The Unitary Orbit of Strongly Irreducible Operators in the Nest Algebra with Well-Ordered Set

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1. Introduction

Let $\mathcal{H}$ be a complex, separable, infinite-dimensional Hilbert space; $\mathcal{L}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ denote (respectively) the algebra of all bounded linear operators acting on $\mathcal{H}$ and the ideal of all compact operators.

Let $\sigma_0(T)$ denote the isolated eigenvalues of T of finite multiplicity. If λ belongs to $\sigma_0(T)$, let $E_T\{\lambda\}$ denote the Riesz projection corresponding to the eigenspace for λ. When X is a compact subset of the plane, let $X^\wedge$ denote the polynomially convex hull of X.

An operator T is *strongly irreducible* if the only idempotent operators in $\{T\}'$ are 0 and I, where $\{T\}'$, denotes the commutant of T. Let Ω be a bounded connected open set in $\mathcal{C}$. Recall that $\mathcal{B}_n(\Omega)$, the set of Cowen–Douglas operators of index n $(1 \le n \le +\infty)$, is the set of those operators B on $\mathcal{H}$ satisfying

(i) $\sigma(B) \supset \Omega$;
(ii) $\operatorname{nul}(\lambda - B) = \operatorname{ind}(\lambda - B) = n, \quad (\lambda \in \Omega)$;
(iii) $\bigvee\{\ker(\lambda - B);\ \lambda \in \Omega\} = \mathcal{H}$.

Note that (iii) can be replaced by

(iii′) $\bigvee\{\ker(\lambda_0 - B)^k : k \ge 1\} = \mathcal{H}$ for some $\lambda_0 \in \Omega$.

A nest $\mathcal{N}$ in $\mathcal{H}$ is a linearly ordered (by inclusion) family of subspaces containing $\{0\}$ and $\mathcal{H}$. The *nest algebra* associated with $\mathcal{N}$ is the family of operators defined by

$$\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{L}(\mathcal{H}) : TN \subset N \text{ for all } N \text{ in } \mathcal{N}\}.$$

In what follows, $N \in \mathcal{N}$ denotes both a subspace and the orthogonal projection onto it; $T \in (\mathrm{SI})$ means that T is a *strongly irreducible* operator on its acting space.

For each $N \in \mathcal{N}$, let

$$N_- = \bigvee\{N' \in \mathcal{N},\ N' \subsetneqq N\}.$$

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If $N_- \neq N$ then $N \ominus N_-$ is called an *atom* of $\mathcal{N}$. If all the atoms of $\mathcal{N}$ are 1-dimensional, $\mathcal{N}$ is called maximal. If $\mathcal{N} = \{0;\ N_n\ (n \geq 1);\ \mathcal{H}\}$, $N_n < N_{n+1}$, and $\dim N_n < +\infty$ $(n = 1, 2, \dots)$, then $\mathcal{N}$ is the nest of type $w + 1$. For more information about nest algebras see [D].

The authors have proved [JJW1] the following result.

THEOREM JJW. *Each nest algebra contains at least one SI operator.*

In the same paper they described the (SI) operator in $\mathcal{T}(\mathcal{N})$ with $\mathcal{N}$ of type $w + 1$.

The following theorem was proved in [JW].

THEOREM JW. *Given an operator $T \in \mathcal{L}(\mathcal{H})$ with connected spectrum $\sigma(T)$, there exists an operator $A \in$ (SI) such that $\Lambda(T) = \Lambda(A)$ and $T \in \overline{S(A)}$, where $\overline{S(A)}$ denotes the closure of the similarity orbit $S(A)$ of A and $\Lambda(T)$ denotes the spectral picture of T, that is, $\sigma_{lre}(T)$, $\rho_{S-F}(T)$ plus the index function.*

$$\rho_{S-F}(T) = \{\lambda \in \mathcal{C} : \lambda - T \text{ is semi-Fredholm}\}; \quad \sigma_{lre}(T) = \sigma(T) \setminus \rho_{S-F}(T).$$

In order to answer a question raised by Arveson in 1981, Herrero [H1] proved the following theorem.

THEOREM H1. *Let $\mathcal{N}$ be a nest in $\mathcal{H}$.*

(i) *If $\mathcal{N}$ is well-ordered and all its atoms are finite-dimensional, then*

$$\mathcal{U}(\mathcal{N}) = \mathcal{U}_a^0(\mathcal{N}) = \mathcal{U}_a(\mathcal{N}) = QT.$$

(ii) *If $\mathcal{N}^\perp$ is well-ordered with finite-dimensional atoms, then*

$$\mathcal{U}(\mathcal{N}) = \mathcal{U}_a^0(\mathcal{N}) = \mathcal{U}_a(\mathcal{N}) = QT^*.$$

(iii) *If neither* (i) *nor* (ii) *holds then let $d = \sum_{A \in \mathcal{A}} \dim A$, where $\mathcal{A}$ denotes the set of atoms of $\mathcal{N}$. It follows that:*

(iiia) *when $d = \infty$, $\mathcal{U}(\mathcal{N}) = \mathcal{U}_a^0(\mathcal{N}) = \mathcal{U}_a(\mathcal{N}) = \mathcal{L}(\mathcal{H})$;*
(iiib) *when $d < \infty$, $\mathcal{U}(\mathcal{N}) = \mathcal{U}_a^0(\mathcal{N}) = \mathcal{L}(\mathcal{H})_d$ and $\mathcal{U}_a(\mathcal{N}) = \mathcal{L}(\mathcal{H})$.*

Here $\mathcal{U}(\mathcal{N})$ denotes the norm closure of $\{UTU^* : T \in \mathcal{T}(\mathcal{N}),\ U$ unitary$\}$, $\mathcal{U}_a(\mathcal{N}) = \{UTU^* + K : T \in \mathcal{T}(\mathcal{N}),\ U$ unitary, K compact$\}$, and $\mathcal{U}_a^0(\mathcal{N}) = \{A \in \mathcal{L}(\mathcal{H})$: for all $\varepsilon > 0$, there are T in $\mathcal{T}(\mathcal{N})$, U unitary, and K compact such that $\|K\| < \varepsilon$ and $A = UTU^* + K\}$. Moreover:

$$\begin{aligned}
\mathcal{N}^\perp &= \{N;\ N^\perp \in \mathcal{T}(\mathcal{N})\};\\
(QT) &= \{T \in \mathcal{L}(\mathcal{H}) : \operatorname{ind}(T - \lambda) \geq 0\ \forall \lambda \in \rho_{S-F}(T)\};\\
(QT)^* &= \{T \in \mathcal{L}(\mathcal{H}) : T^* \in (QB)\}\\
&= \{T \in \mathcal{L}(\mathcal{H}) : \operatorname{ind}(T - \lambda) \leq 0\ \forall \lambda \in \rho_{S-F}(T)\};\\
\mathcal{L}(\mathcal{H})_d &= \left\{T \in \mathcal{L}(\mathcal{H}) : \sum_{\lambda \in \sigma_0(T) \setminus \sigma_e(T)^\wedge} \operatorname{ran} E_T\{\lambda\} \leq d\right\}.
\end{aligned}$$

