Layer Potential Operators and a Space of Boundary Data for Electromagnetism in Nonsmooth Domains

RODOLFO H. TORRES* & GRANT V. WELLAND†

1. Introduction

Several developments in the study of singular integrals and related harmonic analysis techniques have made it possible to apply the classical method of layer potentials to numerous boundary value problems in domains with minimal regularity assumption. Among others, important examples in the case of C^1 domains are [7] and [18], and in the case of Lipschitz domains [17] and [6]. If for a given boundary value problem the classical method of layer potentials in smooth domains provides solutions for continuous boundary data, then one has come to expect that in C^1 domains, the method should also give solutions for data in L^p for the full range $1 < p < \infty$; see for example [7] and [18]. Remarkably, this should still be achieved, as in the classical case, through Fredholm theory. On the other hand, in the case of arbitrary Lipschitz domains, Fredholm theory is not applicable in general. In some cases, however, the method of layer potentials combined with energy estimates like the ones in [9] still provides solutions in the Lipschitz situation, at least for a more restrictive range of p; see for example [17] and [6].

Along these lines, we have studied in [16] several boundary value problems for the scalar Helmholtz equation $(\Delta + k^2)u = 0$ in Lipschitz domains (see also [1]). This equation arises in the study of the scattering of time-harmonic acoustic waves. The purpose of this paper is to study a boundary value problem in the vector-valued case, which naturally occurs in the scattering of electromagnetic waves. We will consider the so-called Maxwell boundary value problem for the case of a perfect conducting surface, extending known results for smooth domains to domains with less regular boundaries. In the case of smooth domains, the classical theory is described in [15] and [4]. More recent developments for the Helmholtz equation in smooth domains, as well as numerous references to related works, can be found in [10] for the scalar case and in [12] for the vector-valued case.

Received August 9, 1995.

*Supported in part by a Rackham Fellowship and NSF grant DMS 9303363.

†Supported under Air Force Grant number 90-0307.

Michigan Math. J. 43 (1996).

The particular form of the boundary conditions, as well as the crucial choice of functional space for the boundary data, is what makes the boundary value problem in this paper technically more complicated. The importance of the problem in applications makes it worthy of consideration.

The treatment of the various singular layer potential operators associated with the vector Helmholtz equation on nonsmooth boundaries rests on the results in [2] and [3] regarding the Cauchy integral operator, from which boundedness results are obtained in a standard way. Another major ingredient in our approach is our previous work on the scalar Helmholtz equation on Lipschitz domains mentioned above.

In the case of time-harmonic waves in $\mathbb{R}^3$ with wave number k ($k \in \mathbb{C}$, $\operatorname{Im} k \geq 0$), and after a suitable normalization, Maxwell equations take the form

$$\operatorname{curl} E - ikH = 0, \qquad \operatorname{curl} H + ikE = 0,$$

where E and H are the electric and magnetic fields representing the wave. It is a well-known fact that solutions of these equations are divergence-free solutions of the vector Helmholtz equation.

Let E' and H' be (given) incoming vector fields, and let E and H be solutions of Maxwell equations corresponding to the scattered fields by a perfectly conducting obstacle. If we represent the object by a bounded domain D, then the tangential component of the total field $E+E'$ should vanish on ∂D. Consequently, if N denotes the outward unit normal to ∂D, then $N \times E = -N \times E'$ on the boundary. This leads to the following boundary value problem:

$$\begin{cases} \operatorname{curl} E - ikH = 0 & \text{on } \mathbb{R}^3 \setminus \bar{D}, \\ \operatorname{curl} H + ikE = 0 & \text{on } \mathbb{R}^3 \setminus \bar{D}, \\ N \times E = A & \text{on } \partial D, \end{cases}$$

where $A = -N \times E'$ is a given tangential vector field. A similar problem can be stated for the interior of D. For C^1 or Lipschitz domains and for noncontinuous boundary data, we shall interpret the boundary values pointwise a.e. in nontangential fashion. Notice that if a divergence-free solution E of the vector Helmholtz equation with wave number k satisfied $N \times E = A$, then defining $H = -(i/k) \operatorname{curl} E$, we have that E and H are solutions of the above problem. If we now let Div denote surface divergence on the boundary of D then we should have, at least for smooth domains,

$$\operatorname{Div}(N \times E) = -\langle N, \operatorname{curl} E \rangle. \tag{1}$$

A consequence of (1) is that the existence of boundary values for the normal component of H implies some extra regularity on the tangential component of E, and vice versa. For this reason some regularity in the boundary data is usually imposed. For example, in the case of smooth domains, the boundary data is often assumed to be in the space of Hölder continuous tangential vector fields with Hölder continuous surface divergence. This regularity of the boundary data yields solutions that are continuous up to the boundary (see [11]). A Sobolev space version of such space of boundary data was studied for C^2 domains in [8], where the boundary values of the solutions were interpreted in an appropriate L^2 sense.

In the case of C^1 and Lipschitz domains, we shall consider the spaces $L_T^{p,\,\mathrm{Div}}(\partial D)$ consisting of all tangential fields in $L^p(\partial D)$ having their surface divergences (in a distributional sense) in $L^p(\partial D)$, $1 < p < \infty$. In addition, we shall impose some boundedness conditions for the nontangential maximal functions E^* and H^*. All this will allow us to make sense of (1) in nonregular domains and, in turn, to guarantee the existence of pointwise boundary values for H. Moreover, the choice of space is optimal in the sense that, for a given tangential vector field A in $L^p(\partial D)$ and a solution E with tangential boundary values A, the companion vector field H has pointwise (nontangentially) boundary values if and only if $\mathrm{Div}\, A$ is also in L^p.

In the above setting, we prove existence and uniqueness of solutions and optimal estimates in $L_T^{p,\,\mathrm{Div}}(\partial D)$ for the full range $1 < p < \infty$, in the case where $\mathrm{Im}\, k > 0$ and when the domain is assumed to be of class C^1 (as usual, some radiating condition at infinity is imposed in the exterior problem). We succeed in extending to the C^1 case the study of the boundary layer operators related to the vector Helmholtz equation, and show that Fredholm theory is still applicable. The vector fields E and H, solutions of the problem, are then expressed as the curl and curl curl of a single layer potential operator. Actually, we also obtain a uniqueness result for an arbitrary Lipschitz domain. The lack of compactness of certain boundary operators, however, prevents us from extending the arguments for the existence result to the Lipschitz situation. As mentioned earlier, a priori energy estimates derived from Rellich-type identities are in many cases an alternative tool to prove invertibility results in the Lipschitz case. Relying in part on the results we present here and on the same choice of boundary data, such an approach has been recently considered in [13] for boundary data in $L_T^{p,\,\mathrm{Div}}(\partial D)$, with p in a small neighborhood of $p = 2$. The case of other values of p for general Lipschitz domains remains open.

