z) = 1$ for all $z \in G$. Assume that there exist $a, b \in G$ such that $\varphi(t,a) = \varphi(t,b)$ for some t. Let $\tau > 0$ be the infimum of these t. By continuity, $\varphi(\tau,a) = \varphi(\tau,b)$. Consider a compact convex set $K \subset G$ that contains $\varphi([0,\tau] \times [a,b])$. By joint continuity of $\varphi(t,z)$ and Lemma 2.1, $\varphi'(t,z)$ is jointly continuous in t and z and hence there exists $t_0 > 0$ such that $\Re\frac{\partial\varphi}{\partial z}(t,z) > 0$ for all $0 \leq t \leq t_0$ and $z \in K$. Thus, by the Noshiro–Warshawski–Wolff theorem (or by the complex Rolle's theorem for more general domains [EJP]), $\varphi(t,\cdot)$ is univalent for each $0 \leq t \leq t_0$ in K. Note that $t_0 < \tau$ and

$$\varphi(t_0, \varphi(\tau - t_0, a)) = \varphi(\tau, a) = \varphi(\tau, b) = \varphi(t_0, \varphi(\tau - t_0, b))$$

by the flow property. This is impossible since $\varphi(\tau - t_0, a) \neq \varphi(\tau - t_0, b)$ are in K and $\varphi(t_0,\cdot)$ is univalent in K. □

We note that the convexity assumption in Lemma 2.2 is unnecessary if we replace the Noshiro–Warshawski–Wolff theorem by the more general theorem of [EJP]. Since a group flow is a semigroup flow, Lemma 2.2 also holds for group flows on $G \subseteq \mathbb{C}$. Using different methods, Berkson and Porta [BP] also note the result of Lemma 2.2 for the case of the open unit disc.

Observe that if $\varphi(t, \cdot)$ is a holomorphic semigroup flow on $\mathbb{C}$, then Lemma 2.2 shows that $\varphi(t, \cdot)$ is univalent in $\mathbb{C}$ and therefore ∞ is the only pole for $\varphi(t, \cdot)$ for each $t \in [0, \infty)$. Hence every holomorphic semigroup flow $\varphi(t, \cdot)$ on $\mathbb{C}$ consists of linear functions; that is, $\varphi(t, z) = a(t)z + b(t)$. In particular $\varphi(t, \cdot)$ maps $\mathbb{C}$ onto $\mathbb{C}$ for every $t \in [0, \infty)$ (also see [BP, p. 110]). For proper subdomains of $\mathbb{C}$ the flow need not be onto, as demonstrated by the following simple example. Let $G = \Delta = \{z \in \mathbb{C} : |z| < 1\}$, and put $\varphi(t, z) = e^{-t}z$. Then, for every $t > 0$, $\mathrm{Im}(\varphi(t, \cdot))$ is a proper subset of Δ.

THEOREM 2.3 (Characterization of semigroup flows on $\mathbb{C}$). *If $\varphi(t, \cdot)$ is a nontrivial holomorphic semigroup flow on $\mathbb{C}$, then φ can have at most one fixed point in $\mathbb{C}$ and one of the following holds:*

(i) *If φ does not have a fixed point in $\mathbb{C}$, then $\varphi(t, z) = z + Kt$ for some $K \in \mathbb{C}$, $K \neq 0$.*
(ii) *If φ has one fixed point at $K \in \mathbb{C}$, then $\varphi(t, z) = e^{\alpha t}z + K(1 - e^{\alpha t})$ for some $\alpha \in \mathbb{C}$, $\alpha \neq 0$.*

In particular, every semigroup flow on $\mathbb{C}$ extends to a group flow on $\mathbb{C}$.

Proof. By Lemma 2.2, $\varphi(t, \cdot)$ is univalent and so $\varphi(t, z) = a(t)z + b(t)$ for some continuous functions $a(t)$ and $b(t)$. Note that $a(t) \neq 0$ since $\varphi(t, z)$ is univalent. Employing the semigroup property of the flow yields

$$a(t+s)z + b(t+s) = a(t)(a(s)z + b(s)) + b(t) = a(s)a(t)z + a(t)b(s) + b(t).$$

Equating the coefficients of the equal powers of z, we have

$$a(t + s) = a(t)a(s) \tag{2.1}$$

and

$$b(t + s) = a(t)b(s) + b(t). \tag{2.2}$$

Clearly, (2.1) and the continuity of a force $a(t) = e^{\alpha t}$, $\alpha \in \mathbb{C}$. If $\alpha = 0$, then $a(t) \equiv 1$, $\varphi(t, z) = z + b(t)$, and (2.2) becomes $b(t + s) = b(s) + b(t)$. Hence $b(t)$ is a continuous linear function in t with $b(0) = 0$; that is, $b(t) = Kt$ for some $K \in \mathbb{C}$. Therefore, in this case $\varphi(t, z) = z + Kt$.

