On the Boundary Orbit Accumulation Set for a Domain with Noncompact Automorphism Group

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Suppose that $D \subset \mathbb{C}^n$ is a smoothly bounded domain, that is, suppose D is bounded and ∂D is C^∞-smooth. We assume that the group $\text{Aut}(D)$ of holomorphic automorphisms of D is noncompact; this means (thanks to a classical result of H. Cartan) that there is a point $q \in \partial D$ such that, for some $p \in D$ and a sequence $\{f_j\} \subset \text{Aut}(D)$, one has $f_j(p) \to q$ as $j \to \infty$. Such a point q is called a *boundary accumulation point for* $\text{Aut}(D)$ (see [GK1] for a discussion of this matter).

Let $S(D)$ denote the set of all boundary accumulation points for $\text{Aut}(D)$. Existing examples of domains with noncompact automorphism groups (see [FIK] for a discussion of the case of Reinhardt domains), for which the set $S(D)$ can be found explicitly, indicate that this set should enjoy some explicit regularity properties.

For instance, let us for the moment restrict attention to smoothly bounded Reinhardt domains. It follows from [FIK] that, for such a domain D, $S(D)$ is always a compact, connected, smooth submanifold of ∂D. For such domains one can also observe other interesting properties of $S(D)$ such as the constancy and minimality of the rank of the Levi form of ∂D along $S(D)$ (see [H] for the genesis of these ideas); there is also a certain relation between this rank and the dimensions of the orbits of the action of $\text{Aut}(D)$ on D. Similarly, the type in the sense of D'Angelo [D1] is constant and maximal along $S(D)$. Many of these properties, when considered for general domains, appear to be related to the conjecture of Greene and Krantz [GK2] which states that every boundary accumulation point for a smoothly bounded domain must be of finite type.

In the present paper we begin a systematic study of the set $S(D)$ for a fairly general class of domains, and obtain foundational results on its topology and the relation of $S(D)$ to other invariant subsets of ∂D. We thank H. Boas, R. Remmert, J. Wolf, and S. Fu for stimulating remarks and suggestions concerning this work. We are also grateful to K. Diederich for a very valuable discussion of the results of this paper.

We say that ∂D is *variety-free* at $q \in \partial D$ if there are no nontrivial germs of complex varieties lying in ∂D and passing through q.

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PROPOSITION 1. *Let D be a bounded domain in $\mathbb{C}^n$. Suppose that ∂D is variety-free at each point of $S(D)$. Then $S(D)$ is compact.*

Proof. We need only prove that $S(D)$ is closed. Let $\{q_k\}$ be a sequence of points from $S(D)$ such that $q_k \to q \in \partial D$ as $k \to \infty$. Since ∂D is variety-free at each point q_k, it follows that for every q_k there is a sequence $\{f_k^j\}$ from $\mathrm{Aut}(D)$ such that f_k^j converges to the constant map q_k in all of D as $j \to \infty$ (see [GK1]).

Fix now a sequence $\{\varepsilon_k\}$, $\varepsilon_k > 0$, $\varepsilon_k \to 0$ as $k \to \infty$. Next, fix a point $p \in D$ and for every k find an index $j(k)$ such that $|f_k^{j(k)}(p) - q_k| < \varepsilon_k$. It is now obvious that $f_k^{j(k)}(p) \to q$ as $k \to \infty$. Hence $q \in S(D)$ and $S(D)$ is closed. □

REMARKS. (1) For smoothly bounded domains, the variety-free assumption in Proposition 1 would follow from the conjecture of Greene and Krantz.

(2) By Proposition 1, if D is smoothly bounded, pseudoconvex, and of finite type then the set $S(D)$ is nowhere dense in ∂D unless D is biholomorphically equivalent to the unit ball. Indeed, since $S(D)$ is closed, it would have interior points if it was not nowhere dense. For pseudoconvex domains of finite type, strictly pseudoconvex points are dense in ∂D (see [K1]), so $S(D)$ would contain a strictly pseudoconvex point and by [R] D would be holomorphically equivalent to the unit ball.

THEOREM 2. *Suppose that $D \subset \mathbb{C}^n$ is a smoothly bounded pseudoconvex domain of finite type. Then, if $S(D)$ contains at least three points, it is a perfect set and thus has the power of the continuum. Moreover, $S(D)$ is in this case either connected or the number of its connected components is uncountable.*

Proof. We note that any automorphism of D extends to a C^∞-automorphism of $\bar{D}$ (see e.g. [D2]).

