The Backward Shift on Weighted Bergman Spaces

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1. Introduction

Let $H^2(\mathbb{D}) = H^2$ denote the *Hardy space* of analytic functions $f = \sum a_n z^n$ on the open unit disk $\mathbb{D} = \{|z| < 1\}$ for which

$$\sup_{0<r<1} \int_{|\zeta|=1} |f(r\zeta)|^2 \frac{|d\zeta|}{2\pi} = \sum_{n=0}^{\infty} |a_n|^2 < +\infty.$$

It is known that the *backward shift operator*

$$Lf = \frac{f - f(0)}{z}$$

is continuous on H^2 and the subspaces (closed linear manifolds) $\mathfrak{M} \subset H^2$ for which

$$L\mathfrak{M} \subset \mathfrak{M}$$

(such $\mathfrak{M}$ will be called *L-invariant* or *backward shift-invariant subspaces*) were completely characterized in [8] by means of duality.

NOTATION. We pause here to set some important notation that will be used throughout the paper. If $\mathcal{B}$ is a Banach space and T is a bounded linear operator on $\mathcal{B}$, we let $\text{Lat}(T, \mathcal{B})$ denote the subspaces $\mathfrak{M} \subset \mathcal{B}$ for which $T\mathfrak{M} \subset \mathfrak{M}$. For a set $S \subset \mathcal{B}$, we let $[S]_{(T,\mathcal{B})}$ denote the smallest T-invariant subspace of $\mathcal{B}$ that contains the set S. In this case, we will say $[S]_{(T,\mathcal{B})}$ is the T-invariant subspace "generated" by S.

The dual of H^2 can be identified with H^2 by means of the pairing

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{|\zeta|=1} f(r\zeta)\overline{g(r\zeta)} \frac{|d\zeta|}{2\pi}, \tag{1.1}$$

and a simple computation with power series reveals

$$\langle Lf, g \rangle = \langle f, zg \rangle \quad \forall f, g \in H^2.$$

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Thus, if $\mathfrak{M} \in \mathrm{Lat}(L, H^2)$ then $\mathfrak{M} = {}^{\perp}\mathcal{K}$, where $\mathcal{K} \in \mathrm{Lat}(M_z, H^2)$ (here M_z: $H^2 \to H^2$, $M_z f = zf$) and ${}^{\perp}\mathcal{K}$ is the preannihilator of $\mathcal{K}$. Beurling's famous characterization of $\mathrm{Lat}(M_z, H^2)$ identifies $\mathcal{K}$ as IH^2, where I is the greatest common inner divisor of $\mathcal{K}$ [4]. The L-invariant subspace $\mathfrak{M} = {}^{\perp}(IH^2)$ was then described using the concept of pseudocontinuation, which was introduced by Shapiro [24], as follows.

Let $\mathbb{D}_e = \mathbb{C}_\infty \setminus \bar{\mathbb{D}}$ denote the complement of the unit disk in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, and let $\mathfrak{N}(\mathbb{D})$ and $\mathfrak{N}(\mathbb{D}_e)$ denote the Nevanlinna class of $\mathbb{D}$ (resp. $\mathbb{D}_e$) (i.e., the quotient of two bounded analytic functions on $\mathbb{D}$ (resp. $\mathbb{D}_e$)). We also let

$$\mathfrak{N}^+(\mathbb{D}) = \{f/g \colon f, g \in H^\infty(\mathbb{D}),\ g \text{ outer}\}$$

$$\mathfrak{N}^+(\mathbb{D}_e) = \{f/g \colon f, g \in H^\infty(\mathbb{D}_e),\ g \text{ outer}\}$$

denote the Smirnov class of $\mathbb{D}$ (resp. $\mathbb{D}_e$).

REMARK. $g \in H^\infty(\mathbb{D}_e)$ is "outer" if $g(1/z)$ is outer (in the usual sense) in $H^\infty(\mathbb{D})$.

DEFINITION. If $G \in \mathfrak{N}(\mathbb{D}_e)$ and $g \in \mathfrak{N}(\mathbb{D})$, then by Fatou's theorem [9, Thm. 1.3] the nontangential limits of G and g exist a.e. on the unit circle $\mathbb{T} = \{|\zeta| = 1\}$. We say that G is a *pseudocontinuation* of g if these limits are equal a.e.

For example, if I is an inner function, then

$$\tilde{I}(z) = \frac{1}{\bar{I}(1/\bar{z})}, \quad \{z \colon |z| > 1, \bar{I}(1/\bar{z}) \neq 0\} \tag{1.2}$$

is a pseudocontinuation of I, while e^z (even though it has an analytic continuation to $\mathbb{C}$) does not have a pseudocontinuation since it has an essential singularity at infinity. By a theorem of Privalov, pseudocontinuations are unique when they exist [16].

THEOREM 1.1 ([8]). *A function $f \in H^2$ belongs to ${}^{\perp}(IH^2)$ if and only if f/I has a pseudocontinuation to $\mathfrak{N}^+(\mathbb{D}_e)$ that vanishes at infinity.*

Moreover, one can easily prove that ${}^{\perp}(IH^2)$ is cyclic and generated by the single vector $f = LI$, that is,

$${}^{\perp}(IH^2) = [f]_{(L, H^2)}.$$

Furthermore, if $f \in {}^{\perp}(IH^2)$ then, using Morera's theorem [11, p. 95], f has an analytic continuation to $\mathbb{C}_\infty \setminus \{z \colon 1/\bar{z} \in \underline{Z}(I)\}$, where $\underline{Z}(I)$ is the "lim-inf zero set of I"; that is,

$$\underline{Z}(I) = \left\{ z \in \bar{\mathbb{D}} \colon \liminf_{\lambda \to z, \lambda \in \mathbb{D}} |I(\lambda)| = 0 \right\}.$$

If S is a set of analytic functions on $\mathbb{D}$, we set

$$\underline{Z}(S) = \bigcap_{g \in S} \underline{Z}(g).$$

We remark that the analog of Theorem 1.1 is true for the Hardy space H^p, $1<p<+\infty$ [8]. The description of the L-invariant subspaces of H^p, $0<p<1$, is more complicated (see Aleksandrov [1]). Here, for $0<p<\infty$, H^p denotes the analytic functions on $\mathbb{D}$ with

$$\sup_{0<r<1}\int_{|\zeta|=1}|f(r\zeta)|^p\frac{|d\zeta|}{2\pi}<+\infty.$$

When $p=+\infty$, H^∞ denotes the bounded analytic functions on $\mathbb{D}$.

In this paper we plan to invoke duality again to examine the backward shift-invariant subspaces of the *weighted Bergman spaces* A^p_α. Here, for $\alpha>-1$ and $1\le p<+\infty$, an analytic function f on $\mathbb{D}$ belongs to A^p_α if

$$\int_{\mathbb{D}}|f(z)|^p(1-|z|)^\alpha\,dA(z)<+\infty,$$

where dA represents area measure on the disk. (The condition $\alpha>-1$ ensures that A^p_α is nonzero.) It is known [10; 14] that the dual of A^p_α can be identified with a space of "smooth" functions $X_{\alpha,p}$ on the unit disk (see Section 3) by means of the pairing

$$\langle f,g\rangle=\lim_{r\to1^-}\int_{|\zeta|=1}f(r\zeta)\overline{g(r\zeta)}\frac{|d\zeta|}{2\pi},\quad f\in A^p_\alpha,\ g\in X_{\alpha,p}.$$

More precisely, for fixed $\alpha>-1$ and $1\le p<+\infty$, let

$$q=p/(p-1),\qquad n_\alpha=\min\{n\in\mathbb{N}\cup\{0\}:n>\alpha\}.$$

Define $X_{\alpha,p}$ to be

$$X_{\alpha,p}=\{h\in\operatorname{Hol}(\mathbb{D}):h^{(n_\alpha+1)}(1-|z|^2)^{n_\alpha-\alpha}\in L^q((1-|z|^2)^\alpha\},\quad 1<p<+\infty;$$

$$X_{\alpha,1}=\left\{h\in\operatorname{Hol}(\mathbb{D}):h^{(n_\alpha+1)}(z)=O\left(\frac{1}{(1-|z|)^{n_\alpha-\alpha}}\right)\right\}.$$

These classes are the well-known Besov, Lipschitz, and Zygmund classes (see Section 3), which all belong to H^1.

Thus, using the identity

$$\langle Lf,g\rangle=\langle f,zg\rangle\quad\forall f\in A^p_\alpha,\ g\in X_{\alpha,p},$$

we conclude that if $\mathcal{M}\subset A^p_\alpha$ is L-invariant then $\mathcal{M}={}^\perp\mathcal{K}$ for some $\mathcal{K}\in\operatorname{Lat}(M_z,X_{\alpha,p})$. The description of $\operatorname{Lat}(M_z,X_{\alpha,p})$ can be quite complicated [5; 22; 23; 25] (see Section 3) and depends not only on the greatest common inner divisor I, as in the H^2 case, but on the zeros of functions in $\mathcal{K}$ (and possibly their derivatives) on the unit circle. Thus our description of $\mathcal{M}={}^\perp\mathcal{K}$ will be reminiscent of the H^2 case in that it will involve the pseudocontinuation of f/I, but will be different in that it will also involve the growth of f near the zero set of $\mathcal{K}$ (and possibly their derivatives) on the circle.

In the case $\alpha=0$ and $p=2$, one can use a result of Richter and Sundberg [19] to derive the following theorem (see Section 2). We first note that if $\mathcal{M}\in\operatorname{Lat}(L,A^2_0)$ then $\mathcal{M}^\perp=[g]_{(M_z,X_{0,2})}$ for some $g\in X_{0,2}$ (see Section 2).

THEOREM 1.2 ([19]). *Let* $\mathcal{M} \in \mathrm{Lat}(L, A_0^2)$, $\mathcal{M} \neq A_0^2$, *and* $g \in X_{0,2}$ *be such that* $\mathcal{M}^\perp = [g]_{(M_z, X_{0,2})}$. *Then:*

(i) $\mathcal{M} \subset \mathfrak{N}(\mathbb{D})$. *In fact, if* $f \in \mathcal{M}$ *then* $fg \in H^p(\mathbb{D})$ *for all* $0 < p < 1$.

(ii) *If* I_g *is the inner divisor of* g, *then* f/I_g *has a pseudocontinuation to* $\mathfrak{N}^+(\mathbb{D}_e)$ *that vanishes at infinity.*

Moreover, every $f \in \mathcal{M}$ *has an analytic continuation to* $\mathbb{C}_\infty \setminus \{z : 1/\bar{z} \in \underline{Z}(\mathcal{M}^\perp)\}$.

REMARKS. (1) Condition (i) says that $\mathcal{M} \subset \mathfrak{N}(\mathbb{D})$, which is important since, in general, A_α^p is not contained in the Nevanlinna class (see Section 3).

(2) In general, the word "pseudocontinuation" in Theorem 1.2 cannot always be replaced by the stronger "analytic continuation". Using a construction of [19] (see Section 2), it is possible to create an $\mathcal{M} \in \mathrm{Lat}(L, A_0^2)$, $\mathcal{M} \neq A_0^2$, such that $\mathbb{T} \subset \underline{Z}(\mathcal{M}^\perp)$ as well as an $f \in \mathcal{M}$ that does not have an analytic continuation across any arc of $\mathbb{T}$. However, by Theorem 1.2, f has a pseudocontinuation across $\mathbb{T}$.

(3) Using a result of Richter [17, p. 215], one can prove, as in the H^2 case, that $\mathcal{M}$ is cyclic (see Section 2).

The main result of this paper gives a complete description of $\mathcal{M}$ in the case where $\mathcal{M}^\perp$ is generated by slightly "smoother" functions. We now state our main theorem.

THEOREM 1.3. *Let* $1 \leq p < +\infty$ *and* $\alpha > -1$ *be fixed. Let* $\mathcal{M} \in \mathrm{Lat}(L, A_\alpha^p)$, $\mathcal{M} \neq A_\alpha^p$, *such that* $\mathcal{M}^\perp = [S]_{(M_z, X_{\alpha,p})}$, *where* S *is a set of functions* $g \in X_{\alpha,p}$ *with*

$$g^{(n_\alpha+1)}(1-|z|^2)^{n_\alpha-\alpha} \log \frac{1}{1-|z|} \in L^q((1-|z|^2)^\alpha \, dA), \quad q = \frac{p}{p-1}. \tag{1.3}$$

Then $\mathcal{M}$ *consists precisely of the functions* $f \in A_\alpha^p$ *with the properties*

(i) $fg \in H^1(\mathbb{D})$ *for all* $g \in S$, *and*

(ii) f/I *has a pseudocontinuation to* $\mathfrak{N}^+(\mathbb{D}_e)$ *that vanishes at infinity* (*where* I *is the greatest common inner divisor of* S).