If $\alpha \neq 0$ then $a(t) = e^{\alpha t}$, and plugging this expression for $a(t)$ into (2.2) gives

$$b(t + s)e^{-\alpha s} = b(s)e^{-\alpha s} + b(t). \tag{2.3}$$

Letting $s = 1$ in (2.3), we have

$$b(t + 1)e^{\alpha} - b(1)e^{\alpha} = b(t) \quad \forall t \geq 0, \tag{2.4}$$

and letting $t = 1$ in (2.3) gives

$$b(1 + s)e^{\alpha s} - b(s)e^{\alpha s} = b(1) \quad \forall s \geq 0. \tag{2.5}$$

We solve (2.5) for $b(s)$ and change s to t to obtain

$$b(t + 1) - b(1)e^{-\alpha t} = b(t) \quad \forall t \geq 0. \tag{2.6}$$

Eliminating $b(t + 1)$ from (2.4) and (2.6) gives $b(t) = K(1 - e^{-\alpha t})$ with $K = b(1)e^{\alpha}/(e^{\alpha} - 1)$. Hence $\varphi(t, z) = e^{\alpha t}z + K(1 - e^{\alpha t})$ and we have (ii).

Finally, we observe that the semigroup flows derived from these calculations are all group flows. Thus every holomorphic semigroup flow on $\mathbb{C}$ is naturally (embedded in) a group flow on $\mathbb{C}$. $\square$

Note that Theorem 2.3 states that flows on $\mathbb{C}$ are either translations or (complex) dilations and conversely. This result can also be obtained by the techniques developed by Berkson and Porta in [BP]. Because all semigroup flows on $\mathbb{C}$ are necessarily group flows, we may refer to semigroup flows on $\mathbb{C}$ simply as flows.

On the extended complex plane, the point at ∞ is a fixed point for both flows described in Theorem 2.3. Thus, in this setting one may assert that flows have either one or two fixed points. Replacing these fixed points by arbitrary points in $\mathbb{C}$, it is easy to describe the flows on S^2 as follows.

(i) Flows with fixed points $a, b \in \mathbb{C}$:

$$\varphi(t,z) = \frac{a(z-b) - e^{\alpha t}(z-a)b}{z - b - e^{\alpha t}(z-a)} = \frac{(be^{\alpha t} - a)z + (1 - e^{\alpha t})ab}{(e^{\alpha t} - 1)z + b - ae^{\alpha t}}. \tag{2.7}$$

(ii) Flows with fixed points at $a \in \mathbb{C}$ and at ∞:

$$\varphi(t,z) = e^{\alpha t}z + a(1 - e^{\alpha t}). \tag{2.8}$$

(iii) Flows with a single fixed point at $a \in \mathbb{C}$:

$$\varphi(t,z) = \frac{z + aKt(z-a)b}{1 + Kt(z-a)} = \frac{(aKt+1)z - a^2bKt}{Ktz - Kta + 1}. \tag{2.9}$$

(iv) Flows with a single fixed point at ∞:

$$\varphi(t,z) = z + Kt, \tag{2.10}$$

where α, b and K are constants.

Observe that the flow $\varphi(t,z) = e^{\alpha t}z + K(1 - e^{\alpha t})$ obtained in Theorem 2.3 is constructed by pre- and post-composition with respect to z of the exponential flow with a translation; namely, if $\psi_1(t,z) = e^{\alpha t}z$ and $\psi_2(t,z) = z - K$ then $\varphi(t,z) = \psi_2^{-1} \circ \psi_1 \circ \psi_2(t,z)$. Also, the flows obtained in Theorem 2.3 do not share any common structure. Composition of flows from different classes do not commute; moreover, addition, multiplication, and/or division of flows from the different classes in this collection do not generate new flows.