It is natural to ask the following questions. (1) Given a $T \in \mathcal{T}(\mathcal{N})$ with connected spectrum $\sigma(T)$, does there exist an operator $A \in \mathcal{T}(\mathcal{N}) \cap$ (SI) such that $\Lambda(A) = \Lambda(T)$? (2) What is the closure of the unitary orbit of the class of (SI) operators in $\mathcal{T}(\mathcal{N})$?

THEOREM 1. *Let $\mathcal{N}$ (or $\mathcal{N}^{\perp}$) be maximal and well-ordered, and let $T \in \mathcal{T}(\mathcal{N})$ with connected spectrum $\sigma(T)$. Then there exists an $A \in \mathcal{T}(\mathcal{N}) \cap$ (SI) such that $\Lambda(A) = \Lambda(T)$ and $T \in \overline{S(A)}$.*

THEOREM 2. (i) *If $\mathcal{N}$ is well-ordered with finite-dimensional atoms, then*

$$\mathcal{U}(\mathcal{T}(\mathcal{N}) \cap (\mathrm{SI})) = (QT)_c \triangleq \{ T \in QT : \sigma(T) \text{ and } \sigma_w(T) \text{ are connected} \},$$

where $\sigma_w(T) = \bigcap_{k \in \mathcal{K}(\mathcal{H})} \sigma(T + K)$ is the Weyl spectrum of T.

(ii) *If $\mathcal{N}^{\perp}$ is well-ordered with finite-dimensional atoms, then $\mathcal{U}(\mathcal{T}(\mathcal{N}) \cap (\mathrm{SI})) = (QT)_c^* \triangleq \{ T : T^* \in (QT)_c \}$.*

Let the nest $\mathcal{N}$ be maximal and of type $w + 1$. That is, $\mathcal{N} = \{0;\ P_n\ (n \geq 1);\ \mathcal{H}\}$, where $P_n \ominus P_{n-1} = \bigvee\{e_n\}$ $(n = 1, 2, \dots)$ and $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis (ONB) of $\mathcal{H}$.

THEOREM 3. *Let Ω be a bounded analytic Jordan domain in $\mathcal{C}$, and let*

$$T = \begin{pmatrix} T_1 & T_{12} & \cdots & * \\ & T_2 & \ddots & \vdots \\ & & \ddots & T_{m-1,m} \\ 0 & & & T_m \end{pmatrix}$$

with respect to decomposition $\mathcal{H} = \bigoplus_{i=1}^{m} \mathcal{H}_i$ $(m < +\infty)$, where $T_i \in \mathcal{B}_1(\Omega)$ with $\sigma(T_i) = \bar{\Omega}$ $(i = 1, 2, \dots, m)$. Then, for each $\varepsilon > 0$, there exists a compact K with $\|K\| < \varepsilon$ such that $T + K \simeq A \in \mathcal{T}(\mathcal{N}) \cap$ (SI).

COROLLARY 4. *Let Ω be a bounded analytic Jordan domain in $\mathcal{C}$, and let $T \in \mathcal{B}_n(\Omega)$ $(n < +\infty)$ with $\sigma(T) = \bar{\Omega}$. Then, for each $\varepsilon > 0$, there exists a compact K with $\|K\| < \varepsilon$ such that $T + K \in$ (SI).*

2. Preparation

In this section, let $\mathcal{N}$ be always maximal and of type $w + 1$, and let $\tau_{A,B}$ be the bounded linear operator on $\mathcal{L}(\mathcal{H})$ such that $\tau_{AB}(X) = AX - XB$.

PROPOSITION 2.1. *Assume $T \in \mathcal{L}(\mathcal{H})$ and $\rho_{S-F}^{s}(T) \neq \emptyset$. Then $T \notin$ (SI), where $\rho_{S-F}^{s}(T)$ is the set of singular points of T.*

Proof. Without loss of generality, we can assume that $0 \in \rho_{S-F}^{s}(T)$. Let

$$T = \begin{pmatrix} T_r & T_{12} & T_{13} \\ 0 & T_0 & T_{23} \\ 0 & 0 & T_l \end{pmatrix} \begin{matrix} H_r \\ H_0 \\ H_l \end{matrix}$$

be the Apostol's triangular representation of T, where

$$H_r = \bigvee\{\ker(\lambda - T) : \lambda \in \rho^r_{S-F}(T)\}, \quad H_l = \{\ker(\lambda - T)^* : \lambda \in \rho^r_{S-F}(T)\},$$
$$\rho^r_{S-F}(T) = \rho_{S-F}(T) \setminus \rho^s_{S-F}(T),$$

and $H_0 = H \ominus (H_r \oplus H_l)$ [H4]. Since 0 is an isolated point of $\sigma(T_0)$, there exist $H_1, H_2 \in \operatorname{lat} T_0$ such that $H_0 = H_1 \dot{+} H_2$, $\sigma(T_1) = \{0\}$, $0 \notin \sigma(T_2)$, and $T_0 = T_1 \dot{+} T_2 \sim T_1 \oplus T_2$ by Riesz's theorem, where $T_1 = T_0|_{H_1}$ and $T_2 = T_0|_{H_2}$. Thus

$$T \sim \begin{pmatrix} T_r & A_{12} & A_{13} & A_{14} \\ 0 & T_1 & 0 & A_{24} \\ 0 & 0 & T_2 & A_{34} \\ 0 & 0 & 0 & T_l \end{pmatrix} \begin{matrix} T_r \\ H_1 \\ H_2 \\ T_l \end{matrix}.$$