Assume that $S(D)$ contains at least three points. We will first show that $S(D)$ cannot have isolated points. Indeed, let $q \in S(D)$ be an isolated point. Let $\{f_j\}$ be a sequence in $\mathrm{Aut}(D)$ such that $f_j \to q$ in all of D as $j \to \infty$. By passing to a subsequence we can also assume that $f_j^{-1} \to r$ in all of D where $r \in S(D)$.

Suppose first that $r = q$. Since $S(D)$ contains at least three points, one can find two distinct points $s, t \in S(D)$, $s, t \neq q$. Then, by Theorem 1 of [B], $f_j(s) \to q$ and $f_j(t) \to q$ as $j \to \infty$. Since q is an isolated point of $S(D)$ and each f_j preserves $S(D)$, we conclude that, for all sufficiently large j, one has $f_j(s) = f_j(t) = q$. This is impossible since every f_j is a one-to-one mapping on $\bar{D}$.

Suppose now that $r \neq q$. Then there is an $s \in S(D)$ such that $s \neq q, r$. Then, by [B], $f_j(q) \to q$ and $f_j(s) \to q$ as $j \to \infty$, which implies as before that for all sufficiently large j, $f_j(q) = f_j(s) = q$; this is again impossible since the f_j are one-to-one on $\bar{D}$. Thus, if $S(D)$ has at least three elements then $S(D)$ does not have isolated points and hence is a perfect set.

Assume now that $S(D)$ is disconnected and that the number of its connected components is not uncountable. Let $S(D) = \bigcup_k S_k(D)$ be the decomposition of

$S(D)$ into the disjoint union of its connected components. We will show that, for every k_0, every $q \in S_{k_0}(D)$, and every neighborhood U of q, there exists $k_1 \neq k_0$ such that $U \cap S_{k_1} \neq \emptyset$. This implies that the number of connected components of $S(D)$ is infinite and that each of the sets $X_m = S(D) \setminus \bigcup_{k=1}^{m} S_k(D)$ is open and dense in $S(D)$. Then, since $S(D)$ is compact (by Proposition 1) and since $\bigcap_{m=1}^{\infty} X_m = \emptyset$, the Baire category theorem gives a contradiction.

Let $S_{k_0}(D)$ be a component of $S(D)$ and let $q \in S_{k_0}$ be such that there exists a neighborhood U of q that does not contain points from $S_k(D)$ with $k \neq k_0$. Since $S(D)$ does not have isolated points, $S_{k_0}(D)$ is not a one-point set. Therefore, by decreasing U if necessary, we can assume that $S_{k_0}(D) \setminus U \neq \emptyset$. Let $\{f_j\} \subset \mathrm{Aut}(D)$ be such that $f_j \to q$ and $f_j^{-1} \to r$ in all of D, with $r \in S(D)$ as $j \to \infty$.

Suppose first that $r \in S_{k_0}(D)$. Then, for any other connected component $S_{k_1}(D)$ of $S(D)$ with $k_1 \neq k_0$, by [B] one has $f_j(S_{k_1}(D)) \subset U$ for all sufficiently large j; this is impossible since U does not contain an entire component of $S(D)$. If $r \notin S_{k_0}(D)$, then $f_j(S_{k_0}(D)) \subset U$ for all sufficiently large j, which is again impossible.

Thus, $S(D)$ either is connected or has uncountably many components. □

REMARK. It follows from [Z] that, if D is a bounded pseudoconvex domain that is also algebraic (i.e., given in the form $D = \{ z \in \mathbb{C}^n : P(z) < 0 \}$, where $P(z)$ is a polynomial such that grad $P \neq 0$ on ∂D), then the set $S(D)$ has only finitely many connected components. Hence, for such domains Theorem 2 implies either that $S(D)$ contains only one or two points or that $S(D)$ is connected and has the power of the continuum.

As noted in the proof of Theorem 2, for a smoothly bounded pseudoconvex domain of finite type, the set $S(D)$ is invariant under the extension of an automorphism of D to the boundary. In the following proposition we show that $S(D)$ is generically the smallest invariant subset of ∂D.