Let G be a simply connected proper domain in $\mathbb{C}$. Since conformal conjugation preserves the flow properties, without loss of generality one can consider G to be the open unit disc Δ. Clearly, if φ is univalent then φ is a homeomorphism from Δ onto $\Im\varphi(\Delta)$. Thus $\Im\varphi(\Delta)$ is simply connected, and hence $\varphi(t,\cdot)$ is a Riemann map from Δ onto a simply connected subset of Δ for every $t > 0$. For the case in which the range is onto, Berkson, Kaufman, and Porta [BKP] give a complete and detailed description of flows consisting of Möbius transformations of the unit disc. Briefly, a family of Möbius transformations

$$\varphi(t,z) = e^{i\theta_t}\frac{z - \alpha_t}{1 - \overline{\alpha_t}z}, \quad |\alpha_t| < 1, \ \ \theta_s \in \mathbb{R},$$

of Δ onto Δ forms a holomorphic flow on Δ if and only if

$$e^{i(\theta_{t+s}-\theta_t-\theta_s)}(1+\alpha_s\bar{\alpha}_t e^{i\theta_s}) = 1+\overline{\alpha_s}\alpha_t e^{-i\theta_s};$$

$$\alpha_{t+s} = \frac{\alpha_t e^{-i\theta_s}+\alpha_s}{1+\overline{\alpha_s}\alpha_t e^{-i\theta_s}}, \quad \alpha_0 = 0, \ \ \theta_0 = 0.$$

For a more detailed description, see [BKP]. In general, we have the following theorem.

THEOREM 2.4 (Semigroup flows on Δ; see [C2] or [S2]). *Let $\phi(t,\cdot)\colon \mathbb{C}\to\mathbb{C}$ be a flow on $\mathbb{C}$. Then, under conjugacy with conformal maps and restrictions of ϕ to invariant subsets of $\mathbb{C}$, $\phi(t,\cdot)$ generates flows on Δ. Conversely, every semigroup flow on Δ is generated in this way.*

It is easy to see that the map

$$\varphi(t,z) = \frac{be^{-t}z}{(e^{-t}-1)z+b}$$

with $|b| = 1$ is a semigroup flow on Δ with fixed points 0 and b, while

$$\varphi(t,z) = \frac{e^{-t}-1+(e^{-t}+1)z}{(e^{-t}-1)z+(e^{-t}+1)}$$

is a semigroup flow on Δ with fixed points at $a = 1$ and $b = -1$. These flows arise from restrictions of the exponential flow on $\mathbb{C}$. On the other hand, if $\alpha > 0$ and

$$\varphi(t,z) = \frac{z(\alpha t-2)-\alpha t}{z(\alpha t)-(\alpha t+2)},$$

then φ is a flow on Δ that is generated from $\phi(t,z) = z+\alpha t$ on $\mathbb{C}$ restricted to the right half plane. This flow, which has a Denjoy–Wolff point at $z = 1$, arises from the fixed-point free flow on $\mathbb{C}$. Likewise, for any real α,

$$\varphi(t,z) = \frac{z-\tanh(\alpha t)}{1-(\tanh(\alpha t))z} = \frac{(e^{2\alpha t}+1)z-e^{2\alpha t}+1}{e^{2\alpha t}+1-(e^{2\alpha t}-1)z}$$

is a flow on Δ that is generated by the flow $\phi(t,z) = e^{-2\alpha t}z+\frac{1}{2}(e^{-2\alpha t}-1)$ on $\mathbb{C}$. This flow has two fixed points at ± 1, where 1 is a repelling fixed point and -1 is an attracting fixed point—the Denjoy–Wolff point.

Finally, it is worthwhile to note that the *unbounded sets* $S_c = \{x+iy : x > -c\}$, $c > 0$ are invariant for the flow $\varphi(t,z) = e^{-\alpha t}z$ ($\alpha > 0$) and $S_{c,d} = \{x+iy : x > c,\ y > d\}$ ($c, d \in \mathbb{R}$) are invariant for the flow $\varphi(t,z) = z+Kt$, $0 \le \arg K \le \pi/2$. By conjugacy, such sets may be used to construct a panorama of flows on Δ. Since flows on subsets of $\mathbb{C}$ are homotopy preserving, it may be interesting to extend these results to flows on multiply connected subsets of $\mathbb{C}$ or to Riemann surfaces.