Note that $\sigma_r(T_r) \cap \sigma_l(T_1) = \sigma_r(T_1) \cap \sigma_l(T_l) = \emptyset$. By Rosenblum's theorem [R], $\tau_{T_r T_1}$ and $\tau_{T_1 T_l}$ are surjective. Thus

$$T \sim \begin{pmatrix} T_r & A_{12} & A_{13} & A_{14} \\ 0 & T_1 & 0 & A_{24} \\ 0 & 0 & T_2 & A_{34} \\ 0 & 0 & 0 & T_l \end{pmatrix} \sim \begin{pmatrix} T_r & 0 & * & * \\ 0 & T_1 & 0 & 0 \\ 0 & 0 & T_2 & * \\ 0 & 0 & 0 & T_l \end{pmatrix}$$
$$\simeq \begin{pmatrix} T_1 & 0 & 0 & 0 \\ 0 & T_r & * & * \\ 0 & 0 & A_2 & * \\ 0 & 0 & 0 & T_l \end{pmatrix} = T_1 \oplus \begin{pmatrix} T_r & * & * \\ 0 & A_2 & * \\ 0 & 0 & T_l \end{pmatrix}. \qquad \square$$

Proposition 2.1 implies that even in $B(\mathcal{H})$, not every fine spectral picture can be realized by (SI) operators.

PROPOSITION 2.2. *Assume that $T \in \mathcal{T}(\mathcal{N})$. Then $\sigma_p(T^*) \cap \rho^r_{S-F}(T^*) = \emptyset$.*

Proof. Since T admits an upper triangular matrix representation with respect to the ONB $\{e_n\}_{n=1}^\infty$, one can choose $\{\lambda_k\}_{k=1}^\infty \subset \rho^r_{S-F}(T) \cap \sigma_p(T)$ such that

$$\bigvee\{\ker(T - \lambda_k)^n : k \geq 1,\ n \geq 1\} = \mathcal{H}.$$

By Apostol's triangular representation, $H_l = 0$; that is, $\sigma_p(T^*) \cap \rho^r_{S-F}(T^*) = \emptyset$. $\square$

COROLLARY 2.3. *Assume that $T \in \mathcal{T}(\mathcal{N}) \cap$ (SI). Then*

$$\min \operatorname{ind}(\lambda - T) = \min(\operatorname{nul}(\lambda - T),\ \operatorname{nul}(\lambda - T)^*) = 0.$$

An operator T is called *almost normal* if T can be written as the sum of a normal operator and a compact operator.

PROPOSITION 2.4 [JJW2]. *Let σ be a connected compact subset of $\mathbb{C}$, and let $\{\lambda_k\}_{k=1}^\infty$ be a dense subset of σ. Then there exists an almost normal operator $T \in \mathcal{T}(\mathcal{N}) \cap$ (SI) such that*

(a) $\sigma(T) = \sigma_{lre}(T) = \sigma$,
(b) $\sigma_p(T) \supset \{\lambda_k\}_{k=1}^\infty$, *and*
(c) $\bigvee\{\ker(T - \lambda_k)^n,\ k \geq 1,\ n \geq 1\} = \mathcal{H}$.

PROPOSITION 2.5. *Assume $T \in \mathcal{T}(\mathcal{N}) \cap \mathcal{B}_1(D)$ with*

$$\Delta(T) := \sum_{n=1}^{\infty}(P_n \ominus P_{n-1})T(P_n \ominus P_{n-1}) = 0 \quad \textit{and} \quad \sigma(T) = \bar{D}.$$

Then

$$\lim_{m\to\infty} \sqrt[m]{\prod_{n=1}^{m}|\alpha_n|} = 1,$$

where $D = \{\lambda \in C : |\lambda| < 1\}$ and $\alpha_n = (Te_{n+1}, e_n)$ $(n = 1, 2, \dots)$.

Proof. Since $T \in \mathcal{B}_1(D)$, it follows that $0 < r < |\alpha_n| \leq \|T\|$ for some positive number r and that T has right inverse B. Computation shows that

$$B = \begin{pmatrix} * & & & * \\ \frac{1}{\alpha_1} & * & & \\ 0 & \frac{1}{\alpha_2} & * & \\ 0 & 0 & \frac{1}{\alpha_3} & \\ \vdots & \vdots & & \ddots \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ \vdots \end{matrix}.$$

For each $\lambda \in D$, $(\lambda - B) = (\lambda T - I)B = \lambda(T - \frac{1}{\lambda})B$. Since $T - \frac{1}{\lambda}$ is invertible, $\lambda \in \rho_{S-F}(B)$ and $\operatorname{ind}(\lambda - B) = -1$. Therefore $\sigma(B) \supset \bar{D}$. If $|\lambda| > 1$ then $(\lambda - B) = \lambda(T - \frac{1}{\lambda})B$. Since $\frac{1}{\lambda} \in D$, $\lambda - B$ is a Fredholm operator and $\operatorname{ind}(\lambda - B) = 0$. Therefore $\sigma_e(B) = \partial D$ and $\sigma_0(B) \subset C \setminus \bar{D}$. Thus there exists a compact K such that $\sigma(B + K) = \bar{D}$ [H4, Prop. 3.45]. For each $\varepsilon > 0$, fix n_0 such that $\|P_{n_0}KP_{n_0} - K\| < \frac{\varepsilon}{2}$. Then $\sigma(A) \subset D_\varepsilon$, where $A = B + P_{n_0}KP_{n_0}$ and $D_\varepsilon = \{\lambda \in C; |\lambda| < 1 + \varepsilon\}$.

Calculation shows that the $(n_0 + m + 1, n_0)$ entry of A^{m+1} is

$$\frac{1}{\alpha_{n_0+1} \cdot \alpha_{n_0+2} \cdots \alpha_{n_0+m+1}}.$$

This implies

$$\overline{\lim_{m\to\infty}} \sqrt[m+1]{\frac{1}{\alpha_{n_0+1} \cdot \alpha_{n_0+2} \cdots \alpha_{n_0+m+1}}} \leq \lim_{m\to\infty} \sqrt[m+1]{\|A^{m+1}\|} \leq 1 + \varepsilon,$$

so

$$\overline{\lim_{m\to\infty}} \sqrt[m]{\frac{1}{\prod_{k=1}^{m}|\alpha_k|}} \leq 1 + \varepsilon.$$