Regularity properties for solutions of Maxwell equations in nonsmooth domains have been considered in [5]. We want to point out that the results of this paper, together with the boundedness of appropriate nontangential maximal functions, can be used in a standard way to show that E and H belong to the Sobolev space $H^{1/2}(D)$. This has been described in a joint work in [14]. We would like to thank M. Mitrea for our conversations on a preliminary version of this paper and related problems.

2. Notation

We recall some basic definitions and facts about Lipschitz domains, and we introduce some spaces of functions defined on the boundary of such domains.

An open, bounded, simply connected domain D of $\mathbb{R}^3$, with connected boundary ∂D, is called *Lipschitz* or C^1 if ∂D is given locally by the graph of a Lipschitz or C^1 function, respectively. Let $d\sigma$ denote surface measure on ∂D. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product in $\mathbb{R}^3$, and let $\times$ denote exterior product.

A vector-valued function A defined $d\sigma$-a.e. on ∂D is said to be *tangential* if $\langle A, N\rangle = 0$ a.e. on ∂D, where N is the unit exterior normal to ∂D. The spaces of all tangential L^p vectors is denoted by $L^p_T(\partial D)$. The L^p–$L^{p'}$ pairing, $1/p+1/p'=1$, used in this paper is given by

$$(A, B) = \int_{\partial D} \langle A(Q), B(Q)\rangle \, d\sigma(Q). \tag{2}$$

The space of L^p scalar functions with tangential gradient in $L^p(\partial D)$ is denoted by $L^{p,1}$. If, for example, ψ is a Lipschitz continuous function on a neighborhood of ∂D, then the tangential gradient vector field of ψ can be computed almost everywhere as $\nabla_T \psi = \nabla\psi - (\partial\psi/\partial N)N$. A vector field A in $L^p_T(\partial D)$ is said to have a *surface divergence* if there exists a (unique) scalar-valued function b in $L^{p'}(\partial D)$ such that

$$\int_{\partial D} \langle \nabla_T \psi, A\rangle \, d\sigma = -\int_{\partial D} \psi b \, d\sigma \tag{3}$$

for all functions ψ that are Lipschitz in a neighborhood of ∂D. The surface divergence b in (3) is denoted by $\operatorname{Div} A$, and the space of all L^p vector fields with surface divergence is denoted $L^{p,\operatorname{Div}}_T(\partial D)$. The norm of the space $L^{p,\operatorname{Div}}_T(\partial D)$ is given by

$$\|A\|_{L^{p,\operatorname{Div}}_T(\partial D)} = \|A\|_{L^p(\partial D)} + \|\operatorname{Div} A\|_{L^p(\partial D)}.$$

Notice that, by density, the formula in (3) holds for all ψ in $L^{p',1}(\partial D)$. For domains with smooth boundaries, the definition of surface divergence can be given in several other equivalent ways—for example, pointwise (see [4]).

We will always assume that at every point Q in the boundary, an open, right circular, doubly truncated cone $\Gamma(Q)$, with vertex at Q and two convex components (one in D and the other in $\mathbb{R}^3\backslash\bar{D}$), has been chosen so that the resulting family of cones is a regular family as described in [17]. The interior component of such cones will be denoted by $\Gamma_+(Q)$ and the exterior by $\Gamma_-(Q)$. As usual, for a function u defined in D or $\mathbb{R}^3\backslash\bar{D}$, the nontangential maximal function u^* is given by

$$u^*(P) = \sup_{X\in\Gamma_+(P)} |u(X)|,$$

or

$$u^*(P) = \sup_{X\in\Gamma_-(P)} |u(X)|,$$

depending on where the function u is defined. Boundary values of functions defined in D or $\mathbb{R}^3\backslash\bar{D}$ are assumed to be taken as nontangential limits almost everywhere with respect to surface measure on the boundary of D. That is, $u|_{\partial D}$ is defined by

$$u(P) = \lim_{\substack{X \to P \\ X \in \Gamma_+(P)}} u(X),$$

or

$$u(P) = \lim_{\substack{X \to P \\ X \in \Gamma_-(P)}} u(X).$$

Similar definitions apply for the normal and partial derivatives of a function, and for each component of a vector-valued function.

3. The Scalar Helmholtz Equation

Let k be a complex number with $\operatorname{Im} k \geq 0$, and consider the radial fundamental solution for the Helmholtz operator $\Delta + k^2$ in $\mathbb{R}^3$,

$$\Phi(X) = -\frac{e^{ik|X|}}{4\pi|X|}.$$

For a function f defined on the boundary of D, the single and the double acoustic layer potentials are given by

$$\mathcal{S}f(X) = \int_{\partial D} \Phi(X-Q) f(Q)\, d\sigma(Q), \quad X \in \mathbb{R}^3,$$

and

$$\mathcal{D}f(X) = \int_{\partial D} \partial_{N_Q} \Phi(X-Q) f(Q)\, d\sigma(Q), \quad X \in \mathbb{R}^3 \backslash \partial D,$$

respectively.

The properties of the single and double layer acoustic potentials and the boundedness results for the corresponding trace operators are essentially the same as those for the case of Laplace's equation $k=0$, and can be obtained from the results in [3] on the Cauchy integral on Lipschitz curves. See [1] and [16].

For any f in $L^p(\partial D)$ we have that $\mathcal{S}f$ and $\mathcal{D}f$ solve the Helmholtz equation in $\mathbb{R}^3 \backslash \partial D$, and also satisfy

$$\|(\mathcal{S}f)^*\|_{L^p(\partial D)} + \|(\nabla \mathcal{S}f)^*\|_{L^p(\partial D)} + \|(\mathcal{D}f)^*\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D)},$$

for any $1 < p < \infty$.

For any $f \in L^p(\partial D)$, the boundary traces of $\mathcal{S}f$ and $\mathcal{D}f$ are given by

$$\lim_{\substack{X \to P \\ X \in \Gamma_+(P)}} \mathcal{S}f(X) = \lim_{\substack{X \to P \\ X \in \Gamma_-(P)}} \mathcal{S}f(X) = Sf(P), \quad P \in \partial D,$$

and

$$\lim_{\substack{X \to P \\ X \in \Gamma_\pm(P)}} \mathcal{D}f(X) = (\pm\tfrac{1}{2}I + K)f(P), \quad P \in \partial D,$$

where

$$Sf(P) = -\frac{1}{4\pi}\int_{\partial D}\frac{e^{ik|Q-P|}}{|Q-P|}f(Q)\,d\sigma(Q), \quad P \in \partial D,$$

and

$$Kf(P) = \text{p.v.}\ \frac{1}{4\pi}\int_{\partial D}\frac{\langle N(Q), Q-P\rangle}{|Q-P|^3}e^{ik|Q-P|}(1-ik|Q-P|)f(Q)\,d\sigma(Q).$$

In addition, for almost every $P \in \partial D$,

$$\lim_{\substack{X\to P\\ X\in\Gamma_\pm(P)}}\frac{\partial \mathcal{S}f}{\partial N}(X) = \lim_{\substack{X\to P\\ X\in\Gamma_\pm(P)}}\langle N(P), \nabla \mathcal{S}f(X)\rangle = (\mp\tfrac{1}{2}I + K^*)f(P),$$

where K^* is the transpose operator of K.