Moreover, every $f \in \mathcal{M}$ *has an analytic continuation to* $\mathbb{C}_\infty \setminus \{z : 1/\bar{z} \in \underline{Z}(S)\}$.

REMARKS. (1) Since $X_{\alpha,p} \subset H^1 \subset \mathfrak{N}^+(\mathbb{D})$ [14, pp. 66–67], S indeed has a greatest common inner divisor I [12, p. 85].

(2) The main theorem continues to hold if we replace $\mathcal{M}^\perp$ by ${}^\perp\mathcal{M}$, the preannihilator of $\mathcal{M}$ in the predual of A_α^p. Of course this is of interest only in the case $p = 1$, when the predual of A_α^1 is different from $X_{\alpha,p}$ (see Section 4).

(3) For certain α and p, we will show (see Theorem 5.1) that condition (1.3) is always satisfied and that in fact the set S will consist of a single function $g \in \mathrm{Hol}(\mathbb{D}) \cap C^\infty(\bar{\mathbb{D}})$. Thus the conditions (i) and (ii) of Theorem 1.3 can be replaced by the conditions

(a) $fg \in H^\infty(\mathbb{D})$, and

(b) f has an analytic continuation to $\mathbb{C}_\infty \setminus \{z : 1/\bar{z} \in \underline{Z}(g)\}$ such that $f/I_g \in \mathfrak{N}^+(\mathbb{D}_e)$ and vanishes at infinity, where I_g is the inner factor of g. (Note that I_g is defined on $\mathbb{D}_e$ by (1.2).) Thus in this case, we will have a complete characterization of the L-invariant subspaces of A^p_α. In general, though, the z-invariant subspaces of $X_{\alpha,p}$ are more complicated and are not always generated by $\mathrm{Hol}(\mathbb{D})\cap C^\infty(\bar{\mathbb{D}})$ functions (see [5, pp. 293–296]).

(4) For these same α and p, we will show, as in the H^2 and A^2_0 cases, that every nontrivial L-invariant subspace $\mathfrak{M}$ is cyclic (see Theorem 5.4); that is, there is a single vector $f\in A^p_\alpha$ such that $\mathfrak{M}=[f]_{(L,A^p_\alpha)}$. Moreover, we can give a specific formula for this vector.

(5) Comparing Theorem 1.3 with Theorem 1.2, one might be tempted to conjecture that conditions (i) and (ii) of Theorem 1.3 completely characterize all the L-invariant subspaces of A^p_α. For certain α and p this is indeed the case, but for other α and p (e.g. $\alpha=0$ and $p=2$) it is not (see Section 2).

We also mention that if the weight $(1-|z|)^\alpha$ is replaced by $w(|z|)$, where w is a positive, continuous, integrable function on $[0,1)$, then every nontrivial L-invariant subspace $\mathfrak{M}$ is contained in $\mathfrak{N}(\mathbb{D})$ and every $f\in\mathfrak{M}$ has a pseudocontinuation to $\mathfrak{N}(\mathbb{D}_e)$ [3]. For general Banach spaces, though, the (nontrivial) backward shift-invariant subspaces need not have pseudocontinuations, even if the space is contained in $\mathfrak{N}(\mathbb{D})$. For example, if one considers the classical Dirichlet space $\{f\in\mathrm{Hol}(\mathbb{D}):\int|f'(z)|^2\,dx\,dy<+\infty\}$, [3] shows that there are nontrivial L-invariant subspaces for which pseudocontinuations need not exist across any portion of $\mathbb{T}$.

In Section 5, we will use Theorem 1.3 to examine the invariant subspaces (under the shift $f\to zf$) of the weighted Bergman space of an annulus, as well as the adjoint of the weighted Dirichlet shift, and the backward shift on $A^{-\infty}$.

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2. Motivation

Our results for the weighted Bergman spaces A^p_α were motivated by several papers of Richter and Sundberg dealing with the adjoint of the Dirichlet shift. In this section, we review some of those ideas and explain how this gives us information about the backward shift on A^2_0.

As mentioned in the introduction, the space A^2_0 is the space of analytic functions $f=\sum a_n z^n$ such that

$$\frac{1}{\pi}\int_{\mathbb{D}}|f(z)|^2\,dA(z)=\sum_{n=0}^{\infty}\frac{|a_n|^2}{n+1}<+\infty. \tag{2.1}$$

The Bergman space A^2_0 is a Hilbert space with the pairing

$$(f,g)_{A^2_0}=\frac{1}{\pi}\int_{\mathbb{D}}f(z)\bar{g}(z)\,dA=\sum_{n=0}^{\infty}\frac{a_n\overline{b_n}}{n+1}.$$

The *Dirichlet space* $\mathfrak{D}$ is the space of analytic functions $g = \sum a_n z^n$ with finite Dirichlet integral

$$D(g) = \frac{1}{\pi}\int_{\mathbb{D}} |g'(z)|^2\, dA = \sum_{n=0}^{\infty} n|a_n|^2. \tag{2.2}$$

By (2.2), $\mathfrak{D} \subset H^2$ and so we can define a norm on $\mathfrak{D}$ by

$$\|g\|_{\mathfrak{D}}^2 = \int_{|\zeta|=1} |g(\zeta)|^2 \frac{|d\zeta|}{2\pi} + D(g) = \sum_{n=0}^{\infty}(1+n)|a_n|^2.$$

This norm induces the inner product

$$(f,g)_{\mathfrak{D}} = \sum_{n=0}^{\infty}(1+n)\hat{f}(n)\overline{\hat{g}(n)}, \tag{2.3}$$

where $\{\hat{f}(n)\}$ are the Fourier coefficients of f (resp. g).

REMARK. A power series computation shows that $g \in \mathfrak{D}$ if and only if $g''(1-|z|^2) \in L^2(dA)$ and so $\mathfrak{D} = X_{0,2}$ (as defined in the introduction).

A computation with power series shows that the operator

$$U\colon \mathfrak{D} \to A_0^2, \quad Ug = (zg)'$$

is unitary. Also notice that, for $f \in \mathfrak{D}$,

$$(Uf)(z) = \left(f, \frac{1}{1-w\bar{z}}\right)_{\mathfrak{D}}$$

and so

$$(LUf)(z) = \left(f, \frac{w}{1-w\bar{z}}\right)_{\mathfrak{D}} = \left(M_z^* f, \frac{1}{1-w\bar{z}}\right)_{\mathfrak{D}} = (UM_z^* f)(z), \tag{2.4}$$

where M_z denotes the adjoint (under the pairing (2.3)) of the Dirichlet shift $M_z\colon \mathfrak{D} \to \mathfrak{D}$, $M_z f = zf$. Thus, if $\mathfrak{M} \in \operatorname{Lat}(L, A_0^2)$ then $U^*\mathfrak{M} \in \operatorname{Lat}(M_z^*, \mathfrak{D})$.

THEOREM 2.1. *Let* $\mathfrak{F} \in \operatorname{Lat}(M_z, \mathfrak{D})$, $\mathfrak{F} \neq (0)$. *Then:*

(1) $\mathfrak{F} \ominus z\mathfrak{F} = \mathfrak{F} \cap (z\mathfrak{F})^\perp$ *is* 1-*dimensional* [18, Thm. 2];
(2) *If* $\phi \in \mathfrak{F} \ominus z\mathfrak{F}$, $\phi \neq 0$, *then* $\mathfrak{F} = [\phi]_{(M_z, \mathfrak{D})}$ [17, p. 215]; *and*
(3) $\mathfrak{F}^\perp = [M_z^*\phi]_{(M_z^*, \mathfrak{D})}$ [17, p. 215].

COROLLARY 2.2. *If* $\mathfrak{M} \in \operatorname{Lat}(L, A_0^2)$, $\mathfrak{M} \neq A_0^2$, *then there is an* $f \in \mathfrak{M}$ *with* $\mathfrak{M} = [f]_{(L, A_0^2)}$.

Proof. Notice that $U^*\mathfrak{M} \in \operatorname{Lat}(M_z^*, \mathfrak{D})$ and so, by Theorem 2.1(3), there is an $h \in U^*\mathfrak{M}$ with $U^*\mathfrak{M} = [h]_{(M_z^*, \mathfrak{D})}$. Using (2.4), we have $\mathfrak{M} = [Uh]_{(L, A_0^2)}$ and so $\mathfrak{M}$ is cyclic. □

As a consequence of our main results we will show that, for certain α and p, every $\mathfrak{M} \in \operatorname{Lat}(L, A_\alpha^p)$ is cyclic (see Theorem 5.4).

Richter and Sundberg also proved the following pseudocontinuation result about the adjoint of the Dirichlet shift. Recall that if $\mathfrak{K} \in \operatorname{Lat}(M_z^*, \mathfrak{D})$ then $\mathfrak{K}^\perp \in \operatorname{Lat}(M_z, \mathfrak{D})$, and by choosing ϕ as in Theorem 2.1(2), $\mathfrak{K}^\perp = [\phi]_{(M_z, \mathfrak{D})}$.

THEOREM 2.3 ([19]). *Let $\mathcal{K} \in \mathrm{Lat}(M_z^*, \mathfrak{D})$, $\mathcal{K} \neq \mathfrak{D}$. Then:*

(1) *$U\mathcal{K} \subset \mathfrak{N}(\mathbb{D})$ and, for all $0 < p < 1$,*

$$\phi Uh \in H^p(\mathbb{D}) \quad \forall h \in \mathcal{K}.$$

(2) *For every $h \in \mathcal{K}$, Uh/I_ϕ has a pseudocontinuation to $\mathfrak{N}^+(\mathbb{D}_e)$ that vanishes at infinity.*

(3) *$\underline{Z}(\phi) = \underline{Z}(\mathcal{K}^\perp)$; moreover, for every $h \in \mathcal{K}$, the function Uh is analytic across the set $\mathbb{C}_\infty \setminus \{1/\bar{z} : z \in \underline{Z}(\phi)\}$.*

REMARKS. I_ϕ denotes the inner factor of ϕ. For $0 < p < +\infty$, we say a function f belongs to $H^p(\mathbb{D}_e)$ if $f(1/z) \in H^p(\mathbb{D})$.

We will very briefly outline a proof of this result since in [19] it is not stated quite in this form.

Proof. The proof of (1) is found in [19, Thm. 2.2]. For the proof of (2), we note that by a construction in [19, p. 869] there is a finite (complex) measure μ on $\mathbb{T}$ such that the Cauchy transform

$$\hat{\mu}(\lambda) = \int_{|\zeta|=1} \frac{d\mu(\zeta)}{\zeta - \lambda}$$

is contained in $H^p(\mathbb{D}) \subset \mathfrak{N}^+(\mathbb{D})$ for all $0 < p < 1$, and $\hat{\mu}(0) = 0$.

Since $\hat{\mu}$, Uh, and ϕ belong to $\mathfrak{N}(\mathbb{D})$ (note that $Uh \in \mathfrak{N}(\mathbb{D})$ by (1)), by Fatou's theorem it follows that the nontangential limits of these three functions exist a.e. on $\mathbb{T}$. They also show [19, p. 869, eq. 2.2] that

$$\hat{\mu}(\zeta) = \phi(\zeta)\overline{(Uh)(\zeta)} \text{ a.e. } \zeta \in \mathbb{T}. \tag{2.5}$$

The function ϕ can be factored as $\phi = I_\phi \mathcal{O}_\phi$, where I_ϕ is inner and $\mathcal{O}_\phi$ is outer.

For $|\lambda| > 1$, define the function

$$G(\lambda) = \frac{\overline{\hat{\mu}(1/\bar{\lambda})}}{\overline{\mathcal{O}_\phi(1/\bar{\lambda})}}.$$

Notice that $\overline{\hat{\mu}(1/\bar{\lambda})} \in H^p(\mathbb{D}_e) \subset \mathfrak{N}^+(\mathbb{D}_e)$ and that $\overline{\mathcal{O}_\phi(1/\bar{\lambda})} \in H^2(\mathbb{D}_e) \subset \mathfrak{N}^+(\mathbb{D}_e)$ is outer. Thus $G \in \mathfrak{N}^+(\mathbb{D}_e)$ and, since $\hat{\mu}(0) = 0$, G vanishes at infinity. Using (2.5), the nontangential limits of G are equal to Uh/I_ϕ a.e. on $\mathbb{T}$. Thus we have shown (2).