PROPOSITION 3. *Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain of finite type with noncompact automorphism group. Suppose that $A \subset \partial D$ is nonempty, compact, and invariant under* $\mathrm{Aut}(D)$. *Assume further that A is not a one-point subset of $S(D)$. Then $S(D) \subset A$.*

In particular, if $\mathrm{Aut}(D)$ *does not have fixed points in ∂D, then $S(D)$ is the smallest compact subset of ∂D invariant under* $\mathrm{Aut}(D)$.

Proof. Since A is closed, it is sufficient to show that every point of $S(D)$ belongs to $\bar{A}$. Let $q \in S(D)$ and $\{f_j\} \subset \mathrm{Aut}(D)$ be such that $f_j \to q$ and $f_j^{-1} \to r$ in all of D as $j \to \infty$, for some $r \in S(D)$. Since A is not a one-point subset of $S(D)$, there is a point $a \in A$, $a \neq r$. Then, by [B], $f_j(a) \to q$ as $j \to \infty$. Since A is invariant under any f_j, we see that $f_j(a) \in A$ for all j and thus q is either an accumulation point for A, or, if $f_j(a) = q$ for some index j, then $q \in A$. □

We now derive from the preceding proposition several corollaries regarding particular sets A.

Fix $0 \leq k \leq n-1$ and denote by $L_k(D)$ the set of all points from ∂D where the rank of the Levi form of ∂D does not exceed k. Clearly, each set $L_k(D)$ is a compact subset of ∂D and is invariant under any automorphism of D. Let l_1 denote the minimal rank of the Levi form on ∂D, and l_2 the minimal rank of the Levi form on $\partial D \setminus L_{l_1}(D)$. For these sets, Proposition 3 gives the following corollary (first proved in [H]). Note that the proof in [H] was also based on the results of [B].

COROLLARY 4. *Let D be as in Proposition 3. Then either*

(i) *$S(D) \subset L_{l_1}(D)$ or*
(ii) *$L_{l_1}(D)$ is a one-point subset of $S(D)$ and $S(D) \subset L_{l_2}(D)$.*

Proof. If $L_{l_1}(D)$ is not a one-point subset of $S(D)$ then, by Proposition 3, $S(D) \subset L_{l_1}(D)$. Suppose now that $L_{l_1}(D)$ is a one-point subset of $S(D)$. Then, since $L_{l_1}(D)$ is strictly contained in $L_{l_2}(D)$, one has $S(D) \subset L_{l_2}(D)$. □

By a similar argument one can endeavor to prove an analogous property of the type $\tau(q)$, $q \in \partial D$, in the sense of D'Angelo. Indeed, denote by $T_k(D)$ the set of all points $q \in \partial D$ where $\tau(q)$ is at least k. We choose t_1 and t_2 such that $T_{t_1}(D) \neq \emptyset$, $t_2 < t_1$, and there exists a point of type t_2 in $\partial D \setminus T_{t_1}(D)$. Since τ is invariant under automorphisms of D, so is every set $T_k(D)$. However, the sets $T_k(D)$ do not have to be closed, as the type function τ may not be upper-semicontinuous on ∂D (see e.g. an example in [D2, p. 136]). Therefore, for the type we have only a somewhat weaker result.

COROLLARY 5. *Let D be as in Proposition 3. Then either*

(i) *$S(D) \subset \overline{T_{t_1}(D)}$ or*
(ii) *$T_{t_1}(D)$ is a one-point subset of $S(D)$ and $S(D) \subset \overline{T_{t_2}(D)}$.*

In place of the type function τ, one can consider the multiplicity function μ on ∂D (see [D2, p. 145] for the definition), which is also invariant under the extensions of automorphisms to ∂D. It should be noted that, for $q \in \partial D$, the number $\tau(q)$ is finite if and only if $\mu(q)$ is finite. In contrast with τ, however, the function μ is upper-semicontinuous on ∂D. Analogously to what we have done above for the function τ, denote by $M_k(D)$ the set of all points $q \in \partial D$ where $\mu(q)$ is at least k, and choose m_1 and m_2 such that $m_1 = \max_{q \in \partial D} \mu(q)$, $m_2 < m_1$, and there exists a point of multiplicity m_2 in $\partial D \setminus M_{m_1}(D)$. Due to the upper-semicontinuity and invariance of μ, each set $M_k(D)$ is a compact subset of ∂D that is invariant under $\mathrm{Aut}(D)$. This observation gives the following analog of Corollary 6 for M_{m_1}, M_{m_2}.