The main properties of the trace operators are summarized in the following theorem.

THEOREM 3.1 [1; 16]. *Let D be a bounded Lipschitz domain in $\mathbb{R}^3$. Then:*

(i) $S: L^p(\partial D) \to L^p(\partial D)$ *is a compact operator for any* $1 < p < \infty$;
(ii) $S: L^p(\partial D) \to L^{p,1}(\partial D)$ *is a bounded operator for any* $1 < p < \infty$;
(iii) $\pm\frac{1}{2}I + K: L^p(\partial D) \to L^p(\partial D)$ *are bounded operators for any* $1 < p < \infty$; *and*
(iv) $\pm\frac{1}{2}I + K: L^{p,1}(\partial D) \to L^{p,1}(\partial D)$ *are bounded operators for any* $1 < p < \infty$.

In addition, if $\operatorname{Im} k > 0$, *then:*

(i) $S: L^2(\partial D) \to L^{2,1}(\partial D)$ *is invertible;*
(ii) $\pm\frac{1}{2}I + K: L^2(\partial D) \to L^2(\partial D)$ *are invertible; and*
(iii) $\pm\frac{1}{2}I + K: L^{2,1}(\partial D) \to L^{2,1}(\partial D)$ *are invertible.*

Moreover, if ∂D is actually of class C^1, then the above invertibility results hold in the analogous L^p settings with $1 < p < \infty$, and the operator K is compact.

We will need to make use on several occasions of the following Dirichlet and regularity boundary value problem for the scalar Helmholtz equation.

THEOREM 3.2 [1; 16]. *Let D be a bounded Lipschitz domain in $\mathbb{R}^3$ and let* $\operatorname{Im} k > 0$. *Then the Dirichlet problem*

$$\begin{cases} (\Delta + k^2)u = 0 \ \ \text{on } D, \\ \|u^*\|_{L^p(\partial D)} < \infty, \\ u\,|_{\partial D} = f \in L^p(\partial D) \end{cases}$$

has a unique solution for $p = 2$. The solution can be written as

$$u = \mathcal{D}((\tfrac{1}{2}I + K)^{-1}f).$$

If ∂D is actually of class C^1, then the result holds for any $1 < p < \infty$.

THEOREM 3.3 [1; 16]. *Under the hypothesis of the previous theorem, the regularity problem*

$$\begin{cases} (\Delta+k^2)u=0 \ \ on\ D, \\ \|u^*\|_{L^p(\partial D)}+\|(\nabla u)*\|_{L^p(\partial D)}<\infty, \\ u\,|_{\partial D}=f\in L^{p,1}(\partial D) \end{cases}$$

has a unique solution for $p=2$. If the domain is C^1 then the result holds for the full range $1<p<\infty$. In either case, the solution can be written as $u=\mathfrak{D}((\frac{1}{2}I+K)^{-1}f)$ and also as $u=\mathcal{S}(S^{-1}f)$.

Similar results are valid in the exterior domain $\mathbb{R}^3\backslash\bar{D}$ provided u is also assumed to have an adequate decay at infinity; this is known as the *Sommerfeld radiation condition.*

REMARK. In the case of an arbitrary Lipschitz domain, the L^2 inversion results in Theorem 3.1 are still true in the L^p settings for appropriate ranges of p. The regularity boundary value problem can actually be solved for p in a (small) neighborhood of $p=2$ which only depends on the Lipschitz character of D. See the remarks in [16, p. 1474]. In what follows, for simplicity in the presentation we state several results for Lipschitz domains only for the case $p=2$. Nevertheless, those results that depend on Theorem 3.3 remain true in a neighborhood of $p=2$.

4. Formulas for the Vector Helmholtz Equation

We shall indicate how to obtain the classical Stratton–Chue integral representation formulas for vector fields satisfying the Helmholtz equation in the case of Lipschitz domains. To do so we need a standard approximation procedure. Given a Lipschitz domain D, we fix a family of smooth approximating domains like the one in [17, Thm. 1.12]. That is, we chose domains Ω_j contained in D such that the following conditions hold.

(i) There is a sequence of Lipschitz diffeomorphisms $\Lambda_j\colon \partial D\to\partial\Omega_j$ such that the Lipschitz constants of Λ_j and its inverse are uniformly bounded in j. Furthermore, $\Lambda_j(Q)\in\Gamma_+(Q)$ for all j and all $Q\in\partial D$, and $\sup_{Q\in\partial D}|Q-\Lambda_j(Q)|\le C/j$.

(ii) There are positive functions $\omega_j\colon\partial D\to\mathbb{R}_+$ bounded away from zero and infinity uniformly in j such that: (a) for any measurable set $F\subset\partial D$, $\int_F \omega_j\,d\sigma=\int_{\Lambda_j(F)} d\sigma_j$; (b) $\omega_j\to 1$ a.e. and every $L^p(\partial D)$, $1\le p<\infty$.

(iii) The sequence of normal vectors to Ω_j, $N_j(\Lambda_j(\cdot))$, converges to N a.e. and in every $L^p(\partial D)$, $1\le p<\infty$.

We also fix an approximating sequence of domains from the outside of D with analogous properties. These approximations will be denoted by $\Omega_j\uparrow D$ and $\Omega_j\downarrow D$, respectively.

Many results known for smooth domains can be extended to the Lipschitz case by using the above approximating domains, provided one works on the right space of functions. We illustrate this with the following result.