Conclusion (3) was shown in [19, Thm. 3.2]. □

REMARKS. (1) By [19, Thm. 3.2], $\mathbb{C}_\infty \setminus \{1/\bar{z} : z \in \underline{Z}(\phi)\}$ is the largest set with the property that whenever $h \in \mathcal{K}$, Uh extends to be analytic in $\mathbb{C}_\infty \setminus \{1/\bar{z} : z \in \underline{Z}(\phi)\}$.

(2) Although $\phi Uh \in H^p$ for all $0 < p < 1$ and all $h \in \mathcal{K}$, we cannot improve this to H^1. One sees this as follows: By [19, Thm. 4.3] there is a $(0) \neq \mathfrak{F} \in \mathrm{Lat}(M_z, \mathfrak{D})$ such that $\underline{Z}(\mathfrak{F}) = \underline{Z}(\phi)$ (where ϕ is chosen as in Theorem 2.1(2)) contains $\mathbb{T}$. Notice that $\mathfrak{F} = [\phi]_{(M_z, \mathfrak{D})} \neq (0)$. Letting $\mathcal{K} = (z\mathfrak{F})^\perp$, note that $\mathcal{K} \in \mathrm{Lat}(M_z^*, \mathfrak{D})$ and $\mathcal{K} \neq \mathfrak{D}$. Notice, by the way ϕ was chosen (i.e.

$\phi \perp z\mathcal{F}$), that $\phi \in \mathcal{K}$. Also notice that $z\phi \in \mathcal{F}$. If it were the case that $\phi U h \in H^1$ for all $h \in \mathcal{K}$, then setting $h = \phi$ we would have

$$z\phi U\phi \in H^1 \Rightarrow (z\phi)(z\phi)' \in H^1.$$

This would mean that $((z\phi)^2)' \in H^1$ which would mean that $(z\phi)^2$ is continuous on $\bar{\mathbb{D}}$ [9, Thm. 3.11]. But since $\underline{Z}(\phi)$ contains $\mathbb{T}$, $\phi \equiv 0$ on $\mathbb{T}$ and so $\phi \equiv 0$ on $\mathbb{D}$, which is impossible.

We can now state the Richter–Sundberg theorems in terms of the backward shift on A_0^2 as follows: It is well known that the dual of A_0^2 is $\mathfrak{D}$ via the pairing

$$\langle f, g\rangle = \lim_{r\to 1^-} \int_{|\zeta|=1} f(r\zeta)\bar{g}(r\zeta)\frac{|d\zeta|}{2\pi} = \lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n \overline{b_n} r^{2n},$$

where $f = \sum a_n z^n \in A_0^2$ and $g = \sum b_n z^n \in \mathfrak{D}$. Hence if $\mathfrak{M} \in \operatorname{Lat}(L, A_0^2)$ then, by (2.4), $U^*\mathfrak{M} \in \operatorname{Lat}(M_z^*, \mathfrak{D})$. A simple computation shows that for $f \in A_0^2$ and $g \in \mathfrak{D}$,

$$\langle f, g\rangle = (U^*f, g)_{\mathfrak{D}}.$$

Thus, if $\mathcal{K} \equiv U^*\mathfrak{M}$,

$$\begin{aligned}\mathfrak{M}^\perp &= \{g \in \mathfrak{D} : \langle f, g\rangle = 0\ \forall f \in \mathfrak{M}\}\\ &= \{g \in \mathfrak{D} : (U^*f, g)_{\mathfrak{D}} = 0\ \forall f \in \mathfrak{M}\}\\ &= \mathcal{K}^{\perp_{\mathfrak{D}}}.\end{aligned}$$

Now apply Theorem 2.3 to obtain Theorem 1.2. We also remark that in statement (i) of Theorem 1.2 we say that $fg \in H^p$ for all $0 < p < 1$. By remark (2) above, this cannot always be improved to H^1.

3. Preliminaries

We first review some basic facts about the weighted Bergman spaces A_α^p. We refer the reader to [7] for further details. By [7, Thm. 1.1], for $f \in A_\alpha^p$ we have

$$|f(z)| = O\left(\frac{1}{(1-|z|)^{(2+\alpha)/p}}\right), \quad z \in \mathbb{D}.$$

Moreover, if $\gamma < (\alpha+1)/p$ and $f \in \operatorname{Hol}(\mathbb{D})$ with

$$|f(z)| = O\left(\frac{1}{(1-|z|)^{\gamma}}\right), \quad z \in \mathbb{D}, \tag{3.1}$$

then a simple estimate says that $f \in A_\alpha^p$. It is a result of [9, Thm. 5.10] that for every $\gamma > 0$ there is a function $f \in \operatorname{Hol}(\mathbb{D})$ satisfying (3.1) and such that the nontangential limits of f do not exist on any set of positive measure. Thus, by Fatou's theorem, such an f does not belong to $\mathfrak{N}(\mathbb{D})$. From this we conclude that A_α^p is not contained in $\mathfrak{N}(\mathbb{D})$. We make this remark because (see [3] and Theorem 1.3) the nontrivial $\mathfrak{M} \in \operatorname{Lat}(L, A_\alpha^p)$ are contained in $\mathfrak{N}(\mathbb{D})$.

We now focus our attention on the dual space of A^p_α, mainly $X_{\alpha,p}$. For fixed $\alpha > -1$ and $1 \le p < +\infty$, let

$$q = p/(p-1), \qquad n_\alpha = \min\{n \in \mathbb{N} \cup \{0\}, n > \alpha\}.$$

Let Hol($\mathbb{D}$) denote the analytic functions on the unit disk $\mathbb{D}$, and let

$$X_{\alpha,p} = \{h \in \mathrm{Hol}(\mathbb{D}) : h^{(n_\alpha+1)}(1-|z|^2)^{n_\alpha-\alpha} \in L^q(1-|z|^2)^\alpha\, dA)\}, \quad p > 1;$$

$$X_{\alpha,1} = \left\{h \in \mathrm{Hol}(\mathbb{D}) : h^{(n_\alpha+1)}(z) = O\left(\frac{1}{(1-|z|)^{n_\alpha-\alpha}}\right)\right\};$$

$$x_{\alpha,1} = \left\{h \in \mathrm{Hol}(\mathbb{D}) : h^{(n_\alpha+1)}(z) = o\left(\frac{1}{(1-|z|)^{n_\alpha-\alpha}}\right) \text{ when } |z| \to 1\right\}.$$

These are well-known classes of functions:

$$X_{\alpha,p} = B^s_{q,q}, \quad p > 1, \ s = \alpha+1-(\alpha+1)/q \quad \text{(the Besov classes);}$$

$$X_{\alpha,1} = \Lambda^{n_\alpha}_{1-(n_\alpha-\alpha)} \ (x_{\alpha,1} = \lambda^{n_\alpha}_{1-(n_\alpha-\alpha)}) \quad \text{if } \alpha \notin \mathbb{Z} \quad \text{(the Lipschitz classes);}$$

$$X_{\alpha,1} = \Lambda^{n_\alpha-1}_* \ (x_{\alpha,1} = \lambda^{n_\alpha-1}_*) \quad \text{if } \alpha \in \mathbb{Z} \quad \text{(the Zygmund classes).}$$

We refer the reader to [10] and [14] for the definitions of these classes and their basic properties. Note that $X_{\alpha,p} \subset H^1$ for all α and p ([14, pp. 66–67]; further references are found on p. 68).

NOTATION. Throughout this paper, the following notation will be in force. For fixed $\alpha > -1$ and $1 \le p < +\infty$,

$$n_\alpha = \min\{n \in \mathbb{N} \cup \{0\}, n > \alpha\} \tag{3.2}$$

$$s = (\alpha+1) - (\alpha+1)(p-1)/p \tag{3.3}$$

$$m = [s], \quad \text{where } [s] \text{ denotes the integer part of } s. \tag{3.4}$$

It is also known [10; 14] that the dual of A^p_α can be identified with $X_{\alpha,p}$ by means of the pairing

$$\langle f, g\rangle = \lim_{r\to 1^-} \sum_{k=0}^{\infty} a_k \overline{b_k} r^{2k} = \lim_{r \to 1^-} \int_{|\zeta|=1} f(r\zeta)\overline{g(r\zeta)}\, \frac{|d\zeta|}{2\pi}, \tag{3.5}$$

where $f = \sum a_k z^k \in A^p_\alpha$ and $g = \sum b_k z^k \in X_{\alpha;p}$. Similarly, with the pairing above, the predual of A^1_α can be identified with $x_{\alpha,1}$. A computation with power series and using the identity

$$\int_0^1 (1-r^2)^n r^{2k+1}\, dr = \frac{1}{2}\frac{k!\, n!}{(k+n+1)!}$$

shows that

$$\langle f, g\rangle = \pi n_\alpha! \int_{\mathbb{D}} f\overline{(z^{n_\alpha+1}g)^{(n_\alpha+1)}}(1-|z|^2)^{n_\alpha}\, dA. \tag{3.6}$$

REMARK. We pause for a moment for a word about topology. When $p > 1$, $X_{\alpha,p}$ is endowed with the norm topology. When $p = 1$, $X_{\alpha,p}$ will be endowed with the weak-star topology that stems from the pairing (3.5).

If $\alpha > p-2+pm$, then (see [14, p. 110] for the appropriate references)

$$C_A^{(m+1)} \subset X_{\alpha,p} \subset C_A^{(m)}, \tag{3.7}$$

where

$$C_A^{(m)} = \{f \in \operatorname{Hol}(\mathbb{D}) : f^{(k)} \in C(\bar{\mathbb{D}})\ \forall 0 \le k \le m\}.$$

Moreover, when $\alpha > p-2+pm$, one also can prove that $X_{\alpha,p}$ is a Banach algebra (see [14, p. 110] for the appropriate references).

As mentioned in the introduction, if $\mathfrak{M} \in \operatorname{Lat}(L, A_\alpha^p)$ then

$$\mathfrak{M}^\perp \in \operatorname{Lat}(M_z, X_{\alpha,p}).$$

For general α and p, the description of $\operatorname{Lat}(M_z, X_{\alpha,p})$ is quite complicated [5], but when $\alpha > p-2+pm$ a complete characterization is known. We first review these results for $p>1$ and treat the case $p=1$ separately.

For $p>1$ one uses the pairing (3.5) and the Hahn–Banach theorem to show that the polynomials are norm dense in $X_{\alpha,p}$. Thus, if $\mathcal{K} \in \operatorname{Lat}(M_z, X_{\alpha,p})$ then $\mathcal{K}$ is a norm closed ideal of $X_{\alpha,p}$. These ideals have been characterized by [22; 23; 25] as follows.

A function $f \in H^1$ can be factored as

$$f = \mathcal{O}_f I_f,$$

where $\mathcal{O}_f$ is the outer factor and I_f is the inner factor. An inner function I can be factored as $I = cBS_\mu$, where c is a constant with $|c|=1$, B is a Blaschke product (with zeros repeated according to multiplicity), and S_μ is a singular inner function with positive singular measure μ. Define

$$\operatorname{spec}(I) = \operatorname{clos} B^{-1}(0) \cup \operatorname{supp} \mu.$$

Fixing $p>1$ and $\alpha > -1$ such that $\alpha > p-2+pm$ and fixing an inner function I, we let $E_0, E_1, \dots, E_m$ be closed subsets of $\mathbb{T}$ with the following properties:

$$E_0 \supset E_1 \supset \cdots \supset E_m \supset \operatorname{spec}(I) \cap \mathbb{T}; \tag{3.8}$$

$$E_0 \backslash E_k, \quad (k = 0, 1, \dots, m) \text{ are isolated points.} \tag{3.9}$$

For $f \in X_{\alpha,p}$ (resp. $x_{\alpha,1}$) and $0 \le k \le m$, let

$$E_k(f) = \{\zeta \in \mathbb{T} : f^{(j)}(\zeta) = 0, 0 \le j \le k\}. \tag{3.10}$$

Finally, let

$$\mathcal{K}(I, E_0, \dots, E_m) = \{f \in X_{\alpha,p}(x_{\alpha,1}) : E_k(f) \supset E_k, 0 \le k \le m, I_f/I \in H^\infty\}.$$

THEOREM 3.1 (Shirokov). *Fix $p>1$ and $\alpha>-1$ that satisfy $\alpha > p-2+pm$. Then $\mathcal{K}(I, E_0, \dots, E_m)$ is a (norm) closed ideal of $X_{\alpha,p}$; moreover, every (norm closed) ideal of $X_{\alpha,p}$ is of this form.*

REMARKS. (1) In order for the ideal $\mathcal{K}(I, E_0, \dots, E_m)$ to be nonzero, it is necessary and sufficient that

$$\int_0^{2\pi} \log \operatorname{dist}(\zeta, E_0 \cup \operatorname{spec}(I))|d\zeta| > -\infty. \tag{3.11}$$

See [26] and [27] for further information.