COROLLARY 6. *Let D be as in Proposition 3. Then either*

(i) *$S(D) \subset M_{m_1}(D)$ or*
(ii) *$M_{m_1}(D)$ is a one-point subset of $S(D)$ and $S(D) \subset M_{m_2}(D)$.*

The proofs of Corollaries 5 and 6 are completely analogous to that of Corollary 4.

REMARKS. (1) It is plausible that Theorem 2 and Corollaries 4–6 hold without the assumptions of pseudoconvexity and finite type.

(2) We note that, in complex dimension 2, the type τ is upper-semicontinuous. As a result, Corollary 5 can be stated in this case without passing to the closures of the T_{t_j}. Also, in complex dimension 2, Corollary 5 is a consequence of the explicit classification of smoothly bounded pseudoconvex domains of finite type with noncompact automorphism groups [BP1].

(3) For a smoothly bounded circular domain, the set $S(D)$ clearly cannot be a one- or two-point set. Thus, Theorem 2 gives that, for smoothly bounded pseudoconvex circular domains of finite type, $S(D)$ is always a perfect set. Next, since for such domains the automorphism group cannot have fixed points on the boundary, Corollary 4 implies that in this case the Levi form of ∂D has constant rank along $S(D)$ and minimizes its rank over ∂D on $S(D)$ (see also [H]). It also should be noted here that, by the results of [BP2], every smoothly bounded convex domain of finite type with noncompact automorphism group is biholomorphically equivalent to a certain polynomially defined domain that admits an action of the 2-dimensional torus $\mathbb{T}^2$. Therefore, for any such a domain, $S(D)$ also is a perfect set, and the rank of the Levi form is constant and minimal on $S(D)$.

(4) The results of [FIK] imply that, for a smoothly bounded Reinhardt domain D, the type is constant along $S(D)$ and maximizes on $S(D)$ the type over ∂D. It is an interesting question whether there exists an analog of this fact for more general domains (cf. Cor. 5, Cor. 6). Note that one can make a statement analogous to Corollary 6 for the multitype introduced in [C], since the multitype function is upper-semicontinuous with respect to lexicographic ordering.

(5) It also follows from [FIK] that, for a smoothly bounded Reinhardt domain D, the real dimension of any orbit of the action of $\mathrm{Aut}(D)$ on D is at least $2(k+1)$, where k is the rank of the Levi form of ∂D along $S(D)$. Moreover, there is precisely one orbit of minimal dimension $2(k+1)$ (see [K2] for a discussion of this phenomenon). Also, the orbit of minimal dimension approaches every point of $S(D)$ nontangentially, whereas any other orbit approaches every point of $S(D)$ only along tangential directions. It would be interesting to know if similar statements hold for more general (e.g. circular) domains. The existence of an orbit that approaches $S(D)$ nontangentially would be very important for a proof of the Greene–Krantz conjecture. It also could be used to show that $S(D)$ is a smooth submanifold of ∂D.

We conclude this paper with a list of immediate open problems arising from our discussion that complement some of the preceding remarks.

OPEN PROBLEMS. (1) For a smoothly (C^∞) bounded domain $D \subset \mathbb{C}^n$, can the set $S(D)$ be a one- or two-point set? Note that the reference [GK3] gives an example of a domain with $C^{1-\varepsilon}$ boundary for which $S(D)$ has only two points. It appears that this example can be modified, using a parabolic group of automorphisms, so that $S(D)$ has just one point. We shall explore this matter further, and investigate increasing the boundary smoothness, in a future paper. Indications are that the case of finite boundary smoothness will be different from the case of infinite boundary smoothness.

(2) For a smoothly bounded domain D, can the set $S(D)$ have uncountably many components? For example, can it be a Cantor-type set?

(3) For a smoothly bounded domain D, is the set $S(D)$ always a smooth submanifold of ∂D? Note that the results of [FIK] imply that, for a smoothly bounded Reinhardt domain, $S(D)$ is always a smooth submanifold of ∂D that is diffeomorphic to a sphere of odd dimension.

(4) Is it always true that the rank of the Levi form is in fact constant and minimal along $S(D)$ and that the type is constant and maximal along $S(D)$?

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