LEMMA 4.1. *Let E be a smooth vector field in D. If E and* curl E *have non-tangential limits for almost any $P \in \partial D$, and if*

$$\|E^*\|_{L^p(\partial D)} + \|(\operatorname{curl} E)^*\|_{L^p(\partial D)} < \infty$$

for some $1 < p < \infty$, then $N \times E$ has a surface divergence in $L^p(\partial D)$. That is, $N \times E \in L_T^{p,\mathrm{Div}}(\partial D)$ and

$$\mathrm{Div}(N \times E) = -\langle N, \operatorname{curl} E \rangle.$$

Proof. By a density argument it is enough to show that

$$\int_{\partial D} \langle \nabla_T \psi, N \times E \rangle \, d\sigma = \int_{\partial D} \psi \langle N, \operatorname{curl} E \rangle \, d\sigma$$

for all ψ that are C^∞ in a neighborhood of $\bar{D}$. This last equality is easily checked for smooth domains and smooth functions up to the boundary. Hence, using $\Omega_j \uparrow D$,

$$\int_{\partial \Omega_j} \langle \nabla \psi, N_j \times E \rangle \, d\sigma_j = \int_{\partial \Omega_j} \psi \langle N_j, \operatorname{curl} E \rangle \, d\sigma_j$$

and

$$\begin{aligned} \int_{\partial D} \langle \nabla_T \psi, N \times E \rangle \, d\sigma &= \int_{\partial D} \langle \nabla \psi, N \times E \rangle \, d\sigma \\ &= \lim_{j \to \infty} \int_{\partial \Omega_j} \langle \nabla \psi, N_j \times E \rangle \, d\sigma_j \\ &= \lim_{j \to \infty} \int_{\partial \Omega_j} \psi \langle N_j, \operatorname{curl} E \rangle \, d\sigma_j \\ &= \int_{\partial D} \psi \langle N, \operatorname{curl} E \rangle \, d\sigma. \end{aligned}$$

To justify the two limits in the above expression, we first observe that

$$\begin{aligned} &\int_{\partial \Omega_j} \langle \nabla \psi, N_j \times E \rangle \, d\sigma_j \\ &\quad = \int_{\partial D} \langle \nabla \psi(\Lambda_j(Q)), N(Q) \times E(\Lambda_j(Q)) \rangle \omega_j \, d\sigma \\ &\qquad + \int_{\partial D} \langle \nabla \psi(\Lambda_j(Q)), (N_j(\Lambda_j(Q)) - N(Q)) \times E(\Lambda_j(Q)) \rangle \omega_j \, d\sigma. \end{aligned}$$

Since $|\nabla \psi|$ is bounded in a neighborhood of D,

$$|\langle \nabla \psi(\Lambda_j(Q)), N(Q) \times E(\Lambda_j(Q)) \rangle| \omega_j \le C |E^*|. \tag{4}$$

By the properties of the approximating domains,

$$\langle \nabla \psi(\Lambda_j(Q)), N(Q) \times E(\Lambda_j(Q))\rangle \omega_j \to \langle \nabla_T \psi, N \times E\rangle$$

at almost every point in the boundary, so—using (4), the fact that $\|E^*\|_{L^p(\partial D)}$ is finite, and the dominated convergence theorem—we can pass to the limit. By the same arguments,

$$\int_{\partial D} \langle \nabla \psi(\Lambda_j(Q)), (N_j(\Lambda_j(Q)) - N(Q)) \times E(\Lambda_j(Q))\rangle \omega_j \, d\sigma \to 0.$$

The limit involving $\operatorname{curl} E$ can be handled in the same way because we are also assuming that $\|(\operatorname{curl} E)^*\|_{L^p(\partial D)}$ is finite. □

For the rest of the paper, we shall no longer present the proof of similar results in such details. Notice that to carry out the above limiting argument we needed the boundedness of appropriate nontangential maximal functions. For this reason we introduce the following spaces of solutions of the vector Helmholtz equation $\Delta E + k^2 E = 0$ (cf. [16]).

DEFINITION 4.1. Let D be a Lipschitz domain. For $1 < p < \infty$, the space $V_i^p(D)$ is the space of all complex-valued vector fields E in $C^2(D)$ that solve the vector Helmholtz equation and for which E, $\operatorname{curl} E$, and $\operatorname{div} E$ exist a.e. on ∂D, while E^*, $(\operatorname{curl} E)^*$, and $(\operatorname{div} E)^*$ are in $L^p(\partial D)$.

As usual, in order to obtain uniqueness results when dealing with the exterior domain, some radiation condition must be imposed.

DEFINITION 4.2. Let D be a Lipschitz domain. For $1 < p < \infty$, the space $V_i^p(D)$ is the space of all complex-valued vector fields E in $C^2(\mathbb{R}^3 \setminus \bar{D})$ that solve the vector Helmholtz equation and satisfy the so-called Silver–Müller radiation condition at infinity,

$$\operatorname{curl} E(X) \times \frac{X}{|X|} + \operatorname{div} E(X) \frac{X}{|X|} - ikE(X) = o(|X|^{-1})$$

for $|X| \to \infty$, uniformly in all directions in $\mathbb{R}^3$, and for which E, $\operatorname{curl} E$, and $\operatorname{div} E$ exist a.e. on ∂D, and E^*, $(\operatorname{curl} E)^*$, and $(\operatorname{div} E)^*$ are in $L^p(\partial D)$.

We can now extend to the Lipschitz situation the following representation formulas.

LEMMA 4.2. *Consider $1 < p < \infty$, and let E be a vector field in $V_i^p(D)$ or in $V_e^p(D)$. Then, for all $X \in D$ or $X \in \mathbb{R}^3 \setminus \bar{D}$, respectively,*

$$\begin{aligned} \pm E(X) = & \int_{\partial D} \operatorname{curl}_X(\Phi(X-Q) N(Q) \times E(Q)) \, d\sigma \\ & - \int_{\partial D} \nabla_X \Phi(X-Q) \langle N(Q), E(Q)\rangle \, d\sigma \\ & + \int_{\partial D} \Phi(X-Q)(N(Q) \times \operatorname{curl} E(Q) - \operatorname{div} E(Q) N(Q)) \, d\sigma. \end{aligned}$$

LEMMA 4.3. (i) *Let E be a vector field in $V_i^2(D)$. Then*

$$\int_{\partial D}(\langle N(Q)\times\bar{E}(Q),\operatorname{curl}E(Q)\rangle+\operatorname{div}E(Q)\langle N(Q),\bar{E}(Q)\rangle)\,d\sigma(Q)$$
$$=\int_D(|\operatorname{curl}E(X)|^2+|\operatorname{div}E(X)|^2-k^2|E(X)|^2)\,dX.$$

(ii) *Let E be a vector field in $V_e^2(D)$. Then*

$$\lim_{r\to\infty}\left(-\int_{|X|=r}|k|^2|E(X)|^2+|\operatorname{curl}E(X)\times N(X)+\operatorname{div}E(X)N(X)|^2\,ds_r\right.$$
$$\left.-2\operatorname{Im}(k)\int_{D_r}|\operatorname{curl}E(X)|^2+|\operatorname{div}E(X)|^2+|k|^2|E(X)|^2\,dX\right)$$
$$=2\operatorname{Im}\left(k\int_{\partial D}(\langle N(Q),E(Q)\times\operatorname{curl}\bar{E}(Q)\rangle\right.$$
$$\left.+\operatorname{div}\bar{E}(Q)\langle N(Q),E(Q)\rangle)\,d\sigma\right),$$

where ds_r is the surface measure on $B_r(0)$, the ball of radius r, and where $D_r=\mathbb{R}^3\setminus\bar{D}\cap B_r(0)$.