(2) It is a result of Khrushchev [13] that given $E_0, \ldots, E_m$ and I as in (3.8) and (3.9), there is a

$$g \in A^\infty \equiv \operatorname{Hol}(\mathbb{D}) \cap C^\infty(\bar{\mathbb{D}})$$

with

$$I_g = I \quad \text{and} \quad E_k(g) = E_k \,\forall 0 \le k \le m. \tag{3.12}$$

Thus, every ideal of $X_{\alpha,p}$ is principal; that is,

$$\mathcal{K}(I, E_0, \ldots, E_m) = [g]_{(M_z, X_{\alpha,p})}.$$

(3) For this generator g, we note that

$$\underline{Z}(g) = E_0 \cup \operatorname{spec}(I_g).$$

(4) If $p = 1$ then, for every $\alpha > -1$, $x_{\alpha,1}$ is a Banach algebra. Using the fact that the dual of A_α^1 is $x_{\alpha,1}$ along with the Hahn–Banach theorem, we get that the polynomials are norm dense in $x_{\alpha,1}$. Thus, as before, if $\mathcal{K} \in \operatorname{Lat}(M_z, x_{\alpha,1})$ then $\mathcal{K}$ is a norm closed ideal and, by [22; 23], $\mathcal{K} = \mathcal{K}(I, E_1, \ldots, E_m)$. Moreover, letting g be chosen as above, we get

$$\mathcal{K}(I, E_0, \ldots, E_m) = [g]_{(M_z, x_{\alpha,1})}.$$

We record this information about $x_{\alpha,1}$ here since it will be used in what follows.

For $X_{\alpha,1}$ the description of $\operatorname{Lat}(M_z, X_{\alpha,1})$ is essentially known but not found in the literature, so we briefly review it here. First note that, for all $\alpha > -1$, $X_{\alpha,1}$ is a Banach algebra. The description of $\operatorname{Lat}(M_z, X_{\alpha,1})$ is as follows.

THEOREM 3.2. *Let* $\alpha > -1$. *Then:*

(1) *For I and E_k $(0 \le k \le m)$ as in* (3.8) *and* (3.9), $\mathcal{K}(I, E_1, \ldots, E_m)$ *is a weak-star closed ideal of* $X_{\alpha,1}$.

(2) *Given any* $\mathcal{K} \in \operatorname{Lat}(M_z, X_{\alpha,1})$, $\mathcal{K}$ *is a weak-star closed ideal of* $X_{\alpha,1}$ *and*

$$\mathcal{K} = \mathcal{K}(I, E_1, \ldots, E_m)$$

for some I and E_k, $0 \le k \le m$.

(3) *If $g \in A^\infty$ satisfies* (3.12) *then*

$$\mathcal{K}(I, E_0, \ldots, E_m) = [g]_{(M_z, X_{\alpha,1})}.$$

To prove this theorem we first need some further information. Note that for $z \in \mathbb{D}$ and $g \in X_{\alpha,1}$,

$$g(z) = \overline{\left\langle \frac{1}{1 - w\bar{z}}, g \right\rangle}.$$

From this we conclude that for fixed $z \in \mathbb{D}$ the functional $g \to g(z)$ is weak-star continuous on $X_{\alpha,1}$. We now appeal to [5, Prop. 2] to get the following lemma.

LEMMA 3.3. *Let $\{g_n\}$ be a sequence in $X_{\alpha,1}$. Then $g_n \to 0$ weak-star if and only if the following two conditions hold:*

(1) *$g_n \to 0$ uniformly on compact subsets of $\mathbb{D}$; and*
(2) *$\|g_n\|_{X_{\alpha,1}} \le M < +\infty$ for all n.*

COROLLARY 3.4. *Let $\{f_n\}$ be a sequence in $X_{\alpha,1}$ that converges to zero weak-star. Then, for each $0 \le k \le m$ and each $\zeta \in \mathbb{T}$, $f_n^{(k)}(\zeta) \to 0$.*

Proof. Fix $0 \le k \le m$ and $\zeta \in \mathbb{T}$. By Lemma 3.3,

$$M \equiv \sup_n \|f_n\|_{X_{\alpha,1}} < +\infty.$$

Notice also that, by our earlier discussion, $X_{\alpha,1} \subset C_A^{(m)}$ with the inclusion being continuous. Thus, for $w_1, w_2 \in \mathbb{T}$ we have

$$|f_n^{(k)}(w_1) - f_n^{(k)}(w_2)| \le MC|w_1 - w_2|^{\beta}, \tag{3.13}$$

where $C > 0$ is independent of w_1, w_2, and n. For $0 < r < 1$,

$$|f_n^{(k)}(r\zeta) - f_n^{(k)}(\zeta)| \le \int_{|w|=1} |f_n^{(k)}(w) - f_n^{(k)}(\zeta)| P_{r\zeta}(w) \frac{|dw|}{2\pi},$$

where $P_{r\zeta}(w)$ is the Poisson kernel. For $\delta > 0$ the above is bounded by

$$\int_{|w-\zeta|<\delta} + \int_{|w-\zeta|>\delta},$$

which by (3.13) is bounded by

$$CM\delta^{\beta} + CM \int_{|w-\zeta|>\delta} P_{r\zeta}(w) \frac{|dw|}{2\pi}.$$

Given $\epsilon > 0$, choose δ_0 so that $CM\delta_0^{\beta} < \epsilon/2$. With this choice of δ_0, choose $0 < r_0 < 1$ so that

$$CM \int_{|w-\zeta|>\delta_0} P_{r_0\zeta}(w) \frac{|dw|}{2\pi} < \frac{\epsilon}{2}$$

(see [12, p. 32]). Thus we have

$$|f_n^{(k)}(r_0\zeta) - f_n^{(k)}(\zeta)| < \epsilon \quad \forall n.$$

From this we conclude that

$$|f_n^{(k)}(\zeta)| \le \epsilon + |f_n^{(k)}(r_0\zeta)| \quad \forall n. \tag{3.14}$$

Since $f_n \to 0$ weak-star we have (Lemma 3.3) that $f_n^{(k)}(r_0\zeta) \to 0$ and so, by (3.14),

$$\liminf_{n\to\infty} |f_n^{(k)}(\zeta)| \le \epsilon$$

and hence $f_n^{(k)}(\zeta) \to 0$. □

Proof of Theorem 3.2. (1) By the Krein–Smulian theorem, to show that $\mathcal{K}(I, E_1, \ldots, E_m)$ is weak-star closed it suffices to show that it is weak-star sequentially closed. To prove this we use Corollary 3.4.

(2) and (3) Recall that $(x_{\alpha,1})^* = A_\alpha^1$ and so $(x_{\alpha,1})^{**} = X_{\alpha,1}$. Hence if $\mathcal{K} \in \operatorname{Lat}(M_z, X_{\alpha,1})$ then ${}^\perp({}^\perp\mathcal{K}) \in \operatorname{Lat}(M_z, x_{\alpha,1})$, and so by the remark (4) above there is a $g \in A^\infty$ with

$$ {}^\perp({}^\perp\mathcal{K}) = [g]_{(M_z, x_{\alpha,1})}. $$

By the Hahn–Banach theorem,

$$ \mathcal{K} = [g]_{(M_z, X_{\alpha,1})}. $$

Letting I be the inner factor of g and $E_k = E_k(g)$ (recall the definition of $E_k(g)$ from (3.10)), we use the fact that $\mathcal{K}(I, E_1, \ldots, E_m)$ is weak-star closed (1) to conclude that

$$ \mathcal{K} = [g]_{(M_z, X_{\alpha,1})} \subset \mathcal{K}(I, E_1, \ldots, E_m). \tag{3.15} $$

Applying the same argument to $\mathcal{K}(I, E_1, \ldots, E_m)$, yields the existence of a $\tilde{g} \in A^\infty$ so that $\mathcal{K}(I, E_1, \ldots, E_m) = [\tilde{g}]_{(M_z, X_{\alpha,1})}$. Using Corollary 3.4, it must be that

$$ I_{\tilde{g}} = I, \qquad E_k(g) = E_k(\tilde{g}) \quad \forall 0 \le k \le m. $$

From the ideal theory for $x_{\alpha,1}$ (remark (4) above) we must have $[g]_{(M_z, x_{\alpha,1})} = [\tilde{g}]_{(M_z, x_{\alpha,1})}$ and so, by the Hahn–Banach theorem, $[g]_{(M_z, X_{\alpha,1})} = [\tilde{g}]_{(M_z, X_{\alpha,1})}$. Combining this with (3.15), we are done. □

4. Proof of the Main Result

LEMMA 4.1. *Let $\beta > 0$ and $u \in L^1((1-|z|)^\beta \, dA)$ such that $|u|$ is subharmonic in $\mathbb{D}$. Then the function F defined for $|\lambda| \ne 1$ by*

$$ F(\lambda) = \int_{\mathbb{D}} \frac{u(1-|z|^2)^\beta}{z-\lambda} \, dA $$

satisfies

$$ \lim_{r \to 1^-} \int_{|\zeta|=1} \left| F(r\zeta) - F\left(\frac{\zeta}{r}\right) \right| |d\zeta| = 0. $$

If $u \in L^1((1-|z|)^\beta \log(1/(1-|z|)) \, dA)$, then F belongs to $H^1(\mathbb{D}_e)$. If $u \in L^1((1-|z|)^{\beta-1} \, dA)$, then F belongs to $H^\infty(\mathbb{D}_e)$.

Proof. From the properties of u and since, for all $|\lambda| \ne 1$, $1/(z-\lambda) \in L^1(\mathbb{D}, dA)$, it follows easily that F is well-defined for $|\lambda| \ne 1$. For $|\zeta| = 1$ and $0 < r < 1$ we have

$$ F(r\zeta) - F\left(\frac{\zeta}{r}\right) = -\zeta\left(\frac{1}{r} - r\right) \int_{\mathbb{D}} \frac{u}{(z-r\zeta)(z-\zeta/r)} (1-|z|^2)^\beta \, dA. $$

From this we have

$$ \int_{|\zeta|=1} \left| F(r\zeta) - F\left(\frac{\zeta}{r}\right) \right| |d\zeta| \le \left(\frac{1}{r} - r\right) \int_{\mathbb{D}} |u|(1-|z|^2)^\beta \, dA \int_{|\zeta|=1} \frac{|d\zeta|}{|z-r\zeta||z-\zeta/r|}. $$

By the Cauchy–Schwartz inequality we get

$$\int_{|\zeta|=1} \frac{|d\zeta|}{|z-r\zeta||z-\zeta/r|} \leq \left(\int_{|\zeta|=1} \frac{|d\zeta|}{|z-r\zeta|^2}\right)^{1/2} \left(\int_{|\zeta|=1} \frac{|d\zeta|}{|z-\zeta/r|^2}\right)^{1/2}. \tag{4.1}$$

Notice that, for all $z \in \mathbb{D}$,

$$\int_{|\zeta|=1} \frac{|d\zeta|}{|z-\zeta/r|^2} = \int_{|\zeta|=1} \frac{|d\zeta|}{|1-rz/\zeta|^2} = 2\pi r^2 \sum_{n=0}^{\infty} r^{2n}|z|^{2n} \leq \frac{2\pi}{1-r^2|z|^2} \leq \frac{2\pi}{1-r|z|}.$$

Also notice that for $|z| < r$ we have

$$\int_{|\zeta|=1} \frac{|d\zeta|}{|z-r\zeta|^2} = \int_{|\zeta|=1} r^{-2}\frac{|d\zeta|}{|1-z/r\zeta|^2} = 2\pi r^{-2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{r^{2n}} = 2\pi \frac{r^{-2}}{1-|z|^2/r^2} \leq \frac{2\pi r^{-2}}{r-|z|}.$$

For $|z| > r$ we have

$$\int_{|\zeta|=1} \frac{|d\zeta|}{|z-r\zeta|^2} = \int_{|\zeta|=1} \frac{|z|^{-2}|d\zeta|}{|1-r\zeta/z|^2} = 2\pi|z|^{-2} \sum_{n=0}^{\infty} \frac{r^{2n}}{|z|^{2n}} = 2\pi \frac{|z|^{-2}}{1-r^2/|z|^2} \leq \frac{2\pi|z|^{-2}}{|z|-r}.$$