3. Group Flows on $\mathbb{C}$, Cocycles, and Coboundaries

In this section we deal with the question of when cocycles of *group* flows on $\mathbb{C}$ are coboundaries. We provide an answer to this question under an integrability assumption on the cocycle and consider various examples to show that the same may be true under more relaxed conditions. We shall treat the same question for semigroup flows on general simply connected domains in $\mathbb{C}$ (including $\mathbb{C}$ itself) in Section 4 and shall present a complete answer.

EXAMPLE 3.1. The following example demonstrates that the answer to the above question is, in general, No. Let $\varphi(t,z) = e^t z$ and let $m(t,z) = \exp(ze^t + t - z)$. It is easy to verify that $\varphi(t,\cdot)$ is a group flow on $\mathbb{C}$, $m(t,\cdot)$ is a cocycle for φ, and $m(t,0) = e^t$ for all $t \in \mathbb{R}$. Suppose $m(t,z)$ is a coboundary for φ and α is such that $m(t,z) = \alpha(\varphi(t,z))/\alpha(z)$. Then $m(t,0) = \alpha(\varphi(t,0))/\alpha(0) = 1$ for all $t \in \mathbb{R}$. This contradicts $m(t,0) = e^t$ for all $t \in \mathbb{R}$.

However, under an integrability assumption, we have the following.

THEOREM 3.2. *Let $m: \mathbb{R} \times G \to \mathbb{C}$ be a cocycle, and suppose that $m(\cdot, z) \in L^1(\mathbb{R})$ for every $z \in G$. Then there exists a $\lambda \in \mathbb{R}$ such that*

$$\frac{e^{i\lambda t}}{m(t,z)} = \frac{\alpha(\varphi(t,z))}{\alpha(z)}$$

for some $\alpha \in H(G)$.

Proof. Let $I_t(f(t))$ denote the Lebesgue integral of $f \in L^1(\mathbb{R})$. Fix $z_0 \in G$. Since $m(\cdot, z) \not\equiv 0$ (as an $L^1(\mathbb{R})$-function) we can *fix* a real λ for which the Fourier transform $I_t(m(t,z_0)e^{-i\lambda t}) \neq 0$. Define $\alpha: G \to \mathbb{C}$ by $\alpha(z) = I_t(m(t,z)e^{-i\lambda t})$. Then, by the cocycle identity and the translation invariance of I_t,

$$\begin{aligned}
\alpha(z) &= I_t(m(t,z)e^{-i\lambda t}) \\
&= I_t(m(t+s,z)e^{-i\lambda(t+s)}) \\
&= I_t(m(s,z)e^{-i\lambda s}m(t,\varphi(s,z))e^{-i\lambda t}) \\
&= m(s,z)e^{-i\lambda s} I_t(m(t,\varphi(s,z))e^{-i\lambda t}) \\
&= m(s,z)e^{-i\lambda s}\alpha(\varphi(s,z)). \qquad (3.1)
\end{aligned}$$

A simple application of Morera's theorem shows that α is holomorphic. □

Observe that the zeros of α are isolated since $\alpha(z_0) \neq 0$. Moreover, since $m(s,z)e^{-i\lambda s}$ is zero-free for each $s \geq 0$, we see from (3.1) that the zeros of α must be fixed points for φ. Hence, on the complement of the fixed points of φ, the cocycle $e^{i\lambda t}/m(t,z) = \alpha(\varphi(t,z))/\alpha(z)$ is a coboundary.

Note that $m(t,z)$ is a coboundary in case α is zero-free and the constant cocycle $e^{i\lambda t}$ is a coboundary. We also note that the construction in Theorem 3.2 is valid if $m(\cdot, z)$ is a nonvanishing, continuous, almost periodic function on $\mathbb{R}$ for each $z \in G$ and if

$$I_t(f(t)) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(s)\,ds$$

is the invariant mean.

Finally, we consider a cocycle that, although it fails the hypotheses of Theorem 3.2, is a coboundary. We use this result as a motivation to prove various cocycles-are-coboundaries theorems for semigroup flows in the next section.

EXAMPLE 3.3. Define a flow $\varphi: \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ by

$$\varphi(t, z) = z - (1+i)t,$$

and let

$$m(t, z) = \exp(t^2 - t(x+y) + it(x-y)) = \exp(t^2 - t(1-i)z).$$

It is easy to verify that m is a cocycle for φ, and that $m(t, z)\exp(it\lambda)$ is neither almost periodic nor integrable in t for any $\lambda \in \mathbb{C}$. Thus, Theorem 3.2 does not apply. However, it is easy to verify that m is a coboundary with

$$\alpha(z) = \exp\left(xy + i\frac{y^2 - x^2}{2}\right) = \exp\left(-i\frac{z^2}{2}\right).$$

The smoothness in t of the cocycle in Example 3.3 plays an essential role here.