Proof. The above formulas hold on any smooth domain (see e.g. [4, pp. 110–120]). The limiting arguments using approximating domains can be applied because of the conditions on E. □

The following corollary is an easy consequence of these lemmas.

COROLLARY 4.4. *Let D be a Lipschitz domain, and assume that* $\operatorname{Im}k>0$. *If E is a vector field in $V_i^2(D)$ and if* $\operatorname{div}E$ *and $N\times E$ vanish almost everywhere on the boundary, then $E\equiv 0$ in D. The analogous result holds for $\mathbb{R}^3\setminus\bar{D}$ if E is in the space $V_e^2(D)$.*

5. Vector-Valued Layer Potential Operators

We will look at solutions to the vector Helmholtz equation defined via layer potentials. The action of the layer potential operators $\mathcal{S}$ and $\mathcal{D}$ on vector fields in $L^p(\partial D)$ is defined componentwise. We have the following.

LEMMA 5.1. *Let A be a vector field in $L^p(\partial D)$, $1<p<\infty$.*

(i) *For almost every $P\in\partial D$,*

$$\lim_{\substack{X\to P\\ X\in\Gamma_\pm(P)}}\operatorname{div}\mathcal{S}A(X)=\mp\frac{1}{2}\langle N,A\rangle(P)+\text{p.v.}\int_{\partial D}\operatorname{div}_P(\Phi(P-Q)A(Q))\,d\sigma.$$

(ii) *For almost every $P\in\partial D$,*

$$\lim_{\substack{X\to P\\ X\in\Gamma_\pm(P)}}\operatorname{curl}\mathcal{S}A(X)=\mp\frac{1}{2}(N\times A)(P)+\text{p.v.}\int_{\partial D}\operatorname{curl}_P(\Phi(P-Q)A(Q))\,d\sigma.$$

(iii) *All the above integral operators are bounded on every $L^p(\partial D)$ for* $1 < p < \infty$.

(iv) *For* $1 < p < \infty$,

$$\|(\operatorname{div} \mathcal{S}A)^*\|_{L^p(\partial D)} + \|(\operatorname{curl} \mathcal{S}A)^*\|_{L^p(\partial D)} \le C\|A\|_{L^p(\partial D)}.$$

Proof. Parts (iii) and (iv) are consequences of the main result of [3]. From this, (i) and (ii) follow from the limiting argument described above and the fact that they hold for smooth domains. □

Notice that if $A \in L^p_T(\partial D)$ then there is no jump for $\operatorname{div} \mathcal{S}A$ across the boundary of D. Notice also that $E = \mathcal{S}A$ is in $V_i^p(D)$ or $V_e^p(D)$. On the other hand, if $E = \operatorname{curl} \mathcal{S}A$ then E still has nontangential boundary values and bounded nontangential maximal function, although $\operatorname{curl} E$ may not have these properties. In fact, using the identity $\operatorname{curl}\operatorname{curl} = -\Delta + \nabla \operatorname{div}$, we see that $\operatorname{curl} E$ is in $V_i^p(D)$ or $V_e^p(D)$ if and only if $(\nabla \operatorname{div} \mathcal{S}A)^*$ is in $L^p(\partial D)$. The following lemma addresses this issue and justifies the choice of boundary data that we shall make for the Maxwell problem.

LEMMA 5.2. *Let D be a bounded Lipschitz domain, and let $1 < p < \infty$. If a vector field A in $L^p_T(\partial D)$ has a surface divergence in $L^p(\partial D)$, then* $\operatorname{div} \mathcal{S}A = \mathcal{S}(\operatorname{Div} A)$. *In particular,* $\|(\nabla(\operatorname{div} \mathcal{S}A))^*\|_{L^p(\partial D)} < \infty$. *If* $\operatorname{Im} k > 0$ *then the converse implication is always true for $p = 2$ and, if the domain is actually C^1, it is also true for the whole range $1 < p < \infty$.*

Proof. Assume first that $\operatorname{Div} A \in L^p(\partial D)$. Because, for any fixed point X in D, the function $\Phi(X - \cdot)$ is Lipschitz on ∂D, we have

$$\begin{aligned}
\operatorname{div} \mathcal{S}A(X) &= \operatorname{div} \int_{\partial D} \Phi(X-Q)A(Q)\,d\sigma(Q) \\
&= \int_{\partial D} \langle \nabla_X \Phi(X-Q), A(Q)\rangle\,d\sigma(Q) \\
&= -\int_{\partial D} \langle \nabla_Q \Phi(X-Q), A(Q)\rangle\,d\sigma(Q) \\
&= -\int_{\partial D} \langle \nabla_{T_Q} \Phi(X-Q), A(Q)\rangle\,d\sigma(Q) \\
&= \int_{\partial D} \Phi(X-Q)(\operatorname{Div} A(Q))\,d\sigma(Q) \\
&= \mathcal{S}(\operatorname{Div} A)(X).
\end{aligned}$$

The converse implication relies on Lemma 3.3. Assume that $A \in L^2_T(\partial D)$ is such that $\|(\nabla(\operatorname{div} \mathcal{S}A))^*\|_{L^2(\partial D)} < \infty$. Since $\operatorname{div} \mathcal{S}A$ solves the scalar Helmholtz equation in D with datum in $L^{2,1}(\partial D)$, it follows from the uniqueness in the regularity problem that $\operatorname{div} \mathcal{S}A = \mathcal{S}b$ for some $b \in L^2(\partial D)$. Now, for any function ψ Lipschitz in a neighborhood of ∂D,

$$\begin{aligned}\int_{\partial D}\langle\nabla_T S\psi, A\rangle\,d\sigma(P) &= \int_{\partial D}\int_{\partial D}\langle\nabla_P\Phi(P-Q)\psi(Q), A(P)\rangle\,d\sigma(Q)\,d\sigma(P)\\ &= -\int_{\partial D}\psi(Q)(\operatorname{div}\mathcal{S}A(Q))\,d\sigma(Q)\\ &= -\int_{\partial D}\psi Sb\,d\sigma\\ &= -\int_{\partial D}S\psi b\,d\sigma.\end{aligned}$$

Since the space of all functions of the form $S\psi$ with ψ as above is dense in $L^{2,1}(\partial D)$, we conclude that $\operatorname{Div}A = Sb$ is in $L^2(\partial D)$, as desired. The change in order of integration in the above calculation can be justified, once again, by a limiting argument using the approximating domains. Finally, notice that if D is actually of class C^1 then the regularity problem for the scalar Helmholtz equation holds for the full range $1 < p < \infty$, and the above arguments can be repeated in the $L^p(\partial D)$ setting if we assume that $\|(\nabla(\operatorname{div}\mathcal{S}A))^*\|_{L^p(\partial D)} < \infty$. $\square$

We consider now the tangential component of the operator obtained by taking the curl of the single layer potential.