That is,

$$\underline{\lim_{m\to\infty}} \sqrt[m]{\prod_{k=1}^{m}|\alpha_k|} \geq \frac{1}{1+\varepsilon}$$

and then, by the arbitrariness of ε,

$$\underline{\lim_{m\to\infty}} \sqrt[m]{\prod_{k=1}^{m}|\alpha_k|} \geq 1.$$

Since the $(1, m+1)$ entry of T^m is $\alpha_1 \dots \alpha_m$, we have $\sqrt[m]{|\alpha_1 \dots \alpha_m|} \le \sqrt[m]{\|T^m\|}$ and

$$\overline{\lim_{m\to\infty}} \sqrt[m]{\prod_{k=1}^{m} |\alpha_k|} \le 1.$$

The proof of Proposition 2.5 is now complete. □

COROLLARY 2.6. *Assume that Ω is an analytic Jordan domain. Let $T_1, T_2 \in \mathcal{T}(\mathcal{N}) \cap \mathcal{B}_1(\Omega)$ with $\sigma(T_1) = \sigma(T_2) = \bar{\Omega}$ and $\Delta(T_1) = \Delta(T_2) = \lambda_0 \in \Omega$. Then*

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} |\alpha_k|^{(i)}} = r \quad (i = 1, 2)$$

for some $r > 0$, where $\alpha_k^{(i)} = (T_i e_{k+1}, e_k)$ $(k = 1, 2, \dots, i = 1, 2)$.

Proof. Let f be the analytic homeomorphism $f\colon \Omega \to D$, with $f(\partial\Omega) = \partial D$ and $f(\lambda_0) = 0$. Then $A_i \in \mathcal{T}(\mathcal{N}) \cap \mathcal{B}_1(D)$, $\Delta(A_i) = 0$, and $\sigma(A_i) = \bar{D}$, where $A_i = f(T_i)$ $(i = 1, 2)$. Let $\beta_n^{(i)} = (A_i e_{n+1}, e_n)$ $(i = 1, 2,\ n = 1, 2, \dots)$. Then

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} |\beta_k|^{(i)}} = 1 \quad (i = 1, 2)$$

by Proposition 2.5. Set $g = f^{-1}$. Since $g(A_i) = T_i$, we have $\alpha_n^{(i)} = g'(0)\beta_n^{(i)}$. Thus

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} |\alpha_k^{(1)}|} = \lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} |\alpha_k^{(2)}|} = |g'(0)| = r. \qquad \square$$

PROPOSITION 2.6. *Let Ω be an analytic Jordan domain, and let*

$$\{T_k\}_{k=1}^{\infty} \subset \mathcal{T}(\mathcal{N}) \cap \mathcal{B}_1(\Omega), \qquad \Delta(T_k) = \lambda_0 \in \Omega,$$

and

$$\sigma(T_k) = \bar{\Omega} \quad (k = 1, 2, \dots).$$

Then, for each $\varepsilon > 0$, there exists $\{C_k\}_{k=1}^{\infty} \subset \mathcal{K}(\mathcal{H})$ with $\|C_k\| < \varepsilon/2^k$ such that

$$B_k = T_k + C_k \in \mathcal{T}(\mathcal{N}) \cap \mathcal{B}_1(\Omega) \quad (k = 1, 2, \dots)$$

and $\ker \tau_{B_i B_j} = \{0\}$ $(i \ne j)$.

Proof. Let $\alpha_n^{(k)} = (T_k e_{n+1}, e_n)$ $(k, n = 1, 2, \dots)$; then, by Corollary 2.5,

$$\lim \sqrt[n]{\prod_{i=1}^{n} |\alpha_i^{(k)}|} = r \quad (k = 1, 2, \dots),$$

so

$$\lim_{n\to\infty}\left(\prod_{i=1}^{n}\left|\frac{\alpha_i^{(k)}}{\alpha_i^{(j)}}\right|\right)^{1/n} = 1 \quad \forall k, j.$$

Claim: There exists a sequence $\{\beta_n^k\}_{n,k=1}^{\infty}$ of complex numbers satisfying (i)

$$\overline{\lim_{n\to\infty}}\prod_{i=1}^{n}\left|\frac{\alpha_i^{(k)}(1-\beta_i^{(k)})}{\alpha_i^{(j)}(1-\beta_i^{(j)})}\right| = \infty \quad (k < j)$$

and

$$\underline{\lim_{n\to\infty}}\prod_{i=1}^{n}\left|\frac{\alpha_i^{(k)}(1-\beta_i^{(k)})}{\alpha_i^{(j)}(1-\beta_i^{(j)})}\right| = 0 \quad (k < j),$$

and (ii) $\lim_{n\to\infty}\beta_n^{(k)} = 0$ and $\sup_n|\beta_n^{(k)}| < \varepsilon/2^k$ $(k = 1, 2, \ldots)$.

We define $\{\beta_n^{(k)}\}$ inductively. Set $\beta_n^{(1)} = 0$ $(n = 1, 2, \ldots)$. Assume that $\{\beta_n^{(k)}\}_{n=1}^{\infty}$ $(k < l)$ have been defined and satisfy (i) and (ii). Set $d_i = 1 - \varepsilon/2^{i+l}$. Since

$$\lim_{n\to\infty}\left(\prod_{i=1}^{n}\left|\frac{\alpha_i^{(k)}(1-\beta_i^{(k)})}{\alpha_i^{(l)}}\right|\right)^{1/n} = 1 \quad (k < l)$$

and

$$\lim_{n\to\infty}\left(\prod_{i=1}^{n}\left|\frac{\alpha_i^{(k)}(1-\beta_i^{(k)})}{\alpha_i^{(l)}d_1}\right|\right)^{1/n} = \frac{1}{d_1} > 1,$$

we can find n_1 such that

$$\prod_{i=1}^{n_1}\left|\frac{\alpha_i^{(k)}(1-\beta_i^{(k)})}{\alpha_i^{(l)}d_1}\right| > 2 \quad (k < l).$$

Define $\beta_n^{(l)} = 1 - d_1$ $(1 \le n \le n_1)$. Since

$$\lim_{n\to\infty}\left(\prod_{i=1}^{n_1}\left|\frac{\alpha_i^{(k)}(1-\beta_i^{(k)})}{\alpha_i^{(l)}(1-\beta_i^{(l)})}\right| \cdot \prod_{i=n_1+1}^{n}\left|\frac{\alpha_i^{(k)}(1-\beta_i^{(k)})}{\alpha_i^{(l)}\frac{1}{d_2}}\right|\right)^{1/n} = d_2 < 1,$$

we can find $n_2 > n_1$ such that

$$\prod_{i=1}^{n_1}\left|\frac{\alpha_i^{(k)}(1-\beta_i^{(k)})}{\alpha_i^{(l)}(1-\beta_i^{(l)})}\right| \cdot \prod_{i=n_1+1}^{n_2}\left|\frac{\alpha_i^{(k)}(1-\beta_i^{(k)})}{\alpha_i^{(l)}\frac{1}{d_2}}\right| < \frac{1}{2} \quad (k < l).$$