4. Semigroup Flows, Cocycles, and Coboundaries

In this section we develop various cocycles-are-coboundaries theorems under fixed-point conditions on the semigroup flow and smoothness conditions on the cocycles. Here, smoothness shall mean differentiability and existence of a limit of the derivative as $t \searrow 0$. We note that the results presented here overlap somewhat with König's results (see [Ko]); however, our conditions and techniques are quite different. We begin with a basic proposition that expresses the singularity condition (i.e., $\frac{\partial\varphi}{\partial t}(0, z_0) = 0$ for some z_0) for flows in terms of fixed points.

PROPOSITION 4.1. *Let $G \subseteq \mathbb{C}$ be a domain, and suppose that φ is a semigroup flow on G. Then φ is singular at $z_0 \in G$ if and only if z_0 is a fixed point of the flow φ.*

Proof. By Theorem 1.1 of [BP], the flow is continuously differentiable in t. If z_0 is a fixed point of φ, then clearly $\frac{\partial\varphi}{\partial t}(t, z_0) = 0$. Hence φ is singular at z_0. Conversely, suppose that $\frac{\partial\varphi}{\partial t}(0, z^\star) = 0$ for some $z^\star$. Differentiating the semigroup property for φ with respect to t, we arrive at

$$\frac{\partial\varphi}{\partial t}(t+s, z) = \frac{\partial\varphi}{\partial z}(s, \varphi(t, z))\frac{\partial\varphi}{\partial t}(t, z),$$

and letting $t = 0$ in this equation yields

$$\frac{\partial\varphi}{\partial t}(s, z) = \frac{\partial\varphi}{\partial z}(s, z)\frac{\partial\varphi}{\partial t}(0, z).$$

For $z = z^\star$ this in turn implies

$$\frac{\partial\varphi}{\partial t}(s, z^\star) = 0 \quad \text{for every } s \geq 0.$$

Thus $\varphi(s, z^\star) = c \in G$. Letting $s = 0$, we get $z^\star = \varphi(0, z^\star) = \varphi(s, z^\star)$; that is, $\varphi(s, z^\star) = z^\star$ for all $s \geq 0$. □

The following theorem demonstrates that, for fixed-point–free flows, smooth cocycles are always coboundaries.

THEOREM 4.2. *Let $G \subseteq \mathbb{C}$ be simply connected. Let $\varphi(t, z)$ be a flow on $G \subseteq \mathbb{C}$ without fixed points (or, equivalently, let $\frac{\partial\varphi}{\partial t}(0, z) \neq 0$ for all $z \in G$). Then every smooth additive cocycle $a(t, z)$ is an additive coboundary with the unique decomposition*

$$a(t, z) = \beta(\varphi(t, z)) - \beta(z), \tag{4.1}$$

with $\beta \in H(G)$ given by

$$\beta(z) = \int_{z_0}^{z} \frac{\frac{\partial a}{\partial t}(0, w)}{\frac{\partial \varphi}{\partial t}(0, w)}\, dw + C, \tag{4.2}$$

where z_0 is a point in G and C is an arbitrary constant.

Proof. Let $a(t, z)$ be a smooth additive cocycle, and let β be defined by (4.2). Consider the coboundary $b(t, z) = \beta(\varphi(t, z)) - \beta(z)$. By the fundamental theorem of calculus,

$$b(t, z) = \int_{z}^{\varphi(t,z)} \beta'(w)\, dw = \int_{z}^{\varphi(t,z)} \frac{\frac{\partial a}{\partial s}(0, w)}{\frac{\partial \varphi}{\partial s}(0, w)}\, dw.$$

Then, for every $z \in G$ we have

$$\frac{\partial b}{\partial t}(t, z) = \frac{\frac{\partial a}{\partial s}(0, \varphi(t, z))}{\frac{\partial \varphi}{\partial s}(0, \varphi(t, z))} \frac{\partial \varphi}{\partial t}(t, z).$$