Let A be a vector field in $L^p(\partial D)$. Then, at almost every point P on ∂D, one has

$$\lim_{\substack{X\to P\\ X\in\Gamma_\pm(P)}} N(P)\times\operatorname{curl}\mathcal{S}A(X) = (\pm\tfrac{1}{2}I+M)A(P),$$

where

$$MA(P) = \text{p.v.}\int_{\partial D} N(P)\times\operatorname{curl}_P(\Phi(P-Q)A(Q))\,d\sigma(Q).$$

Clearly, the vector field MA is always tangential and, by the results in [3], M is a well-defined, bounded operator from $L^p_T(\partial D)$ into itself for all $1 < p < \infty$. Furthermore, routine calculations show that its formal transpose, $M^*\colon L^p_T(\partial D)\to L^p_T(\partial D)$, has the form

$$M^*B = N\times M(N\times B),\quad B\in L^p_T(\partial D).$$

THEOREM 5.3. *Let D be a bounded Lipschitz domain in $\mathbb{R}^3$, and let* $\operatorname{Im}k > 0$. *Then, the operators* $\pm\frac{1}{2}I+M\colon L^p_T(\partial D)\to L^p_T(\partial D)$ *are injective for* $2\le p<\infty$ *and have dense ranges for* $1 < p \le 2$.

Proof. We first treat the case $p = 2$. Suppose that $(\frac{1}{2}I+M)A = 0$ a.e. on ∂D for a tangential vector field A in $L^2_T(\partial D)$, and set $U = \mathcal{S}A$ in $\mathbb{R}^3\setminus\partial D$. As a single acoustic layer potential, U belongs to both $V_i^2(D)$ and $V_e^2(D)$. Also, going to ∂D from inside D, we have

$$N\times\operatorname{curl}U = (\tfrac{1}{2}I+M)A = 0.$$

By Lemma 4.2, we have that for any X in D,

$$\begin{aligned} U(X) = \operatorname{curl}_X \int_{\partial D} & \Phi(X-Q)N(Q)\times U(Q)\,d\sigma(Q) \\ -\int_{\partial D} & \nabla_X\Phi(X-Q)\langle N(Q), U(Q)\rangle\,d\sigma(Q) \\ -\int_{\partial D} & \Phi(X-Q)(\operatorname{div} U(Q))N(Q)\,d\sigma(Q). \end{aligned}$$

Taking the divergence of both sides of this equation and then going to the boundary, on ∂D we obtain

$$\operatorname{div} U = k^2 S(\langle N, U\rangle) + (\tfrac{1}{2}I + K)(\operatorname{div} U),$$

or

$$(\tfrac{1}{2}I - K)(\operatorname{div} U) = k^2 S(\langle N, U\rangle).$$

Now, since S maps $L^2(\partial D)$ into $L^{2,1}(\partial D)$ and $\frac{1}{2}I-K$ is invertible in the latter space, it follows that $\operatorname{div} U \in L^{2,1}(\partial D)$. Thus, by the uniqueness in the regularity problem for the scalar Helmholtz equation, there exists a scalar-valued function u in $L^2(\partial D)$ such that $\operatorname{div} U = Su$ on ∂D. Furthermore, since $\operatorname{div} U$ is continuous across ∂D—in this case by the uniqueness in the Dirichlet problem for the scalar Helmholtz equation with L^2-data (Theorem 3.2)—it follows that $\operatorname{div} U = \mathcal{S}u$ in all $\mathbb{R}^3$.

Now let $E = \operatorname{curl} \mathcal{S}A$ on $\mathbb{R}^3 \setminus \partial D$. Note that

$$\operatorname{curl} E = \operatorname{curl}\operatorname{curl} U = -\Delta U + \nabla(\operatorname{div} U) = k^2 U + \nabla(\operatorname{div} U);$$

hence E satisfies the Helmholtz equation in D and is in $V_e^2(D)$. Since both $\operatorname{div} E$ and $E\times N$ are zero on the boundary of D, it follows by Corollary 4.4 that E must be identically zero in the interior of D. Let $H = (1/ik)\operatorname{curl} E$; that is, let

$$H = -ikU - \frac{i}{k}\nabla(\operatorname{div} U) = -ik\mathcal{S}A - \frac{i}{k}\nabla(\mathcal{S}u) \quad \text{on } \mathbb{R}^3\setminus\partial D.$$

Because $\mathcal{S}A$ and the tangential component of $\nabla(\mathcal{S}u)$ are continuous across the boundary, it follows that the same is true for $N\times H$. Consequently, these boundary values, taken from the exterior of D, are zero since $H\equiv 0$ in D.

Furthermore, H is divergence-free, satisfies the Helmholz equation in $\mathbb{R}^3\setminus\bar{D}$, and is also in $V_e^2(D)$. Corollary 4.4 finally yields that H must vanish in the exterior of D as well. However, in $\mathbb{R}^3\setminus\bar{D}$,

$$k^2 E = -\Delta E = \operatorname{curl}\operatorname{curl} E + \nabla \operatorname{div} E;$$

therefore $E\equiv 0$ in the exterior of D also. Taking this to the boundary, it then follows that

$$(-\tfrac{1}{2}I + M)A = 0,$$

and finally that

$$A = (\tfrac{1}{2}I+M)A-(-\tfrac{1}{2}I+M)A = 0.$$

The arguments for $-\frac{1}{2}I+M$ are virtually the same. Next we show that the ranges of $\pm\frac{1}{2}I+M$ are dense. Toward this end, we shall check that $\pm\frac{1}{2}I+M^*$ are also injective on $L^2_T(\partial D)$. Since this is an immediate consequence of $\pm\frac{1}{2}I+M = N\times((\pm\frac{1}{2}I+M^*)(N\times\cdot))$ on $L^2_T(\partial D)$, the proof of the case $p=2$ is complete.