Define $\beta_n^{(l)} = 1 - 1/d_2$ $(n_1 \le n \le n_2)$. Continue the process, defining

$$\beta_n^{(l)} = \begin{cases} 1 - d_n, & n_{2k-2} < n \le n_{2k-1}, \\ 1 - 1/d_n, & n_{2k-1} < n \le n_{2k}, \end{cases}$$

such that

$$\prod_{i=1}^{n_{2h-1}}\left|\frac{\alpha_i^{(k)}(1-\beta_i^{(k)})}{\alpha_i^{(l)}(1-\beta_i^{(l)})}\right| > 2^h$$

and

$$\prod_{i=1}^{n_{2h}} \left| \frac{\alpha_i^{(k)}(1-\beta_i^{(k)})}{\alpha_i^{(l)}(1-\beta_i^{(l)})} \right| < 2^{-h} \quad (k < l),$$

where $h = 1, 2, \dots$. Therefore $\{\beta_n^{(j)}\}_{n=1}^{\infty}$ $(j = 1, \dots)$ satisfy (i) and (ii).

Define

$$C_k e_n = -\alpha_n^{(k)} \beta_n^{(k)} e_{n-1} \quad (n = 1, 2, \dots, \ k = 1, 2, \dots).$$

Then C_k is compact and $\|C_k\| < \varepsilon/2^k$ $(k = 1, 2, \dots)$. Therefore, $B_k = T_k + C_k \in \mathcal{T}(\mathcal{N}) \cap B_1(\Omega)$.

If $X \in \ker \tau_{B_k, B_j}$ (i.e., if $B_k X = X B_j$), then computation shows that X admits a representation by an upper triangular matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & \\ & x_{22} & \\ 0 & & \ddots \end{pmatrix}$$

with respect to the ONB $\{e_n\}_{n=1}^{\infty}$. Calculations indicate that

$$|x_{m,m}| = \prod_{n=1}^{m-1} \left| \frac{\alpha_n^{(j)}(1-\beta_n^{(j)})}{\alpha_n^{(k)}(1-\beta_n^{(k)})} \right| |x_{11}| \quad (m = 1, 2, \dots).$$

Thus $x_{mm} = 0$ $(m = 1, 2, \dots)$, by (i). Similarly,

$$|x_{n,n+l}| = \prod_{i=1}^{l} \left| \frac{\alpha_{n+i-1}^{(j)}(1-\beta_{n+i-1}^{(j)})}{\alpha_i^{(k)}(1-\beta_i^{(k)})} \right| \cdot \prod_{i=l+1}^{n-1} \left| \frac{\alpha_i^{(j)}(1-\beta_i^{(j)})}{\alpha_i^{(k)}(1-\beta_i^{(k)})} \right| x_{1,l},$$

and $x_{n,n+l} = 0$ $(n = 1, 2, \dots, \ l = 1, 2, \dots)$ by (i). That is, $X = 0$ and $\ker \tau_{B_k, B_j} = \{0\}$ $(k \neq j)$. □

PROPOSITION 2.7. *Let $T \in B_\infty(\Omega)$; then there exists $A \in \mathcal{T}(\mathcal{N})$ with $A \simeq T$.*

Proof. Without loss of generality, we can assume that $0 \in \Omega$ and set $H_n = \ker T^n \ominus \ker T^{n-1}$ $(n = 1, 2, \dots)$. Then T admits the representation by an upper triangular matrix

$$T = \begin{pmatrix} 0 & T_{12} & T_{13} & \dots \\ & 0 & T_{23} & \\ & & 0 & \\ & & & \ddots \end{pmatrix} \begin{matrix} H_1 \\ H_2 \\ H_3 \\ \vdots \end{matrix}$$

with respect the decomposition $H = \bigoplus_{i=1}^{\infty} H_i$ of the space. Let $\{e_n^{(i)}\}_{n=1}^{\infty}$ be an ONB of H_i $(i = 1, 2, \dots)$, and let B be the right inverse of T. Set $N_1 = \bigvee e_1^{(1)}$ and $x_1^{(2)} = Be_1^{(1)}$. Since $TB = I$, $x_1^{(2)} \notin N_1$. Set $N_2 = \bigvee\{N_1, x_1^{(2)}\}$, $N_3 = \bigvee\{N_2, e_2^{(1)}\}$, and $x_2^{(2)} = Be_2^{(1)}$. Since $TB = I$, $x_2^{(2)} \notin N_3$. Set $N_4 = \bigvee\{N_3, x_2^{(2)}\}$. Let $x_1^{(3)} = B^2 e_1^{(1)}$; similarly, $x_1^{(3)} \notin N_4$. Define $N_5 = \bigvee\{N_4, x_1^{(3)}\}$ and $N_6 = \bigvee\{N_5, e_3^{(1)}\}$. Set $x_3^{(2)} = Be_3^{(1)}$, $N_7 = \bigvee\{N_6, x_3^{(2)}\}$; $x_2^{(3)} = B^2 e_2^{(1)}$, $N_8 = \bigvee\{N_7, x_2^{(3)}\}$; $x_1^{(4)} = B^3 e_1^{(1)}$, $N_9 = \bigvee\{N_8, x_1^{(4)}\}$; .... Thus $\mathcal{M} = \{0;\ N_n\ (n \geq 1);\ \mathcal{H}\}$. Hence there is a unitary U such that $UTU^* \in \mathcal{T}(\mathcal{N})$. □

3. Proof of the Main Theorem

Proof of Theorem 1. First, we assume that $\mathcal{N}$ is maximal and of type $\omega+1$. By Proposition 2.4, we can assume that $\rho_{S-F}(T)\cap\sigma(T)\neq\emptyset$ and that $\{\Omega_k\}_{k=1}^{l}$ $(1\leq l\leq\infty)$ is the class of the connected components of $\rho_{S-F}(T)\cap\sigma(T)$. By Proposition 2.2, $\min\{\mathrm{ind}(T-\lambda),\ \lambda\in\Omega_k\}>0$. Set $\Phi_k=(\bar{\Omega}_k)^0$ $(k=1,2,\ldots,l)$. Let $\{\lambda_k\}_{k=1}^{p_1}$ $(0\leq p_1\leq\infty)$ and $\{\mu_k\}_{k=1}^{p_2}$ $(p_2\leq\infty)$ be dense subsets of $\bigcup_{k=1}^{l}\Phi_k\setminus\bigcup_{k=1}^{l}\Omega_k$ and $\sigma(T)\setminus\bigcup_{k=1}^{l}\Phi_k$, respectively. Set $B_k=M_+^*(\Phi_k^*)$ $(k=1,2,\ldots,l)$, where $M_+(\Phi_k^*)$ is the Bergman operator on $L_a^2(\Phi_k^*)$ and where $\Phi_k^*=\{\lambda;\ \bar{\lambda}\in\Phi_k\}$ $(k=1,2,\ldots)$. Thus $B_k\in B_1(\Phi_k)$ and $\sigma(B_k)=\bar{\Omega}_k$.