By smoothness of φ (cf. [BP, Thm. 1.1]),

$$\frac{\partial \varphi}{\partial s}(0, \varphi(t, z)) = \frac{\partial \varphi}{\partial t}(t, z) \quad \text{and} \quad \frac{\partial a}{\partial s}(0, \varphi(t, z)) = \frac{\partial a}{\partial t}(t, z),$$

since

$$\begin{aligned} \frac{\partial a}{\partial s}(0, \varphi(t, z)) &= \lim_{s\to 0} \frac{\partial a}{\partial s}(s, \varphi(t, z)) \\ &= \lim_{s\to 0} \frac{\partial}{\partial s}(a(s + t, z) - a(t, z)) \\ &= \lim_{s\to 0} \frac{\partial a}{\partial s}(s + t, z) = \frac{\partial a}{\partial t}(t, z). \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial b}{\partial t}(t, z) &= \frac{\frac{\partial a}{\partial s}(0, \varphi(t, z))}{\frac{\partial \varphi}{\partial s}(0, \varphi(t, z))} \frac{\partial \varphi}{\partial t}(t, z) \\ &= \frac{\frac{\partial a}{\partial t}(t, z)}{\frac{\partial \varphi}{\partial t}(t, z)} \frac{\partial \varphi}{\partial t}(t, z) = \frac{\partial a}{\partial t}(t, z). \end{aligned}$$

Thus $\frac{\partial b}{\partial t}(t,z) = \frac{\partial a}{\partial t}(t,z)$ and $b(t,z) = a(t,z) + \gamma(z)$. Since b is an additive coboundary, β is an additive cocycle. Hence $\gamma(z) \equiv 0$. So $b(t,z) = a(t,z)$ and therefore $a(t,z) = \beta(\varphi(t,z)) - \beta(z)$ is an additive coboundary.

Now let $a(t,z)$ be a smooth additive coboundary, and let β be as in (4.1). Then

$$\frac{\partial a}{\partial t}(t,z) = \beta'(\varphi(t,z))\frac{\partial \varphi}{\partial t}(t,z) \quad \text{for every } t \geq 0 \text{ and } z \in G.$$

At $t = 0$,

$$\frac{\partial a}{\partial t}(0,z) = \beta'(\varphi(0,z))\frac{\partial \varphi}{\partial t}(0,z) = \beta'(z)\frac{\partial \varphi}{\partial t}(0,z).$$

By $\frac{\partial \varphi}{\partial t}(0,z) = \frac{\partial \varphi}{\partial t}(t,z)\big|_{t=0} \neq 0$, we have

$$\beta'(z) = \frac{\frac{\partial a}{\partial t}(0,z)}{\frac{\partial \varphi}{\partial t}(0,z)},$$

which readily implies (4.2). □

Consequently, if $\frac{\partial \varphi}{\partial t}(0,z) \neq 0$ for all $z \in G$, then every smooth φ-cocycle is a coboundary. For example, in the disc all smooth cocycles associated with flows having only a Denjoy–Wolff point on the boundary of the disc are coboundaries. On the other hand, the flow $\varphi(t,z) = e^t z$ occurring in Example 3.1 fails to be nonsingular since $\frac{\partial \varphi}{\partial t}(0,0) = 0$.

The following theorem provides a necessary and sufficient condition for cocycles to be coboundaries when the flow has a fixed point.

THEOREM 4.3. *Let $G \subseteq \mathbb{C}$ be simply connected, and let $\varphi\colon [0,\infty) \times G \to G$ be a flow with a fixed point at $z_0 \in G$. Let $a(t,z)$ be a smooth additive cocycle. Then a is an additive coboundary if and only if*

(i) *there is a holomorphic function $h \in H(G)$ such that*

$$a(t,z) = \int_0^t h(\varphi(s,z))\frac{\partial \varphi}{\partial s}(s,z)\,ds \tag{4.3}$$

or equivalently,

(ii) *there exists an $h \in H(G)$ such that*

$$\frac{\partial a}{\partial t}(0,z) = \frac{\partial \varphi}{\partial t}(0,z)h(z). \tag{4.4}$$

Note that (ii) is the same as saying that the order of zero of $\frac{\partial a}{\partial t}(0,z)$ at z_0 is no less than the order of zero of $\frac{\partial \varphi}{\partial t}(0,z)$ at z_0.