We are considering bounded domains, so the above reasoning implies that $\pm\frac{1}{2}I+M$ are also injective on $L^p_T(\partial D)$ for $2\le p<\infty$. Since $\pm\frac{1}{2}I+M = N\times((\pm\frac{1}{2}I+M^*)(N\times\cdot))$ on $L^p_T(\partial D)$, it follows that so do $\pm\frac{1}{2}I+M^*$. Finally, using the duality of M and M^* given by (2), we infer that $\pm\frac{1}{2}I+M$ have dense ranges for $1<p\le 2$. $\square$

We now turn our attention to the action of M on $L^{p,\mathrm{Div}}_T(\partial D)$. The main result in this respect is the following.

THEOREM 5.4. *Let D be a Lipschitz domain in $\mathbb{R}^3$, and let $1<p<\infty$. The operator M is bounded from $L^{p,\mathrm{Div}}_T(\partial D)$ into itself. Moreover, if A is a vector field, initially in $L^2_T(\partial D)$, for which $(\pm\frac{1}{2}I+M)A$ is in $L^{2,\mathrm{Div}}_T(\partial D)$, then A actually belongs $L^{2,\mathrm{Div}}_T(\partial D)$. If the domain is of class C^1, then the last statement remains true for the whole range $1<p<\infty$.*

Proof. We already know that $\pm\frac{1}{2}I+M$ are bounded on $L^p_T(\partial D)$. In estimating $\mathrm{Div}((\frac{1}{2}I+M)A)$, we shall use Lemma 4.1. First note that, for A in $L^{p,\mathrm{Div}}_T(\partial D)$, the vector field $E=\operatorname{curl}\mathcal{S}A$ in D satisfies the hypotheses of that lemma, as

$$\operatorname{curl}E = \operatorname{curl}\operatorname{curl}\mathcal{S}A = (-\Delta+\nabla\operatorname{div})\mathcal{S}A = k^2\mathcal{S}A+\nabla\mathcal{S}(\mathrm{Div}\,A),$$

by Lemma 5.2. Therefore, since

$$\mathrm{Div}((\tfrac{1}{2}I+M)A) = \mathrm{Div}(N\times E) = -\langle N, \operatorname{curl}E\rangle,$$

we finally obtain

$$\|\mathrm{Div}((\tfrac{1}{2}I+M)A)\|_{L^p} \le C\|\operatorname{curl}E\|_{L^p} \le C(\|A\|_{L^p}+\|\mathrm{Div}\,A\|_{L^p}).$$

Similar arguments apply to $-\frac{1}{2}I+M$.

Consider now A in $L^p_T(\partial D)$ so that $B=(\frac{1}{2}I+M)A\in L^{p,\mathrm{Div}}_T(\partial D)$, where $p=2$ for a general Lipschitz domain and $1<p<\infty$ if the domain is C^1. Clearly $\mathcal{S}A$ falls within the range of Lemma 4.2. Thus, after taking the divergence in D and going to the boundary, we have

$$\operatorname{div}SA = k^2S(\langle N, SA\rangle)+(\tfrac{1}{2}I+K)(\operatorname{div}SA)+\operatorname{div}SB.$$

Using once again Lemma 5.2, we can write

$$(\tfrac{1}{2}I-K)(\operatorname{div}SA) = S(k^2\langle N, SA\rangle+\mathrm{Div}\,B).$$

By Theorem 3.1, it follows then that the boundary trace of $\operatorname{div}SA$ is in $L^{p,1}(\partial D)$. Hence, by the uniqueness in the regularity problem for the scalar

Helmholtz equation in D, there exists a $b \in L^p(\partial D)$ such that $\operatorname{div} \mathcal{S}A = \mathcal{S}b$ in D. Finally, we obtain $\operatorname{Div} A = b \in L^p(\partial D)$. $\square$

THEOREM 5.5. *If D is a bounded C^1 domain, then the operator M is compact on every $L^p_T(\partial D)$, $1 < p < \infty$.*

Proof. As has been observed many times (see e.g. [4, p. 60]), for a tangential vector field A the integrand $N(P) \times \operatorname{curl}_P(\Phi(P-Q)A(Q))$ in MA can also be written as

$$\nabla_P \Phi(P-Q)\langle N(P)-N(Q), A(Q)\rangle - \partial_{N_P}\Phi(P-Q)A(Q).$$

The second term is precisely the integrand of K^*A (recall that K^* is the transpose of the singular double acoustic layer potential operator). Furthermore, if the domain D is of class C^1, the gain in the first term is $N(P)-N(Q) = o(1)$ as $|P-Q|$ goes to zero. From these and well-known techniques [2; 7], the theorem follows. $\square$

The following is an immediate consequence of Theorem 5.5, Theorem 5.3, and the Fredholm theory for compact operators.

COROLLARY 5.6. *Let D be a bounded C^1 domain and let $\operatorname{Im} k > 0$. Then $\pm\frac{1}{2}I + M$ are invertible on $L^p_T(\partial D)$ for any $1 < p < \infty$.*

If we use Theorem 5.4 then we also have the following.

COROLLARY 5.7. *Let D be a C^1 domain and let $\operatorname{Im} k > 0$. Then the opertors $\pm\frac{1}{2}I + M$ are invertible on $L^{p,\operatorname{Div}}_T(\partial D)$ for any $1 < p < \infty$.*

6. The Maxwell Boundary Value Problems

In this section we solve the boundary value problem for the Maxwell equations mentioned in the introduction. The uniqueness of solution for $2 \le p < \infty$ is a simple consequence of Corollary 4.4, even in the Lipschitz case, because we are dealing with bounded domains. More generally, we have the following lemma.

LEMMA 6.1. *Let D be a bounded C^1 domain and assume that $\operatorname{Im} k > 0$. Suppose that E is a solution of the vector Helmholtz equation in D that is in $V_i^p(D)$ for some $1 < p < \infty$. If $\operatorname{div} E$ and $N \times E$ vanish at almost any point of ∂D, then actually E vanishes identically inside D. A similar result is valid for vector fields in $V_e^p(D)$.*

Proof. The proof once again involves the representation formulas of Lemma 4.2. Suppose that E satisfies the hypotheses of the lemma inside D (the reasoning for the complementary domain is completely analogous).

First, note that $\operatorname{div} E$ is a solution of the homogeneous Dirichlet problem for the scalar Helmholtz equation in D. Therefore, by the uniqueness part in Theorem 3.2, $\operatorname{div} E$ is identically zero inside D.