Thus the left side of (4.1) is bounded by

$$\leq \frac{2\pi r^{-2}}{||z|-r|^{1/2}(1-r|z|)^{1/2}}.$$

Hence

$$\int_{|\zeta|=1} \left|F(r\zeta) - F\left(\frac{\zeta}{r}\right)\right| |d\zeta| \leq C(1-r) \int_{\mathbb{D}} \frac{|u|(1-|z|)^{\beta}}{||z|-r|^{1/2}(1-r|z|)^{1/2}}\, dA.$$

Note that on $\{|z| < r^2\} \cup \{\sqrt{r} < |z| < 1\}$ we have

$$||z|-r|^{-1/2}(1-r|z|)^{-1/2} \leq c_1(1-r)^{-1},$$

and so, by the dominated convergence theorem, it follows that

$$\lim_{r\to 1^-} (1-r) \int_{\{|z|<r^2\}\cup\{\sqrt{r}<|z|<1\}} \frac{|u|(1-|z|)^{\beta}}{||z|-r|^{1/2}(1-r|z|)^{1/2}}\, dA = 0.$$

For $0 < \rho < 1$, define

$$U(\rho) = \int_{|\zeta|=1} |u(\rho\zeta)||d\zeta|.$$

Then U is an increasing function of ρ and

$$(1-r) \int_{\{r^2<|z|<\sqrt{r}\}} \frac{|u|(1-|z|)^{\beta}}{||z|-r|^{1/2}(1-r|z|)^{1/2}}\, dA$$
$$\leq c_2(1-r)^{\beta+1/2}U(\sqrt{r}) \int_{r^2}^{\sqrt{r}} \rho|\rho-r|^{-1/2}\, d\rho \leq c_3(1-r)^{\beta+1}U(\sqrt{r})$$

and, from

$$(1-r)^{\beta+1}U(\sqrt{r}) \le c_4 \int_{r^{1/2}}^{r^{1/3}} U(\rho)\,d\rho,$$

we deduce that the limit in the statement is zero.

If $u \in L^1((1-|z|)^\beta \log(1/(1-|z|)\,dA)$, we use Fubini's theorem and the well-known estimate

$$\int_{|\lambda|=r} \frac{|d\lambda|}{|1-\lambda|} = O\left(\log \frac{1}{1-r}\right), \quad r \to 1^-,$$

to obtain, for all $\rho > 1$,

$$\begin{aligned}\int_{|\lambda|=\rho} |F(\lambda)|\,|d\lambda| &\le \int_{\mathbb{D}} |u|(1-|z|^2)^\beta \int_{|\lambda|=\rho} \frac{|d\lambda|}{|z-\lambda|}\,dA \\ &\le c \int_{\mathbb{D}} |u|(1-|z|^2)^\beta \log \frac{1}{\rho-|z|}\,dA,\end{aligned}$$

which shows that these integrals remain bounded when $\rho \to 1^+$, that is, $F \in H^1(\mathbb{D}_e)$. In the case when $u \in L^1((1-|z|^2)^{\beta-1}\,dA)$, we use the fact that $|z-\lambda| > 1-|z|$ for $|\lambda| > 1$ to conclude that

$$|F(\lambda)| \le 2^\beta \int_{\mathbb{D}} |u|(1-|z|)^{\beta-1}\,dA;$$

that is, F belongs to $H^\infty(\mathbb{D}_e)$. □

Before proving the main result, we make the following remarks.

REMARKS. (1) If $f(\zeta)$ ($\zeta \in \mathbb{T}$) is the boundary function for an $f \in H^1(\mathbb{D})$, then $\overline{f(\zeta)}$ is the boundary function for a function in $H^1(\mathbb{D}_e)$ (the function being $\overline{f(1/\bar{z})}$). A similar situation holds for a function in $H^1(\mathbb{D}_e)$.

(2) We remind the reader that if $f \in \mathfrak{N}^+(\mathbb{D})$ and $f(\zeta) \in L^p(\mathbb{T})$ for some $0 < p \le +\infty$, then $f \in H^p(\mathbb{D})$ [9, Thm. 2.11].

(3) Recall from the introduction that if I is an inner function then I has a pseudocontinuation to $\mathfrak{N}(\mathbb{D}_e)$ given by

$$\tilde{I}(z) = \frac{1}{\bar{I}(1/\bar{z})}, \quad |z| > 1, \; 1/\bar{z} \notin \underline{Z}(I). \tag{4.2}$$

(4) The spectral radius of L is equal to unity and hence $(1-\lambda L)^{-1}L$ exists for $\lambda \in \mathbb{D}$. A computation shows that, for $f \in A^p_\alpha$,

$$(1-\lambda L)^{-1}Lf = \frac{f-f(\lambda)}{z-\lambda}. \tag{4.3}$$

Proof of Theorem 1.3. As usual, for a function h on $\mathbb{D}$ and $0 < r < 1$, we let h_r denote the function $h_r(z) = h(rz)$. Suppose that $f \in A^p_\alpha$ satisfies conditions (i) and (ii) of Theorem 1.3. Then for $g \in S$ we have $f\bar{g} \in L^1(\mathbb{T})$. If we write $g = I_g \mathcal{O}_g$, with I_g inner and $\mathcal{O}_g$ outer, then I divides I_g.

We first argue that f/I_g has a pseudocontinuation $\tilde{f}_g \in \mathfrak{N}^+(\mathbb{D}_e)$ that vanishes at infinity. Toward this end, we notice that by (4.2) the function I/I_g has a pseudocontinuation

$$\overline{I_g(1/\bar{\lambda})}/\overline{I(1/\bar{\lambda})}, \quad |\lambda| > 1. \tag{4.4}$$

But since $I_g/I \in H^\infty(\mathbb{D})$, it follows that $I_g(\zeta)/I(\zeta)$, $\zeta \in \mathbb{T}$, are the boundary values of an $H^\infty(\mathbb{D})$ function. Thus by remark (1) above $\overline{I_g(\zeta)/I(\zeta)}$ are the boundary values of an $H^\infty(\mathbb{D}_e)$ function. Thus, by (4.4), I/I_g has a pseudocontinuation to $H^\infty(\mathbb{D}_e)$. Because f/I has a pseudocontinuation to $\mathfrak{N}^+(\mathbb{D}_e)$ that vanishes at infinity,

$$\frac{f}{I_g} = \frac{f}{I}\frac{I}{I_g}$$

has a pseudocontinuation $\tilde{f}_g \in \mathfrak{N}^+(\mathbb{D}_e)$ that vanishes at infinity.

Then on $\mathbb{T}$ we have a.e.

$$f\bar{g}(\zeta) = \tilde{f}_g(\zeta)\overline{\mathcal{O}_g}(1/\bar{\zeta}) \in H^1(\mathbb{D}_e),$$

which vanishes at infinity; hence

$$\int_{|\zeta|=1} f\overline{\zeta^k g}\frac{|d\zeta|}{2\pi} = 0 \quad \forall k \in \mathbb{N}\cup\{0\}.$$

Note also that $f_r g_r$ converges to fg in $L^1(\mathbb{T})$ and hence, by the dominated convergence theorem,

$$f_r\overline{g_r} - f\bar{g} = (f_r g_r - fg)\overline{g_r}/g_r + fg(\overline{g_r}/g_r - \bar{g}/g) \tag{4.5}$$

converges to 0 in $L^1(\mathbb{T})$ when $r \to 1^-$. Thus, for $g \in S$ and $k \in \mathbb{N}\cup\{0\}$ we have

$$\langle f, z^k g\rangle = \lim_{r\to 1^-}\int_{|\zeta|=r} f\overline{\zeta^k g}\frac{|d\zeta|}{2\pi} = \int_{|\zeta|=1} f\overline{\zeta^k g}\frac{|d\zeta|}{2\pi} = 0, \tag{4.6}$$

which shows that $f \in \mathfrak{M}$.

Conversely, let $f \in \mathfrak{M}$. By the remark above, $(1-\lambda L)^{-1}L\mathfrak{M} \subset \mathfrak{M}$ for all $\lambda \in \mathbb{D}$. For $g \in S$ use (3.6) and (4.2) to obtain

$$\begin{aligned} 0 &= \langle (1-\lambda L)^{-1}Lf, g\rangle \\ &= \lim_{r\to 1^-}\int_{|\zeta|=r} \frac{f-f(\lambda)}{\zeta-\lambda}\bar{g}\frac{|d\zeta|}{2\pi} \\ &= \pi n_\alpha! \int_{\mathbb{D}} \frac{f-f(\lambda)}{z-\lambda}\overline{(z^{n_\alpha+1}g)^{(n_\alpha+1)}}(1-|z|^2)^{n_\alpha}\,dA. \end{aligned}$$

By Lemma 41, we can define the functions G and H on $\mathbb{C}\backslash\mathbb{T}$ by the formulas

$$G(\lambda) = \pi n_\alpha! \int \frac{\overline{(z^{n_\alpha+1}g)^{(n_\alpha+1)}}(1-|z|^2)^{n_\alpha}}{z-\lambda}\,dA,$$

$$H(\lambda) = \pi n_\alpha! \int \frac{f\overline{(z^{n_\alpha+1}g)^{(n_\alpha+1)}}(1-|z|^2)^{n_\alpha}}{z-\lambda}\,dA,$$

and notice from the above that

$$f(\lambda)G(\lambda) = H(\lambda) \quad \forall \lambda \in \mathbb{D}. \tag{4.7}$$

For $|\zeta| = 1$ and $0 < r < 1$, a computation using power series and (3.6) gives

$$G\left(\frac{\zeta}{r}\right) = \pi n_\alpha! \int \frac{\overline{(z^{n_\alpha+1}g)^{(n_\alpha+1)}(1-|z|^2)^{n_\alpha}}}{z-\zeta/r}\,dA$$
$$= \lim_{\rho\to 1^-}\int_{|z|=\rho} \frac{\bar g}{z-\zeta/r}\frac{|dz|}{2\pi} = r\zeta\bar g(r\zeta). \tag{4.8}$$

Another computation reveals

$$G_r(\zeta) = \pi n_\alpha! \int_{|z|\le r} \frac{\overline{(z^{n_\alpha+1}g)^{(n_\alpha+1)}(1-|z|^2)^{n_\alpha}}}{z-r\zeta}\,dA \in H^1(\mathbb{D}_e)$$

or equivalently, by the above remarks, $\overline{G_r} \in H^1(\mathbb{D})$.

By Lemma 4.1 and (4.8) we have

$$G_r \to \zeta\bar g(\zeta) \text{ in } H^1(\mathbb{D}_e) \text{ as } r\to 1^- \tag{4.9}$$

and

$$H_r \to H \text{ in } L^1(\mathbb{T}) \text{ as } r\to 1^-. \tag{4.10}$$

Then, by (4.7) and (4.10), $f_r G_r = H_r \to H$ in $L^1(\mathbb{T})$; as in (4.5), it follows that $f_r \overline{G_r}$ converges in $L^1(\mathbb{T})$, hence also in $H^1(\mathbb{D})$ to $h = H\bar G/G$. Since for $\lambda\in\mathbb{D}$

$$h(\lambda) = \lim_{r\to 1^-} f_r\overline{G_r}(\lambda) = f(\lambda)\lim_{r\to 1^-}\overline{G_r}(\lambda) = f(\lambda)\lambda g(\lambda),$$

we get that $fg\in H^1$. Moreover, on $\mathbb{T}$ we have a.e. $f\overline{zg} = H \in H^1(\mathbb{D}_e)$, which implies

$$f/I_g(\zeta) = \zeta H(\zeta)/\overline{\mathcal{O}_g(1/\bar\zeta)} \tag{4.11}$$

a.e. on $\mathbb{T}$. Then f/I_g has a pseudocontinuation $\tilde f_g \in \mathfrak{N}^+(\mathbb{D}_e)$ with

$$\tilde f_g(\infty)\overline{\mathcal{O}_g(0)} = \int_{\mathbb{D}} f\overline{(z^{n_\alpha+1}g)^{(n_\alpha+1)}}(1-|z|^2)^{n_\alpha}\,dA = 0. \tag{4.12}$$

This shows that f has a pseudocontinuation in $\mathfrak{N}(\mathbb{D}_e)$ such that the inner factor in the denominator of the canonical Nevanlinna factorization divides $1/\tilde I_g$ for arbitrary $g\in S$. Then this inner factor also divides $1/\tilde I$, where I is the greatest common inner divisor of the inner functions I_g $(g\in S)$ and $\tilde I$ is given by (4.2); that is, f/I has a pseudocontinuation $\tilde f$ in $\mathfrak{N}^+(\mathbb{D}_e)$ satisfying

$$\tilde f = \tilde f_g \tilde I_g/\tilde I.$$

Finally, if we choose $g\in S$ such that $I_g/I(0)\neq 0$, then $\tilde I_g/\tilde I$ has no pole at ∞. Hence from (4.12) we obtain $\tilde f(\infty) = 0$.