In [H3], Herrero gave the following example. Define $\nu_1=1$, $\nu_2=\frac{1}{4},\ldots,$ $\nu_n=(\nu_1\ldots\nu_{n-1})^n$, and let $\{\alpha_n\}$ be the sequence

$$\begin{gathered}\nu_1,\nu_2,\ldots,\nu_9,\\ \nu_1,\nu_2,\ldots,\nu_{90},\\ \nu_1,\nu_2,\ldots,\nu_{900},\\ \cdots\cdots\cdots\cdots\cdots\end{gathered}$$

Let V be the backward unilateral weighted shift with weights $\{\alpha_n\}$. Then V is not compact quasinilpotent and V^k is not compact for any power $k\geq 1$. Define $B_{\lambda_k}=\lambda_k+V$ and $B_{\mu_j}=\mu_j+V$ $(k=1,\ldots,p_1,\ j=1,\ldots,p_2)$.

Define

$$A_1=\begin{pmatrix}B_1\oplus\left(\bigoplus_{k=2}^{l}B_k^{(n_k)}\right) & 0\\ 0 & \left(\bigoplus_{k=1}^{p_1}B_{\lambda_k}\right)\oplus\left(\bigoplus_{j=1}^{p_2}B_{\mu_j}\right)\end{pmatrix}.$$

Thus A_1 is an upper triangular operator with $\sigma_w(A_1)=\sigma(A_1)$ connected, where $\sigma_w(A_1)$ denotes the Weyl spectrum of A_1, that is, $\sigma_w(A_1)=\bigcap\{\sigma(A_1+K),\ K\in\mathcal{K}(\mathcal{H})\}$.

By [H2], for each $\varepsilon>0$ there exists a compact K with $\|K\|<\varepsilon$ such that $G=A_1+K\in\mathcal{B}_1(\Omega_1)$. Since $G,B_1\in\mathcal{B}_1(\Omega_1)$, they admit upper triangular matrix representations

$$G=\begin{pmatrix}\lambda_0 & g_1 & & *\\ & \lambda_0 & g_2 & \\ & & \lambda_0 & \ddots\\ 0 & & & \ddots\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}\lambda_0 & b_1 & & & *\\ & \lambda_0 & b_2 & & \\ & & \lambda_0 & b_3 & \\ 0 & & & \ddots & \ddots\end{pmatrix}$$

for some $\lambda_0\in\Omega_1$ with respect to some ONBs of their acting spaces, and $0<r<|g_n|<R$ and $0<r<|b_n|<R$ $(n=1,2,\ldots)$ for some r and R. Assume that

$$\varlimsup_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}|g_k|}\geq\varlimsup_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}|b_k|}.$$

(The proof is similar for the opposite inequality.)

By arguments similar to those used in the proof of Proposition 2.6, we can find $\{\beta_k^{(i)}\}_{k=1}^{\infty}$ $(i=1,2,\ldots)$ satisfying (i)

$$\overline{\lim_{n\to\infty}}\prod_{k=1}^{n}\frac{|g_k|}{|b_k(1-\beta_k^{(i)})|}=\infty \quad\text{and}\quad \overline{\lim_{n\to\infty}}\prod_{i=1}^{n}\frac{|1-\beta_k^{(j)}|}{|1-\beta_k^{(i)}|}=0 \quad (i\neq j)$$

and (ii) $\lim_{k\to\infty}|\beta_k^{(j)}|=0$ and $\sup_k|\beta_k^{(i)}|<\varepsilon/2^i$ $(i=1,2,\dots)$. Define compact operators $C_1, C_2, \dots, C_{n_1-1}$ with $\|C_i\|<\varepsilon/2^i$ such that $T_i=B_1+C_i\in B_1(\Omega_1)$ $(i=1,2,\dots,n_1-1)$, $\ker\tau_{T_iG}=\{0\}$ and $\ker\tau_{T_iT_j}=\{0\}$ $(i\neq j)$. Since $\sigma_r(G)\cap\sigma_l(T_i)\neq\emptyset$, there exist compact operators $D_1, D_2, \dots, D_{n_1-1}$ such that $D_i\notin\operatorname{ran}\tau_{GT_i}$ and $\|D_i\|<\varepsilon/2^i$ (see [F]).

Case I: $n_1=\infty$. Define

$$\bar{A}=\begin{pmatrix} G & D_1 & D_2 & D_3 & \dots \\ & T_1 & & & \\ & & T_2 & & \\ & & & T_3 & \\ 0 & & & & \ddots \end{pmatrix}.$$

If $P\in\{\bar{A}\}'$ then $P^2=P$. Assume that

$$P=\begin{pmatrix} P_{00} & P_{01} & P_{02} & \dots \\ P_{10} & P_{11} & P_{12} & \dots \\ P_{20} & P_{21} & P_{22} & \dots \end{pmatrix}$$

with respect to the same decomposition of the space. Then, since $\ker\tau_{T_iG}=\ker\tau_{T_iT_j}=\{0\}$ $(i\neq j)$, we have $P_{ij}=0$ $(i\geq 1,\ j\geq 0,\ i\neq j)$. Since $G, T_i\in\mathcal{B}_1(\Omega_1)$, it follows that $G, T_i\in(\mathrm{SI})$ $(i=1,2,\dots)$; see [FJ]. Since P_{ii} $(i=0,1,\dots)$ is idempotent and $P_{00}\in\{G\}'$, we have $P_{ii}\in\{T_i\}'$ $(i=1,2,\dots)$ and $P_{ii}=\delta_i$ ($\delta_i=0$ or I). Assume that $\delta_0=0$ (otherwise, consider $I-P$). Since $D_i\notin\operatorname{ran}\tau_{G_iT_i}$, $P_{ii}=0$ and $P_{0i}=0$ $(i=1,2,\dots)$, that is, $P=0$. Therefore $\bar{A}\in(\mathrm{SI})$. It is not difficult to see that $\bar{A}\in\mathcal{B}_\infty(\Omega_1)$. By Proposition 2.7, $\bar{A}\simeq A\in\mathcal{T}(\mathcal{N})\cap(\mathrm{SI})$. Furthermore, $\Lambda(A)=\Lambda(T)$.