Next, by Lemma 4.2 we can write

$$E(X) = -\int_{\partial D} \nabla_X \Phi(X-Q)\langle N(Q), E(Q)\rangle \, d\sigma(Q) + \int_{\partial D} \Phi(X-Q) N(Q) \times \operatorname{curl} E(Q) \, d\sigma(Q).$$

Taking the curl of both sides and going to the boundary, we obtain

$$N \times \operatorname{curl} E = -(\tfrac{1}{2}I + M)(N \times \operatorname{curl} E)$$

or

$$(-\tfrac{1}{2}I + M)(N \times \operatorname{curl} E) = 0,$$

so that $N \times \operatorname{curl} E = 0$.

After this, we again go to the boundary in the integral representation of E and take the inner product with N. We obtain

$$\langle N, E\rangle = (\tfrac{1}{2}I - K^*)(\langle N, E\rangle)$$

or

$$(\tfrac{1}{2}I + K^*)(\langle N, E\rangle) = 0.$$

Therefore $\langle N, E\rangle = 0$ and the same integral representation formula finally yields $E \equiv 0$ on D. □

THEOREM 6.2. *Let D be a bounded C^1 domain in $\mathbb{R}^3$, and assume that $\operatorname{Im} k > 0$. The interior Maxwell boundary value problem*

$$(M_i) \begin{cases} \operatorname{curl} E - ikH = 0 \text{ on } D, \\ \operatorname{curl} H + ikE = 0 \text{ on } D, \\ N \times E = A \in L_T^{p,\operatorname{Div}}(\partial D) \end{cases}$$

has a unique solution in $V_i^p(D)$ for all $1 < p < \infty$. Moreover, the solution can be written as

$$E(X) = \operatorname{curl}_X \int_{\partial D} \Phi(X-Q)(\tfrac{1}{2}I + M)^{-1} A(Q) \, d\sigma(Q),$$

and $H = (1/ik) \operatorname{curl} E$. In particular,

$$C^{-1}\|A\|_{L_T^{p,\operatorname{Div}}(\partial D)} \le \|E^*\|_{L^p(\partial D)} + \|H^*\|_{L^p(\partial D)} \le C\|A\|_{L_T^{p,\operatorname{Div}}(\partial D)}. \tag{5}$$

Similarly, the exterior Maxwell boundary value problem

$$(M_e) \begin{cases} \operatorname{curl} E - ikH = 0 \text{ on } \mathbb{R}^3 \backslash \bar{D}, \\ \operatorname{curl} H + ikE = 0 \text{ on } \mathbb{R}^3 \backslash \bar{D}, \\ N \times E = A \in L_T^{p,\operatorname{Div}}(\partial D) \end{cases}$$

has a unique solution in $V_e^p(D)$. The solution can be written as

$$E(X) = \operatorname{curl}_X \int_{\partial D} \Phi(X-Q)(-\tfrac{1}{2}I+M)^{-1}A(Q)\, d\sigma(Q),$$

and $H = (1/ik)\operatorname{curl} E$. *Hence* (5) *holds in this case also.*

Proof. Obviously, E given above is a divergence-free vector field for which E^* is in $L^p(\partial D)$. Since

$$H = -ik\mathcal{S}((\pm\tfrac{1}{2}I+M)^{-1}A) - \frac{i}{k}\nabla\mathcal{S}(\operatorname{Div}((\pm\tfrac{1}{2}I+M)^{-1}A)),$$

we also see that $\|H^*\|_{L^p(\partial D)} < \infty$. The fact that E and H given by the above formulas satisfy (M_i) and (M_e), respectively, is already contained in the previous sections; Lemma 6.1 gives the uniqueness of solutions. As for (5), we note that $|N\times E| \le |E|$ while, by Lemma 4.1, $|\operatorname{Div} A| = |k\langle N, H\rangle| \le C|H|$ and everything follows. □

References

[1] R. Brown and Z. Shen, *The initial-Dirichlet problem for a fourth-order parabolic equation in Lipschitz cylinders,* Indiana Univ. Math. J. 39 (1990), 1313–1353.

[2] A. Calderón, *Cauchy integral on Lipschitz curves and related operators,* Proc. Nat. Acad. Sci. U.S.A. 74 (1977), 1324–1327.

[3] R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes Lipschitiziennes,* Ann. of Math. (2) 116 (1982), 361–387.

[4] D. Colton and R. Kress, *Integral equation methods in scattering theory,* Wiley, New York, 1983.

[5] M. Costabel, *A remark on the regularity of Maxwell's equations on Lipschitz domains,* Math. Methods Appl. Sci. 12 (1990), 365–368.

[6] B. Dahlberg and C. Kenig, *Hardy spaces and the Neumann problem in L^p for Laplace's equation in a Lipschitz domain,* Ann. of Math. (2) 125 (1987), 437–465.

[7] E. Fabes, M. Jodeit, and N. Rivière, *Potential techniques for boundary value problems on C^1 domains,* Acta Math. 141 (1978), 165–186.

[8] P. Hähner, *An exterior boundary value problem for the Maxwell's equations with boundary data in a Sobolev space,* Proc. Roy. Soc. Edinburgh Sect. A 109 (1988), 213–224.

[9] D. Jerison and C. Kenig, *The Neumann problem on Lipschitz domains,* Bull. Amer. Math. Soc. (N.S.) 4 (1981), 203–207.

[10] R. Kleinman and P. Martin, *On single integral equations for the transmission problem of acoustic,* SIAM J. Appl. Math. 48 (1988), 307–325.

[11] R. Kress, *On the boundary operator in electromagnetic scattering,* Proc. Roy. Soc. Edinburgh Sect. A 103 (1986), 91–98.

[12] P. Martin and P. Ola, *Boundary integral equations for the scattering of electromagnetic waves by a homogeneous dielectric obstacle,* Proc. Roy. Soc. Edinburgh Sect. A 123 (1993), 185–208.

[13] M. Mitrea, *The method of layer potentials in electromagnetic scattering theory on nonsmooth domains,* preprint.
[14] M. Mitrea, R. Torres, and G. Welland, *Regularity and approximation results for the Maxwell problem in C^1 and Lipschitz domains,* Clifford Algebras in analysis and related topics (J. Ryan, ed.), pp. 297–308, CRC Press, Boca Raton, FL, 1996.
[15] C. Müller, *Foundations of the mathematical theory of electromagnetic waves,* Springer, Berlin, 1969.
[16] R. Torres and G. Welland, *The Helmholtz equation and transmission problems with Lipschitz interfaces,* Indiana Univ. Math. J. 42 (1993), 1457–1485.
[17] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains,* J. Funct. Anal. 59 (1984), 572–611.
[18] ———, *The Dirichlet problem for the biharmonic equation in C^1 domains,* Indiana Univ. Math. J. 36 (1987), 867–895.

R. H. Torres
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109

G. V. Welland
Department of Mathematics
and Computer Science
University of Missouri—St. Louis
St. Louis, MO 63121