Proof. (i) Let $a(t,z)$ be an additive coboundary; that is, let

$$a(t,z) = \beta(\varphi(t,z)) - \beta(z)$$

for some $\beta \in H(G)$. Then, for every $z \in G$, we have

$$a(t,z) = \beta(\varphi(t,z)) - \beta(\varphi(0,z)) = \int_0^t \frac{\partial}{\partial s}\beta(\varphi(s,z))\,ds$$
$$= \int_0^t \beta'(\varphi(s,z))\frac{\partial\varphi}{\partial s}(s,z)\,ds = \int_0^t h(\varphi(s,z))\frac{\partial\varphi}{\partial s}(s,z)\,ds,$$

where $h(z) = \beta'(z)$.

Conversely, if $h \in H(G)$ and β is a primitive of h on G, then reversing the above argument gives the conclusion. It is easy to check that $\int_0^t h(\varphi(s,z))\frac{\partial\varphi}{\partial t}(s,z)\,ds$ is a smooth additive cocycle for every $h \in H(G)$.

(ii) Note that, by (i),

$$\frac{\partial a}{\partial t}(0,z) = \frac{\partial}{\partial t}\left(\int_0^t h(\varphi(s,z))\frac{\partial\varphi}{\partial s}(s,z)\,ds\right)_{t=0} = \left(h(\varphi(t,z))\frac{\partial\varphi}{\partial t}(t,z)\right)_{t=0}$$
$$= h(\varphi(0,z))\frac{\partial\varphi}{\partial t}(0,z) = h(z)\frac{\partial\varphi}{\partial t}(0,z).$$

That is, (4.4) holds for every additive coboundary $a(t,z)$.

Conversely, if $\frac{\partial a}{\partial t}(0,z) = \frac{\partial\varphi}{\partial t}(0,z)h(z)$ then

$$\frac{\partial a}{\partial t}(t,z) = \frac{\partial a}{\partial s}(0,\varphi(t,z)) = \frac{\partial\varphi}{\partial s}(0,\varphi(t,z))h(\varphi(t,z))$$
$$= h(\varphi(t,z))\frac{\partial\varphi}{\partial t}(t,z) \quad \text{for every } t \geq 0.$$

Hence $a(t,z) = \int_0^t h(\varphi(s,z))\frac{\partial\varphi}{\partial s}(s,z)\,ds + \gamma(z)$ and, as in Theorem 4.2, $\gamma \equiv 0$. Hence $a(t,z)$ is an additive coboundary. □

Observe that, according to Theorem 4.2, the identity (4.3) gives a description of every additive coboundary (and therefore for every cocycle) of a flow $\varphi(t,z)$ without fixed points. We also note that the condition of *smoothness* on cocycles in Theorems 4.2 and 4.3 may be eliminated. This topic will be treated elsewhere.

Revisiting the additive analog of Example 3.1 in the context of Theorem 4.3 illustrates a useful point.

EXAMPLE 4.4. Let $\varphi(t,z) = e^t z$ be a flow on $\mathbb{C}$, and let $a(t,z) = ze^t + t - z$ be an additive φ-cocycle. Then

$$\frac{\frac{\partial a}{\partial t}(0,z)}{\frac{\partial\varphi}{\partial t}(0,z)} = \frac{z+1}{z} \notin H(\mathbb{C})$$

and so this cocycle fails to be a coboundary. Theorem 4.3(ii) asserts that if $a(t,z) = \int_0^t h(\varphi(s,z))\,ds$ with $h \in H(G)$, then $a(t,z)$ is a coboundary if and only if

$$\frac{\frac{\partial a}{\partial t}(0,z)}{\frac{\partial\varphi}{\partial t}(0,z)} = \frac{h(\varphi(0,z))}{\frac{\partial\varphi}{\partial t}(0,z)} = \frac{h(z)}{\frac{\partial\varphi}{\partial t}(0,z)} \in H(G),$$

which is equivalent to stating that the order of zero of h at z_0 is no less than the order of zero of $\frac{\partial\varphi}{\partial t}(0,z_0)$.

In view of Theorem 3.2, the question arises of which constant cocycles are coboundaries. If a is an additive cocycle for φ and if φ has no fixed points, then every constant cocycle $a(t) = ct$ is a coboundary by Theorem 4.2. The particular form of the function β is given by (4.2). If φ has a fixed point z_0 and if a is a constant additive coboundary, then at $z = z_0$ we have

$$a(t) = \beta(\varphi(t, z_0)) - \beta(z_0) = \beta(z_0) - \beta(z_0) = 0 \quad \forall t.$$

Thus, for flows with fixed points, the zero cocycle is the only constant cocycle that is a coboundary.