Case II: $n_1<\infty$. Define

$$\bar{A}=\begin{pmatrix} G & D_1 & D_2 & \dots & D_{n_1-1} \\ & T_1 & & & \\ & & T_2 & & \\ & & & \ddots & \\ & & & & T_{n_1-1} \end{pmatrix}.$$

Then $\Lambda(\bar{A})=\Lambda(T)$. By the same argument used in case I, $\bar{A}\in(\mathrm{SI})$. Since $\bar{A}\in\mathcal{B}_{n_1}(\Omega_1)$, $\bar{A}$ admits an upper triangular matrix representation

$$\bar{A}=\begin{pmatrix} \lambda_0 & & \\ & \lambda_0 & \\ & & \ddots \end{pmatrix}\begin{matrix} \ker(\bar{A}-\lambda_0) \\ \ker(\bar{A}-\lambda_0)^2\ominus\ker(\bar{A}-\lambda_0) \\ \vdots \end{matrix}.$$

Since $\dim\ker(\bar{A}-\lambda_0)^k<\infty$ $(k=1,2,\dots)$, $\bar{A}\simeq A\in\mathcal{T}(\mathcal{N})$.

Second, we assume that $\mathcal{N}$ is well-ordered with 1-dimensional atoms. Then $\mathcal{N}=\bigoplus_{\alpha=1}^{\beta}\mathcal{N}_\alpha$, where $\mathcal{N}_\alpha$ has order type $w+1$ and β is a finite or countable ordinal.

Without loss of generality, we assume that β is a limit ordinal. Let $T \in \mathcal{T}(\mathcal{N})$; then $\rho_{S-F}^{-}(T) = \emptyset$. By the arguments used in the first step, we can find an (SI) operator $\bar{A} \in \mathcal{T}(\mathcal{N}_1) \cap \mathcal{B}_n(\Omega)$ that satisfies $\Lambda(\bar{A}) = \Lambda(T)$ and $\rho_{S-F}^{+}(T)$. Pick β pairwise distinct points $\{\lambda_\alpha\}_{\alpha=1}^{\beta-1}$ in $\sigma_{lre}(T)$ and let

$$\Lambda_\alpha = \begin{pmatrix} \lambda_\alpha & 1 & & & \\ & \lambda_\alpha & \frac{1}{2} & & \\ & & \ddots & \frac{1}{3} & \\ & & & \ddots & \ddots \end{pmatrix} \begin{matrix} e_1^{(\alpha)} \\ e_2^{(\alpha)} \\ e_3^{(\alpha)}, \\ \vdots \end{matrix}$$

where $e_n^{(\alpha)}$ is an ONB of $\mathcal{N}_\alpha$ and each $\bigvee\{e_n^{(\alpha)}\}$ is an atom of $\mathcal{N}_\alpha$. Then A_α belongs to $\mathcal{T}(\mathcal{N}_\alpha) \cap (\mathrm{SI})$, $\sigma(A_\alpha) = \sigma_{lre}(A_\alpha) = \lambda_\alpha$, and $\sigma_r(\bar{A}) \cap \sigma_l(A_\alpha) \neq \emptyset$. Thus there exists a compact J_α such that $J_\alpha \notin \operatorname{ran} \tau_{\bar{A}A_\alpha}$ and $\sum_\alpha \|J_\alpha\| < +\infty$. Since $\bar{A} \in \mathcal{B}_n(\Omega)$ and $\Omega \cap \{\lambda_\alpha\}_{\alpha=1}^{\beta-1} \subset \Omega \cap \sigma_{lre}(T) = \emptyset$, we have $\ker \tau_{A_\alpha \bar{A}} = \{0\}$. Set

$$A = \begin{pmatrix} \bar{A} & J_1 & J_2 & \cdots \\ & A_1 & & \\ & & A_2 & \\ & & & \ddots \end{pmatrix}.$$

As in the proof of the first step, we can deduce $A \in \mathcal{T}(\mathcal{N}) \cap (\mathrm{SI})$ and $\Lambda(\bar{A}) = \Lambda(T)$.

Finally, we assume that $\mathcal{N}^\perp$ is well-ordered with 1-dimensional atoms. According to the above proof, we can find an (SI) operator $A \in \mathcal{T}(\mathcal{N}^\perp)$ such that $\Lambda(A) = \Lambda(T^*)$; furthermore, $A^* \in \mathcal{T}(\mathcal{N})$ and $\Lambda(A^*) = \Lambda(T)$. From the construction of A and by the similarity orbit theorem [AFHV, Thm. 9.2], it is not difficult to see that $T \in \overline{S(A)}$. The proof of Theorem 1 is now complete. □

Proof of Theorem 2. (1) For each $T \in \mathcal{T}(\mathcal{N})$ with connected spectrum $\sigma(T)$, by Theorem 1 there exists $A \in \mathcal{T}(\mathcal{N}) \cap (\mathrm{SI})$ such that $\Lambda(A) = \Lambda(T)$ and $T \in \overline{S(A)}$; that is, there exists a sequence $\{X_n\}_{n=1}^{\infty}$ of invertible operators such that

$$B_n = X_n A X_N^{-1} \to T.$$

Since B_n is an upper triangular operator, there exists a unitary U_n such that

$$C_n = U_n B_n U_n^* \in \mathcal{T}(\mathcal{N}) \quad (n = 1, 2, \dots),$$

that is, $C_n \in \mathcal{T}(\mathcal{N}) \cap (\mathrm{SI})$ and $U_n^* C_n U_n \to T$. Hence the closure of the unitary orbit of the class of (SI) operators in $\mathcal{T}(\mathcal{N})$ contains all the operators in $\mathcal{T}(\mathcal{N})$ with connected spectrum.