To finish, we need to show that f has an analytic continuation to $\mathbb{C}_\infty\backslash\{z: 1/\bar z\in \underline{Z}(S)\}$. Using the fact that f has a pseudocontinuation $\tilde f$ to $\mathbb{D}_e$, we need only show that if $\zeta\in\mathbb{T}\backslash\underline{Z}(S)$ then f extends to be analytic in a neighborhood of ζ.

If $\zeta\in\mathbb{T}\backslash\underline{Z}(S)$, then there exists a $g\in S$ and an open set U containing ζ such that

$$|g(z)|\ge\delta>0 \quad \forall z\in U\cap\mathbb{D}. \tag{4.13}$$

But since $fg \in H^1(\mathbb{D})$, we have $f \in H^1(U \cap \mathbb{D})$. From (4.11) we observe that the pseudocontinuation $\tilde{f}$ of f satisfies

$$\tilde{f}(z) = \frac{zH(z)}{\bar{g}(1/\bar{z})}, \quad |z| > 1.$$

By Lemma 4.1, $H \in H^1(\mathbb{D}_e)$ and thus, by (4.13), $\tilde{f} \in H^1(U \cap \mathbb{D}_e)$. Using Morera's theorem [11, p. 95], we get that f has an analytic continuation across $\mathbb{T} \cap U$. □

REMARKS. (1) If the functions $g \in S$ in Theorem 1.3 satisfy the stronger condition

$$g^{(n_\alpha+1)}(1-|z|^2)^{n_\alpha-\alpha-1} \in L^q((1-|z|^2)^\alpha), \quad q = p/(p-1), \tag{4.14}$$

then, by Lemma 4.1, the condition (i) in Theorem 1.3 can be replaced by

(i′) *$fg \in H^\infty$ for all $f \in \mathfrak{M}$ and all $g \in S$.*

(2) Let S be a subset of $x_{\alpha,1}$ such that

$$|g^{(n_\alpha+1)}(z)| = O\left(\frac{1}{(1-|z|^2)^{n_\alpha}} \log \frac{1}{1-|z|}\right) \quad \forall g \in S. \tag{4.15}$$

Then exactly the same argument shows that $[S]^\perp$ consists of all functions $f \in A^1_\alpha$ with the properties (i) and (ii) of Theorem 1.3.

5. Applications

In this section we will give the following applications of our results:

(1) L-invariant subspaces of certain Bergman spaces;
(2) cyclic vectors;
(3) the adjoint of the weighted Dirichlet shift;
(4) weighted Bergman spaces on an annulus; and
(5) the backward shift on $A^{-\infty}$.

5.1. L-invariant Subspaces of Certain Bergman Spaces

By combining Theorem 1.3, Theorem 3.2, and (4.14), we can completely characterize the L-invariant subspaces of A^p_α for fixed $\alpha > -1$ and $1 \le p < +\infty$ that satisfy

$$\alpha > p - 2 + mp. \tag{5.1}$$

We remind the reader that the definition of m is found in (3.4).

NOTATION. For a set $A \subset \mathbb{C}$, let

$$A^* = \{1/\bar{a} : a \in A\}.$$

Notice that if A contains the origin, then A^* contains the point at infinity.

Before stating Theorem 5.1, we remind the reader that any inner function I has a pseudocontinuation given by $1/\overline{I(1/\bar{z})}$, $z \in \mathbb{D}_e$. Moreover, this pseudocontinuation is an analytic continuation to $\mathbb{C}_\infty \setminus \operatorname{spec}(I)^*$ [12, p. 68].

THEOREM 5.1. *Let α and p satisfy* (5.1). *If $\mathfrak{M} \in \operatorname{Lat}(M_z, A^p_\alpha)$ with $\mathfrak{M} \neq A^p_\alpha$, then $\mathfrak{M} = {}^\perp[g]_{(M_z, A^{p'}_\alpha)}$ for some $g \in A^\infty$; moreover, $f \in A^p_\alpha$ belongs to $\mathfrak{M}$ if and only if*

(1) *$fg \in H^\infty(\mathbb{D})$ and*

(2) *f has an analytic continuation to $\mathbb{C}_\infty \setminus \underline{Z}(g)^*$ such that f/I_g belongs to $\mathfrak{N}^+(\mathbb{D}_e)$ and vanishes at infinity.*

5.2. Cyclic Vectors

In this section we will show that when α and p satisfy (5.1), every nontrivial L-invariant subspace $\mathfrak{M} \subset A^p_\alpha$ is *cyclic*, that is,

$$\mathfrak{M} = [f]_{(L, A^p_\alpha)}$$

for some vector f. We will also give a specific formula for this vector. In order to do this, we need a few preliminaries.

For a finite (complex) measure μ on $\mathbb{T}$, let

$$\hat{\mu}(z) = \int_{|\zeta|=1} \frac{d\mu(\zeta)}{\zeta - z}$$

be the Cauchy transform of μ. Note that $\hat{\mu} \in \operatorname{Hol}(\mathbb{D})$ and

$$\hat{\mu}^{(k)}(z) = k! \int_{|\zeta|=1} \frac{d\mu(\zeta)}{(\zeta - z)^{k+1}}, \quad k = 0, 1, 2, \ldots .$$

LEMMA 5.2. *Fix $\alpha > -1$ and $1 \le p < +\infty$ satisfying* (5.1). *If μ is a finite* (*complex*) *measure on $\mathbb{T}$ then*

$$\hat{\mu}^{(k)} \in A^p_\alpha \quad \forall 0 \le k \le m, \tag{5.2}$$

and for every $g \in X_{\alpha,p}$ we have

$$\langle \hat{\mu}^{(k)}, g \rangle = \int_{|\zeta|=1} \bar{\zeta}^{k+1} \overline{(\zeta^k g)^{(k)}}\, d\mu(\zeta). \tag{5.3}$$

Proof. For α and p satisfying (5.1) we have $X_{\alpha,p} \subset C_A^{(m)}$; in particular, $x_{\alpha,1} \subset X_{\alpha,1} \subset C_A^{(m)}$ and the inclusion maps are continuous. Then the rule

$$g \to \int_{|\zeta|=1} \zeta^{k+1} (\zeta^k g)^{(k+1)}\, d\mu(\zeta)$$

defines a continuous linear functional on $X_{\alpha,p}$ if $p > 1$ (or $x_{\alpha,1}$). Hence there is an $f \in A^p_\alpha$ such that

$$\langle f, g \rangle = \int_{|\zeta|=1} \bar{\zeta}^{k+1} \overline{(\zeta^k g)^{(k)}}\, d\mu(\zeta)$$

for all $g \in X_{\alpha,p}$, $p>1$ (or $x_{\alpha,1}$). For $g(z)=z^n$ we obtain

$$\frac{f^{(n)}(0)}{n!}=\langle f,z^n\rangle=\frac{(n+k)!}{n!}\int_{|\zeta|=1}\bar{\zeta}^{n+k+1}\,d\mu(\zeta)=\frac{\hat{\mu}^{(k+n)}(0)}{n!},$$

which means that $f=\hat{\mu}^{(k)}$ and that (5.3) holds for all $g\in X_{\alpha,p}$, $p>1$ (or $x_{\alpha,1}$).

To finish, note that $x_{\alpha,1}^{**}=X_{\alpha,1}$ and so (5.3) holds for all $g\in X_{\alpha,1}$. □

REMARK. For the proof of this next lemma, we recall some facts.

(1) If μ is a finite measure on $\mathbb{T}$, then $\hat{\mu}(\lambda)$, $|\lambda|<1$, belongs to $H^p(\mathbb{D})$ for all $0<p<1$ [9, p. 39]. Note also that $\hat{\mu}(\lambda)$, $|\lambda|>1$, belongs to $H^p(\mathbb{D}_e)$ for all $0<p<1$. Furthermore ([9, p. 39]; see also [20, Prop. 1]),

$$\lim_{r\to 1^+}\hat{\mu}(r\zeta)-\lim_{r\to 1^-}\hat{\mu}(r\zeta)=2\pi\zeta\mu'(\zeta)\ \text{ a.e. }\zeta\in\mathbb{T}.$$

(2) If $f'\in H^p$ for some $0<p<1$ then $f\in H^q$, where $q=p/(1-p)$. This is a result of Hardy and Littlewood; see [9, p. 88] for further references.

(3) If $f'\in H^1$, then $f\in C(\bar{\mathbb{D}})$ [9, Thm. 3.11].

(4) If $f\in\mathfrak{N}^+(\mathbb{D})$ and $f(\zeta)\in L^p(\mathbb{T},|d\zeta|)$, then $f\in H^p(\mathbb{D})$ [9, Thm. 2.11].

(5) If $f\in \operatorname{Hol}(\mathbb{D})$ and $(Af)(z)=\int_0^z f(w)\,dw$ (i.e., the antiderivative that vanishes at zero), then using integration by parts one shows that, for all $n\in\mathbb{N}$,

$$A^n(z^nf^{(n)})=z^nf+\sum_{k=1}^{n}c_kA^kf+p,$$

where c_k are constants and p is a polynomial.

LEMMA 5.3. *Let $\mu,\nu_1,\dots,\nu_n$ be finite (complex) measures on $\mathbb{T}$ such that $\nu_1,\dots,\nu_n$ are singular with respect to $|dz|$. If*

$$\hat{\mu}(z)+\sum_{k=1}^{n}z^k\hat{\nu}_k^{(k)}(z)=0\quad\forall z\in\mathbb{D},\tag{5.4}$$

then $\nu_1=\cdots=\nu_n=0$ and $d\mu=\bar{f}\,|dz|$ for some $f\in H^1$.

Proof. Assume $\nu_1,\dots,\nu_n$ are not all zero. Let $r\le n$ be the greatest index with $\nu_r\ne 0$. Notice from (5.4) that

$$z^r\hat{\nu}_r^{(r)}=-\hat{\mu}-\sum_{k=1}^{r-1}z^k\hat{\nu}_k^{(k)}.$$

Now notice that

$$A^r(z^r\hat{\nu}_r^{(r)})=-A^r(\hat{\mu})-\sum_{k=1}^{r-1}A^{r-k}A^k(z^k\hat{\nu}_k^{(k)}).$$

But from remark (5) above we have

$$A^k(z^k\hat{\nu}_k^{(k)})=z^k\hat{\nu}_k+\sum_{j=1}^{k}c_jA^j\hat{\nu}_k+p_k,$$

and so by remarks (1) and (2) above we have

$$A^r(z^r\hat{\nu}_r^{(r)}) = h \in H^1(\mathbb{D}).$$

Now apply remark (5) again to obtain

$$A^r(z^r\hat{\nu}_r^{(r)}) = z^r\hat{\nu}_r + \sum_{s=1}^{r} c_s A^s \hat{\nu}_r + p_r = h \in H^1.$$

Again by remarks (1) and (2),

$$z^r\hat{\nu}_r \in H^1(\mathbb{D}) \Rightarrow \hat{\nu}_r \in H^1(\mathbb{D}).$$

Notice that $\hat{\nu}_r(\lambda)$, $|\lambda| < 1$, belongs to $H^1(\mathbb{D})$, that $\hat{\mu}_r(\lambda)$, $|\lambda| > 1$, belongs to $H^p(\mathbb{D}_e)$ for all $0 < p < 1$, and that since ν_r is singular with respect to $|dz|$ it follows (by remark (1) above) that

$$\lim_{s\to 1^-} \hat{\mu}_r(s\zeta) - \lim_{s\to 1^+} \hat{\mu}_r(s\zeta) = 0. \tag{5.5}$$

This says that $\hat{\mu}_r(\lambda)$, $|\lambda| > 1$, belongs to $H^p(\mathbb{D}_e)$ and has $L^1(\mathbb{T}, |dz|)$ boundary values, which means (by remark (4) above) $\hat{\mu}_r(\lambda)$, $|\lambda| > 1$ belongs to $H^1(\mathbb{D}_e)$. By (5.5) and Morera's theorem, $\hat{\nu}_r$ is entire. However, $\hat{\nu}_r(\infty) = 0$ and so, by Liouville's theorem, $\hat{\nu}_r \equiv 0$ on $\mathbb{C}_\infty$. This implies that

$$\int_{|\zeta|=1} \zeta^j \, d\nu_r(\zeta) = 0 \quad \forall j \in \mathbb{Z},$$

which means that $\nu_r \equiv 0$. Thus by the definition of ν_r we have $\nu_1 = \cdots = \nu_n = 0$.