We obtain the same result from Theorem 4.3 by letting $z = z_0$ in the equation

$$ct = a(t) = \int_0^t h(\varphi(s, z)) \frac{\partial \varphi}{\partial s}(s, z)\, ds.$$

5. Projective Multipliers and Coboundaries

The cocycle exhibited in Example 3.3 has its source in the following question. Suppose that u, v is a harmonic conjugate pair of functions on G, and define A, K by

$$A(t, z) = \exp(itv(z)) \tag{5.1}$$

and

$$K(t, z) = \exp(tu(z)) \tag{5.2}$$

for $(t, z) \in [0, \infty) \times G$. If A is a cocycle, does it follow that K and hence $(AK)(t, z) = \exp(t(u(z) + iv(z)))$ is a cocycle?

The following example shows that K need not be a cocycle but, for this particular example, a suitable modification of K is a cocycle.

Example 5.1. For the flow $\varphi(t, z) = (x - t) + i(y - ct)$ and for conjugate harmonic functions $u(x, y) = -x - cy$ and $v(x, y) = cx - y$ of Example 3.3, it is easy to verify that A is a cocycle and that K is a *projective cocycle* in the sense that

$$K(s + t, z) = w(s, t)K(t, z)K(s, \varphi(t, z)), \tag{5.3}$$

where $w(s, t) = \exp(-(1 + c^2)st)$. Thus K fails to be a cocycle, since the multiplier w is not identically 1 for any real c. However, w is *trivial* in the sense that

$$w(s, t) = \frac{p(s)p(t)}{p(s + t)}, \tag{5.4}$$

where $p: \mathbb{R} \to \mathbb{R}$ is given by $p(t) = \exp((1 + c^2)t^2/2)$ for $t \in \mathbb{R}$. Note that (5.3) and (5.4) imply that $B(t, z) = p(t)K(t, z)$ satisfies the cocycle identity. Consequently, $m(t, z) = A(t, z)B(t, z)$ is a φ-cocycle given explicitly by

$$\begin{aligned} m(t, z) &= \exp\left(\frac{(1 + c^2)t^2}{2}\right) \exp(t(-x - cy)) \exp(it(cx - y)) \\ &= \exp\left(\frac{(1 + c^2)t^2}{2}\right) \exp(-t(1 - ci)z). \end{aligned}$$

For $c = 1$, we have

$$m(t, z) = e^{t^2} \exp(t(-x - y)) \exp(it(x - y)) = e^{t^2} \exp(-t(1 - i)z).$$

If we let $\alpha(z) = \exp(-i(z^2/2))$, then $m(t, z) = \alpha(\varphi(t, z))/\alpha(z)$ is a coboundary.

The function w is an example of a *projective multiplier* or a 2-cocycle, and the statement that w is trivial is simply the statement that a certain second cohomology group is trivial.

Our considerations suggest the following general question: Given $h \in H(G)$, can one find a *weight* $p: [0, \infty) \to \mathbb{C} \setminus \{0\}$ such that $m(t, z) = p(t) \exp(th(z))$ is a φ-cocycle? We do not have a complete answer to this question. However, weights of a quadratic type can always be found if the flow φ and the holomorphic function h are strongly related as in Example 5.1. We have the following result.

PROPOSITION 5.2. *Let $h \in H(G)$, and suppose that $c = [h(\varphi(s, z)) - h(z)]/2s$ is constant on $[0, \infty) \times G$. Then $p(t) \exp(th(z))$ is a φ-cocycle if and only if $p(t) = \exp(ct^2 + dt)$ for some $d \in \mathbb{C}$.*

Proof. Let $p(t) = \exp(q(t))$, and suppose that $q(t) + th(z)$ is an additive cocycle. From the additive cocycle identity we obtain

$$q(t + s) - q(s) - q(t) = t(h(\varphi(s, z)) - h(z)) = 2cst = c[(s + t)^2 - s^2 - t^2],$$

and so

$$q(s + t) - c(s + t)^2 = [q(s) - cs^2] + [q(t) - ct^2].$$

Hence $q(s) - cs^2$ is additive and consequently linear (since $q(0) = 0$). Therefore, $q(s) - cs^2 = ds$ for some $d \in \mathbb{C}$. The converse is a straightforward verification. □

We note that, if φ has a fixed point at z_0 then $c = [h(\varphi(s, z_0)) - h(z_0)]/2s = 0$ and so $q(s) = ds$.

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