(2) For each quasitriangular operator B on $\mathcal{H}$ with connected spectrum $\sigma(B)$ and Weyl spectrum $\sigma_w(T)$, and for each $\varepsilon > 0$, there exists a compact K_0 with $\|K_0\| < \varepsilon$ such that $\sigma(B + K_0) = \sigma_w(B + K_0)$. Since $B + K_0$ is quasitriangular, there exists a compact K_1 with $\|K_1\| < \varepsilon$ such that

$$B + K_0 + K_1 = \begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{pmatrix} \begin{matrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{matrix}$$

with respect to an ONB $\{f_i\}_{i=1}^{\infty}$ of $\mathcal{H}$, and $\sigma(B + K_0 + K_1) \subset \sigma(B)_\varepsilon$. Since $\sigma_0(B + K_0 + K_1) \subset \{\lambda_i\}_{i=1}^{\infty}$, we can "adjust" the diagonal—that is, we can find a compact K_2 with $\|K_2\| < \varepsilon$ such that $\sigma_0(B + K_0 + K_1 + K_2) \subset \sigma(B)$. Thus $C = B + K_0 + K_1 + K_2$ admits an upper triangular matrix representation with respect to the ONB $\{f_i\}_{i=1}^{\infty}$ with connected spectrum $\sigma(C) = \sigma_w(B)$ and $\|B - C\| < 3\varepsilon$. Therefore, the closure of the unitary orbit of the class of operators with connected spectrum in $\mathcal{T}(\mathcal{N})$ contains all the quasitriangular operators with connected spectrum and Weyl spectrum.

Parts (1) and (2) imply that the closure of the unitary orbit of the class of (SI) operators containing $\mathcal{T}(\mathcal{N})$ is the class of all quasitriangular operators on $\mathcal{H}$ with connected spectrum and Weyl spectrum.

(3) Suppose that A belongs to the closure of the unitary orbit of the class of (SI) operators in $\mathcal{T}(\mathcal{N})$. Then there are A_n in $\mathcal{T}(\mathcal{N}) \cap$ (SI) and U_n unitary ($n = 1, 2, \dots$) such that $\lim_n U_n^* A_n U_n = A$. It is easy to see that $\sigma(U_n^* A_n U_n) = \sigma_w(U_n^* A_n U_n)$ and that they are connected. Since A_n ($n = 1, 2, \dots$) are all quasitriangular, it is not difficult to show that A is quasitriangular and that $\sigma(A)$ and $\sigma_w(A)$ are connected. Thus, Theorem 2 is proved. □

Proof of Theorem 3. Without loss of generality, we can assume that $0 \in \Omega$. Thus T_k admits the representation

$$T_k = \begin{pmatrix} 0 & & & * \\ & 0 & & \\ & & 0 & \\ 0 & & & \ddots \end{pmatrix}$$

with respect to some ONB of H_k. Thus $T_k \in \mathcal{T}(\mathcal{N}_k) \cap B_1(\Omega)$, where $\mathcal{N}_k$ is the maximal nest of type $w + 1$ related to the ONB. For each $\varepsilon > 0$, there exists a compact C_k with $\|C_k\| < \varepsilon/2^k$ such that $B_k = T_k + C_k \in \mathcal{T}(\mathcal{N}_k) \cap B_1(\Omega)$ and $\ker \tau_{B_k B_j} = \{0\}$ $(k \neq j)$. Since $\sigma_r(B_{k-1}) \cap \sigma_l(B_k) \neq \emptyset$ $(k > 1)$, there exists D_k with $\|D_k\| < \varepsilon/2^k$ such that $B_{k-1,k} = D_k + T_{k-1,k} \notin \operatorname{ran} \tau_{B_{k-1} B_k}$ $(k = 2, \dots, m)$.

Set

$$K = \begin{pmatrix} C_1 & D_2 & & 0 \\ & C_2 & \ddots & \\ & & \ddots & D_m \\ 0 & & & C_m \end{pmatrix};$$

then $K \in \mathcal{K}(\mathcal{H})$ and $\|K\| < \varepsilon$. Define

$$A = T + K = \begin{pmatrix} B_1 & B_{12} & \dots & * \\ & B_2 & \ddots & \vdots \\ & & \ddots & B_{m-1,m} \\ 0 & & & B_m \end{pmatrix}.$$

By the same argument used in the proof of Theorem 1, $A \in (\mathrm{SI})$. It is not difficult to prove that $A \in \mathcal{B}_m(\Omega)$. Let $N_1 = \ker A$, $N_2 = \ker A^2, \dots, N_k = \ker A^k, \dots$. Then $\bigvee\{N_k : k = 1, 2, \dots\} = \mathcal{H}$ and $\dim N_k = mk$, and A admits the representation

$$A = \begin{pmatrix} 0 & A_{12} & A_{13} & \dots \\ & 0 & A_{23} & \dots \\ & & 0 & \\ 0 & & & \ddots \end{pmatrix} \begin{matrix} N_1 \\ N_2 \ominus N_1 \\ N_3 \ominus N_2 \\ \vdots \end{matrix} .$$

Let $\mathcal{M}$ denote the maximal nest refined from $\mathcal{M}' = \{0;\ N_k\ (k \geq 1);\ \mathcal{H}\}$. Then $A \in \mathcal{T}(\mathcal{M})$. Thus we can find a unitary U such that $UAU^* \in \mathcal{T}(\mathcal{N})$. □

Proof of Corollary 4. Assume that $0 \in \Omega$. Set $H_1 = \bigvee_{k=0}^{\infty} B^k e$, where B is the right inverse of T and $e \in \ker T$. Then $\mathcal{H}_1 \in (\operatorname{Lat} T) \cap (\operatorname{Lat} B)$ and T has the representation

$$T = \begin{pmatrix} T_1 & * \\ 0 & L_1 \end{pmatrix} \begin{matrix} H_1 \\ H_1^{\perp} \end{matrix} .$$

It is not difficult to prove that $T_1 \in B_1(\Omega)$, $\sigma(T_1) = \bar{\Omega}$, and $L_1 \in B_{n-1}(\Omega)$. Repeating this argument, T can be expressed as

$$T = \begin{pmatrix} T_1 & & & * \\ & T_2 & & \\ & & \ddots & \\ & & & T_n \end{pmatrix} \begin{matrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{matrix} ,$$

where $T_k \in B_1(\Omega)$ with $\sigma(T_k) = \bar{\Omega}$ $(k = 1, 2, \dots)$. By Theorem 3, for each $\varepsilon > 0$ there exists a compact K with $\|K\| < \varepsilon$ such that $T + K \in (\mathrm{SI})$. □

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