Finally, from (5.4) we get that $\hat{\mu}(z) = 0$ for all $z \in \mathbb{D}$, which means that

$$\int_{|\zeta|=1} \zeta^j \, d\mu(\zeta) = 0 \quad \forall j < 0;$$

by the theorem of F. and M. Riesz [9, Thm. 3.8], this gives the desired result. □

THEOREM 5.4. *Let α and p satisfy* (5.1), *and let $\mathcal{K}(I, E_0, \ldots, E_m)$ be a nontrivial ideal of $X_{\alpha,p}$. Let $\mu_0, \mu_1, \ldots, \mu_m$ be finite measures with* $\operatorname{supp}\mu_k = E_k$, $0 \le k \le m$, *and let*

$$f_0 = L\left(I + \sum_{k=0}^{m} \hat{\mu}_k\right). \tag{5.6}$$

Then

$$^{\perp}\mathcal{K}(I, E_0, \ldots, E_m) = \operatorname{span}\{L^j f_0, \, j \ge 0\}.$$

Proof. If $g \in X_{\alpha,p}$, then

$$\begin{aligned} &\langle L^j f_0, g\rangle = 0 \;\forall j \ge 0 \\ &\Leftrightarrow \langle f_0, z^j g\rangle = 0 \;\forall j \ge 0 \\ &\Leftrightarrow \left\langle f_0, \frac{1}{1-\bar{a}z} g\right\rangle = 0 \;\forall a \in \mathbb{D} \\ &\Leftrightarrow \int \frac{1}{1-a\bar{z}} I\bar{z}\bar{g}|dz| + \sum_{k=0}^{m} \int \bar{z}^{k+1} \overline{\left(\frac{z^k(zg)}{1-\bar{a}z}\right)^{(k)}} \, d\mu_k(z) = 0 \;\forall a \in \mathbb{D}. \end{aligned}$$

The last equality is an application of Lemma 5.2.

Suppose that $\langle L^j f_0, g\rangle = 0$ for all $j \geq 0$. Letting $h = zg$, we have

$$\int \frac{1}{1-a\bar{z}} I\bar{h}|dz| + \sum_{k=0}^{m} \int \bar{z}^{k+1} \overline{\left(\frac{z^k h}{1-\bar{a}z}\right)^{(k)}} \, d\mu_k(z) = 0 \quad \forall a \in \mathbb{D}. \tag{5.7}$$

Notice that

$$\left(\frac{z^k h}{1-\bar{a}z}\right)^{(k)} = \sum_{j=0}^{k} \frac{k!}{j!} \bar{a}^{k-j} \frac{(z^k h)^{(j)}}{(1-\bar{a}z)^{k-j+1}},$$

and so (5.7) becomes

$$\begin{aligned}
&\int_{|z|=1} \frac{1}{1-a\bar{z}} I\bar{h}|dz| + \int_{|z|=1} \sum_{k=0}^{m} \bar{z}^{k+1} \left(\sum_{j=0}^{k} \frac{k!}{j!} a^{k-j} \frac{\overline{(z^k h)^{(j)}}}{(1-a\bar{z})^{k-j+1}} \right) d\mu_k(z) \\
&\quad = \int_{|z|=1} \frac{1}{1-a\bar{z}} I\bar{h}|dz| \\
&\qquad + \int_{|z|=1} \sum_{k=0}^{m} \bar{z}^{k+1} \left(\sum_{j=0}^{k} \frac{k!}{j!} \frac{a^{k-j} z^{k-j+1} \overline{(z^k h)^{(j)}}}{(z-a)^{k-j+1}} \right) d\mu_k(z) \\
&\quad = \int_{|z|=1} \frac{1}{1-a\bar{z}} I\bar{h}|dz| + \int_{|z|=1} \sum_{k=0}^{m} \left(\sum_{j=0}^{k} \frac{k!}{j!} \frac{a^{k-j} \bar{z}^j \overline{(z^k h)^{(j)}}}{(z-a)^{k-j+1}} \right) d\mu_k(z) \\
&\quad = \int_{|z|=1} \frac{1}{1-a\bar{z}} I\bar{h}|dz| + \sum_{r=0}^{m} a^r \int_{|z|=1} \sum_{\substack{k-j=r \\ 0 \leq j \leq k \leq m}} \frac{k!}{j!} \frac{\bar{z}^j \overline{(z^k h)^{(j)}}}{(z-a)^{r+1}} \, d\mu_k(z),
\end{aligned}$$

which we rewrite as

$$\int \frac{z}{z-a} I\bar{h}|dz| + \sum_{r=0}^{m} a^r \hat{\nu}_r^{(r)}(a), \tag{5.8}$$

where ν_r are the finite singular measures (note that the measures μ_k, $0 \leq k \leq m$, are singular)

$$d\nu_r = \sum_{\substack{k-j=r \\ 0 \leq j \leq k \leq m}} \frac{k!}{j!(k-j)!} \bar{z}^j \overline{(z^k h)^{(j)}} \, d\mu_k. \tag{5.9}$$

By Lemma 5.3, this implies that $g \in IH^1$ and $d\nu_r \equiv 0$, $r \leq m$. But from (5.9) we have

$$d\nu_m = \overline{z^m h} \, d\mu_m \equiv 0$$

and hence $h = 0$ on E_m. Analogously, from $d\nu_{m-1} = 0$ we obtain

$$d\nu_{m-1} = \overline{z^m h} \, d\mu_{m-1} + m\bar{z}\overline{(z^m h)'} \, d\mu_m \equiv 0.$$

But since $h = 0$ on E_m, we conclude that $h = 0$ on E_{m-1} and $h' = 0$ on E_{m-1}. Clearly this can be continued to show that $E_k(h) \supset E_k$ for all $0 \leq k \leq m$ (recall the definition of $E_k(h)$ from (3.10)). Thus $g \in \mathcal{K}(I, E_0, \ldots, E_m)$.

Of course, if $g \in \mathcal{K}(I, E_0, \ldots, E_m)$ then (5.7) is satisfied and so $\langle L^j f_0, g\rangle = 0$ for all $j \geq 0$; thus, by an application of the Hahn-Banach theorem, we are done. □

5.3. The Adjoint of the Weighted Dirichlet Shift

In this section we will apply our results to examine the invariant subspaces for the adjoint of the weighted Dirichlet shift. For $-\infty < \beta < +\infty$, let D_β be the space of $f = \sum a_k z^k \in \mathrm{Hol}(\mathbb{D})$ with norm

$$\|f\|_\beta^2 = \sum_{k=0}^{\infty} (1+k)^\beta |a_k|^2,$$

where $f = \sum a_k z^k$. D_β is called the *weighted Dirichlet space.* We refer the reader to a paper of Brown and Shields [5] for further information, but for now we note that

$$D_\beta = \begin{cases} A^2_{-1-\beta}, & \beta < 0 \\ X_{-1+\beta,2}, & \beta > 0 \end{cases}$$
$$= \{g \in \mathrm{Hol}(\mathbb{D}) : g^{(n_{-1+\beta}+1)}(1-|z|^2)^{n_{-1+\beta}+1-\beta} \in L^2(1-|z|^2)^{-1+\beta}\},$$

with equivalent norms.

D_β is a Hilbert space with the inner product

$$(f,g)_{\mathfrak{D}_\beta} = \sum_{k=0}^{\infty} (1+k)^\beta a_k \overline{b_k},$$

and the operator $M_z : D_\beta \to D_\beta$, $M_z f = zf$ ("multiplication by z") is continuous. In this section we wish to obtain information about the invariant subspaces for M_z^*, the adjoint of M_z. This problem has been examined before in the special cases $\beta = 0$ and $\beta = 1$ [8; 19].

For $-\infty < \beta < +\infty$ and $f \in D_\beta$, let

$$(B_\beta f)(z) = \left(f, \frac{1}{1-\bar{z}w}\right) = \sum a_k (1+k)^\beta z^k, \quad f = \sum a_k z^k.$$

$B_\beta f \in \mathrm{Hol}(\mathbb{D})$, and a calculation similar to (2.4) shows that $B_\beta : D_\beta \to D_{-\beta}$ is unitary with

$$LB_\beta = B_\beta M_z^*,$$

where M_z^* is the adjoint of M_z on D_β and L is the backward shift (i.e., $Lf = z^{-1}(f - f(0))$) on $D_{-\beta}$. Also note that

$$\langle B_\beta f, g\rangle = (f,g)_{\mathfrak{D}_\beta} \quad \forall f, g \in \mathfrak{D}_\beta.$$

Applying Theorem 1.3 yields the following.

THEOREM 5.5. *Let $\beta > 0$ and $\mathfrak{M} \in \mathrm{Lat}(M_z^*, \mathfrak{D}_\beta)$, $\mathfrak{M} \neq \mathfrak{D}_\beta$, such that $\mathfrak{M}^\perp = [S]_{(M_z, \mathfrak{D}_\beta)}$* (note: $\perp$ is with respect to the inner product in the Dirichlet space), *where S is a set of functions $g \in D_\beta$ with*

$$g^{(n_{-1+\beta}+1)}(1-|z|^2)^{n_{-1+\beta}+1-\beta} \log \frac{1}{1-|z|} \in L^2((1-|z|^2)^{-1+\beta}).$$

Then $\mathfrak{M}$ consists exactly of the functions $f \in D_\beta$, where

(1) $gB_\beta f \in H^1(\mathbb{D})$ *for all* $g \in S$, *and*
(2) $(B_\beta f)/I$ *has a pseudocontinuation to* $\mathfrak{N}^+(\mathbb{D}_e)$ *that vanishes at infinity (where* I *is the greatest common inner divisor of* S*).*

Moreover, $B_\beta f$ *has an analytic continuation to* $\mathbb{C}_\infty \setminus \underline{Z}(S)^*$.

Condition (5.1) with $\alpha = -1+\beta$ can be written as

$$-1+\beta > 2m, \tag{5.10}$$

so applying Theorem 5.1 yields the following corollary.

COROLLARY 5.6. *If* β *satisfies* (5.10) *and* $\mathfrak{M} \in \operatorname{Lat}(M_z, \mathfrak{D}_\beta)$, $\mathfrak{M} \neq \mathfrak{D}_\beta$, *then*

$$\mathfrak{M} = [g]^{\perp}_{(M_z, \mathfrak{D}_\beta)}$$

for some $g \in A^\infty$. *Moreover,* $f \in D_\beta$ *belongs to* $\mathfrak{M}$ *if and only if*

(1) $gB_\beta f \in H^\infty(\mathbb{D})$, *and*
(2) $B_\beta f$ *has an analytic continuation to* $\mathbb{C}_\infty \setminus \underline{Z}(g)^*$ *such that* $B_\beta f / I_g \in \mathfrak{N}^+(\mathbb{D}_e)$ *and vanishes at infinity.*

Furthermore, if f_0 *is defined by* (5.6) *then*

$$\mathfrak{M} = [B_{-\beta} f_0]_{(M_z^*, \mathfrak{D}_\beta)}.$$

REMARK. Corollary 5.6 says that for β satisfying (5.10), every M_z^*-invariant subspace of D_β is cyclic (i.e., generated by a single vector). Recall from Section 2 that this was discovered by Richter for $\beta = 1$.

5.4. Bergman Spaces on an Annulus

In this section, we will apply Theorem 1.3 to invariant subspace problems for weighted Bergman spaces on an annulus and the exterior disk.

Let $w_{\alpha,p}$ be a positive, continuous function on $(1,\infty)$ with

$$w_{\alpha,p}(\rho) \sim \begin{cases} (\rho-1)^\alpha & \text{if } \rho \sim 1, \\ \rho^{-4} & \text{if } \rho \sim +\infty. \end{cases}$$

For $1 \le p < +\infty$ and $\alpha > -1$, let

$$L_a^{p,\alpha}(\mathbb{D}_e) = \{f \in \operatorname{Hol}(\mathbb{D}_e) : f \in L^p(\mathbb{D}_e, w_{\alpha,p}(|z|)\,dA)\}$$

be the weighted Bergman space on the exterior disk. A straightforward calculation reveals that the operator

$$U: A_\alpha^p \to L_a^{p,\alpha}(\mathbb{D}_e), \quad (Uf)(z) = f(1/z)$$

is continuous and invertible with $L_\infty U = UL$, where

$$L_\infty : L_a^{p,\alpha}(\mathbb{D}_e) \to L_a^{p,\alpha}(\mathbb{D}_e), \quad L_\infty g = z(g(z) - g(\infty)).$$

(Note that since $w_{\alpha,p}$ is a finite measure, L_∞ is continuous.) Applying Theorem 5.1, we have the following.

THEOREM 5.7. *Fix* $\alpha > -1$ *and* $1 \le p < +\infty$ *satisfying* (5.1). *If*

$$\mathfrak{N} \in \operatorname{Lat}(L_\infty, L_a^{\alpha,p}(\mathbb{D}_e)), \quad \mathfrak{N} \neq L_a^{\alpha,p}(\mathbb{D}_e),$$

then $\mathfrak{N} = U^{\perp}[g]_{(M_z, X_{\alpha,p})}$ *for some* $g \in A^{\infty}$. *Moreover,* $f \in L_a^{p,\alpha}(\mathbb{D}_e)$ *belongs to* $\mathfrak{N}$ *if and only if*

(1) $fUg \in H^{\infty}(\mathbb{D}_e)$ *and*

(2) f *has an analytic continuation to* $\mathbb{C}_{\infty}\setminus\{\bar{z}: z \in \underline{Z}(g)\{$ *such that* $\overline{I_g}(\bar{z})f \in z\mathfrak{N}^{+}(\mathbb{D})$.

Furthermore, if f_0 *is defined as in* (5.6) *then*

$$\mathfrak{N} = [Uf_0]_{(L_{\infty}, L_a^{\alpha,p})}.$$

We can apply Theorem 5.7 to examine the invariant subspaces under the forward shift $M_z f = zf$ of the weighted Bergman space on the annulus. Let $G = \{z: 1 < |z| < R\}$ be an annulus, and let

$$L_a^{\alpha,p}(G) = \{f \in \mathrm{Hol}(G): f \in L^p(G, w_{\alpha,p}(|z|)\,dA)\}$$

be the weighted Bergman space on the annulus. It is well known that $\mathrm{Lat}(M_z, L_a^{\alpha,p}(G))$ is very complicated, and a complete characterization is unknown. We can, however, characterize the subspaces $\mathcal{K} \in \mathrm{Lat}(M_z, L_a^{\alpha,p}(G))$ with the additional condition

$$1 \in \mathcal{K},$$

as was done by Royden [20] for the Hardy space. For further information on the Hardy space of an annulus, we refer the reader to [2; 20; 21].

For a function $f \in L_a^{\alpha,p}(G)$ we write the Laurent series in the following way:

$$f = \sum_{k=1}^{\infty} a_k z^k + \sum_{k=0}^{\infty} \frac{b_k}{z^k};$$

let f_- be the second sum above. For a subspace $1 \in \mathcal{K} \subset L_a^{\alpha,p}(G)$ with $z\mathcal{K} \subset \mathcal{K}$, we have

$$\mathcal{K} = \mathrm{span}\{z^k: k \geq 1\} + \mathcal{K}_-,$$

where $\mathcal{K}_- = \{f_-: f \in \mathcal{K}\}$. Now, using the fact that $1 \in \mathcal{K}_-$, we have

$$L_{\infty}\mathcal{K}_- \subset \mathcal{K}_-.$$

Apply Theorem 5.7 with $\mathfrak{N} = \mathcal{K}_-$ to obtain the following.

COROLLARY 5.8. *Fix* $\alpha > -1$ *and* $1 \leq p < +\infty$ *satisfying* (5.1). *If* $\mathcal{K} \in \mathrm{Lat}(M_z, L_a^{\alpha,p}(G))$, $\mathcal{K} \neq L_a^{\alpha,p}(G)$, *and* $1 \in \mathcal{K}$, *then*

$$\mathcal{K}_- = U^{\perp}[g]_{(M_z, X_{\alpha,p})}$$

for some $g \in A^{\infty}$. *Moreover,* $f \in L_a^{p,\alpha}(G)$ *belongs to* $\mathcal{K}$ *if and only if*

(1) $f_- Ug \in H^{\infty}(\mathbb{D}_e)$ *and*

(2) f_- *has an analytic continuation to* $\mathbb{C}_{\infty}\setminus\{\bar{z}: z \in \underline{Z}(g)\}$ *such that* $\overline{I_g}(\bar{z})f_- \in z\mathfrak{N}^{+}(\mathbb{D})$.

As a corollary to this result, we get that $\mathcal{K}$ is generated by the two vectors 1 and Uf_0.

COROLLARY 5.9. *Fix $\alpha > -1$ and $1 \le p < +\infty$ satisfying* (5.1). *If* $\mathcal{K} \in \mathrm{Lat}(M_z, L_a^{\alpha,p}(G))$, $\mathcal{K} \ne L_a^{\alpha,p}(G)$, *and* $1 \in \mathcal{K}$, *then*

$$\mathcal{K} = [1, Uf_0]_{(M_z, L_a^{\alpha,p})}.$$

5.5. *The Backward Shift on* $A^{-\infty}$

Let $t > 0$ and define

$$A^{-t} = \{f \in \mathrm{Hol}(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1-|z|)^t |f(z)| < +\infty\},$$
$$A^{-\infty} = \bigcup_{t>0} A^{-t}.$$

$A^{-\infty}$ is a topological algebra when endowed with the inductive limit topology [15] and the backward shift operator L is continuous.

REMARK. A set $U \subset A^{-\infty}$ is *open* if, given $f \in U$, there exist sequences $\{t_k\}$, $t_k \uparrow +\infty$, and $\{\epsilon_k\}$, $\epsilon_k > 0$, with

$$\bigcup_k \{g \in A^{-t_k} : \|f-g\|_{A^{-t_k}} < \epsilon_k\} \subset U.$$

In this section, we employ duality again to characterize the L-invariant subspaces of $A^{-\infty}$ and show, unlike the weighted Bergman spaces, they are not all cyclic.

The dual of $A^{-\infty}$ can be identified with A^{∞} with the usual pairing (1.1). (For a description of the topology of A^{∞} see [26, p. 1267].) Thus, as before, if $\mathfrak{M}$ is an L-invariant subspace of $A^{-\infty}$ then $\mathfrak{M} = {}^{\perp}\mathcal{K}$; here $\mathcal{K}$ is an ideal of A^{∞}, which by a result of Taylor and Williams [26] is of the form

$$\mathcal{K} = \mathcal{K}(I, E_0, E_1, E_2, \dots) = \{h \in A^{\infty} : E_k(h) \supset E_k \ \forall 0 \le k < +\infty, I_h/I \in H^{\infty}\}$$

(same as before except there are now an infinite number of E_js). Again by [13], this ideal is generated by a single function $g \in A^{\infty}$; that is,

$$I_g = I, \qquad E_k(g) = E_k \quad \forall 0 \le k < +\infty. \tag{5.11}$$

Notice that if $f \in A^{-\infty}$ then $f \in A^{-t}$ for some $t > 0$, which means that $f \in A_t^1$. Also, if $g \in A^{\infty}$ as in (5.11), then g certainly belongs to $X_{\alpha,1}$ and, moreover, easily satisfies (4.14). If one follows the proof of Theorem 1.3 (using the Hahn–Banach theorem for topological vector spaces) one obtains the following.

THEOREM 5.10. *Let* $\mathfrak{M} \in \mathrm{Lat}(L, A^{-\infty})$, $\mathfrak{M} \ne A^{-\infty}$. *Then*

$$\mathfrak{M} = {}^{\perp}\mathcal{K}(I, E_0, E_1, E_2, \dots) = {}^{\perp}[g]_{(M_z, A^{\infty})}$$

for some $g \in A^{\infty}$. *Moreover,* $f \in A^{-\infty}$ *belongs to* $\mathfrak{M}$ *if and only if*

(1) $fg \in H^{\infty}(\mathbb{D})$ *and*
(2) f *has an analytic continuation to* $\mathbb{C}_{\infty} \setminus Z(g)^*$ *such that* f/I_g *belongs to* $\mathfrak{N}^+(\mathbb{D}_e)$ *and vanishes at infinity.*

Moreover, if $\mathfrak{M} = {}^{\perp}\mathcal{K}(I, E_0, \ldots, E_k)$ *and* f_0 *is defined as in* (5.6), *then*

$$\mathfrak{M} = [f_0]_{(L, A^{-\infty})}.$$

However, unlike the weighted Bergman spaces, not all L-invariant subspaces of $A^{-\infty}$ are cyclic.

COROLLARY 5.11. *Let* $\mathcal{K}(I, E_0, \ldots)$ *be a nontrivial ideal of* A^{∞}. *Then* ${}^{\perp}\mathcal{K}(I, E_0, \ldots)$ *is cyclic if and only if* $\bigcap_j E_j = \emptyset$.

Proof. By the previous theorem, it suffices to show that if $\bigcap_j E_j \neq \emptyset$ then ${}^{\perp}\mathcal{K}(I, E_0, \ldots)$ is not cyclic. So suppose that

$$[f]_{(L, A^{-\infty})} = {}^{\perp}\mathcal{K}(I, E_0, \ldots).$$

Since $f \in A^{-t}$ for some $t > 0$, it follows that $f \in A_t^1$. Since

$$[f]_{(L, A^{-\infty})} \neq A^{-\infty},$$

there is a nonzero $h \in A^{\infty}$ with

$$\langle f, z^j h \rangle = 0 \quad \forall j \geq 0.$$

But h also belongs to $X_{t,1}$, so f generates a nontrivial L-invariant subspace of A_t^1 that, by our earlier discussion, is of the form

$$\{h \in A_t^1 : \langle h, z^j g \rangle = 0 \ \forall j \geq 0\}, \tag{5.12}$$

where $g \in A^{\infty}$ and more importantly (by the result of Khrushchev [13]) $g \notin \mathcal{K}(I, E_0, \ldots)$ (i.e., all the derivatives of g do not vanish on $\bigcap_j E_j$). But (5.12) implies

$$g \in [f]_{(L, A^{-\infty})}^{\perp} = \mathcal{K}(I, E_0, \ldots),$$

a contradiction. □

6. Final Remarks

6.1. H^p, $0 < p < 1$

Knowing that, for $0 < p < 1$, the dual of H^p is $X_{1/p-2,1}$ (with the pairing (1.1)) [10], one might be tempted to use our techniques to obtain a characterization of the backward shift-invariant subspaces of H^p. One should avoid this temptation because our techniques involve heavy use of duality and in particular the Hahn–Banach theorem. For $0 < p < 1$, H^p is not a Banach space and the Hahn–Banach theorem may fail [10]. The L-invariant subspaces of H^p are more complicated and were handled (using much different techniques) in [1].

6.2. Cyclic Vectors

By [8], a vector $f \in H^2$ is L-cyclic for H^2 (i.e., $[f]_{(L, H^2)} = H^2$) if and only if f does not have a pseudocontinuation to $\mathfrak{N}(\mathbb{D}_e)$. For the weighted Bergman spaces, lack of a pseudocontinuation is still a sufficient condition for L-cyclicity but it is not a necessary one. For example, an inner function I has a pseudocontinuation. One proves, using the theorem of F. and M. Riesz, that I is not L-cyclic for A_α^p if and only if I is an inner divisor for some nonzero $X_{\alpha,p}$ function. Note that when α and p satisfy (5.1), this is equivalent to

$$\int \log \operatorname{dist}(\zeta, \operatorname{spec}(I))|d\zeta| > -\infty. \tag{6.1}$$

Thus, some inner functions are L-cyclic for the Bergman spaces. Note also that in the Hardy space, L-cyclicity is independent of the choice of $1 < p < +\infty$. For the Bergman space, however, L-cyclicity does depend on the parameters α and p. For example, one can create an inner function that divides a nonzero $X_{0,2}$ function and with $\operatorname{spec}(I) = \mathbb{T}$ [6]. By (6.1), this function would be L-cyclic for A_0^p, $1 \le p < 2$, but not for A_0^2.

6.3. A^{-t}

For each $t > 0$, one can choose an $f \in A^{-t} \setminus \mathfrak{N}(\mathbb{D})$ [9, p. 86]. Since A^{-t} is not separable, $[f] \neq A^{-t}$ and so the analog of Theorem 1.2 and Theorem 1.3 cannot possibly hold here. However, if one endows A^{-t} with the weak-star topology, then A^{-t} is separable and one can prove that every (nontrivial) weak-star closed L-invariant subspace $\mathfrak{M}$ belongs to $\mathfrak{N}(\mathbb{D})$ and that every $f \in \mathfrak{M}$ has a pseudocontinuation to $\mathfrak{N}(\mathbb{D}_e)$ [3]. A complete characterization of these $\mathfrak{M}$ remains open.

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