A New Construction of Isospectral Riemannian Nilmanifolds with Examples

RUTH GORNET

1. Introduction

The *spectrum* of a closed Riemannian manifold (M, g), denoted spec(M, g), is the collection of eigenvalues with multiplicities of the associated Laplace–Beltrami operator acting on smooth functions. Two Riemannian manifolds (M, g) and (M', g') are said to be *isospectral* if spec$(M, g) =$ spec(M', g'). A basic question in spectral geometry is determining what geometric information is contained in the spectrum of a Riemannian manifold.

Despite considerable research in the area, only a few geometric properties are known to be spectrally determined: for example, dimension, volume, and total scalar curvature. Examples of isospectral manifolds provide us with the only means for identifying properties not determined by the spectrum.

The primary goal of this paper is the development of a new construction for producing pairs of isospectral nilmanifolds of arbitrary step, and a comparison of the properties of resulting new examples. The new construction is a generalization of the one used by Gordon and Wilson to construct pairs of isospectral Heisenberg manifolds. Two-step nilmanifolds in general, and Heisenberg manifolds in particular, have been a rich source of examples of isospectral manifolds, and their geometry has been studied in some detail [DG1; BG; O; P1; P2; P4; GW1; GW2; G1; G2; E]. The higher-step examples that we introduce here have a much richer geometry, however, exhibiting many interesting and important properties not previously found. In particular, we present new examples of manifolds that are isospectral on functions but not isospectral on 1-forms. The techniques used to compare the 1-form spectrum are new, as previous techniques could not be applied to the higher-step examples.

Almost all known examples of isospectral manifolds can now be constructed by Sunada's theorem [S] or its generalizations [GW1; DG2]. (See [B2] for a general overview.) Sunada's method and its generalizations are based on representation theory, and all manifolds constructed by these methods are *strongly isospectral;* that is, all natural, strongly elliptic, self-adjoint

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operators on the manifolds are isospectral. In particular, these manifolds share the same p-form spectrum. The *p-form spectrum* of a manifold is the eigenvalue spectrum of the Laplace–Beltrami operator extended to act on smooth p-forms for p a positive integer. (See Section 2 or the Appendix for details.)

The new, higher-step construction uses techniques from Riemannian geometry, Lie groups, and representation theory to produce pairs of isospectral nilmanifolds of arbitrary step. While representation theory is used as a tool, this construction differs from previous ones in that the resulting pairs of isospectral manifolds need not be isospectral on 1-forms and so do not fall under the traditional Sunada set-up. This property was previously exhibited by pairs of isospectral Heisenberg manifolds constructed by Gordon and Wilson [GW2; G2]. Moreover, for any choice of P, Ikeda [I2] has constructed examples of isospectral lens spaces that are isospectral on p-forms for $p = 0, 1, \ldots, P$ but not isospectral on $(P+1)$-forms. These are the only known examples.

The only other examples of isospectral manifolds that do not fall under the traditional Sunada construction are bounded domains of Urakawa [U] and nonlocally isometric examples of Szabo [Sz2], Gordon [G3; G4], and Gordon and Wilson [GW3]. Recent results of Pesce [P3] now explain the Ikeda and Urakawa examples in a Sunada-like setting. This setting requires a genericity assumption that excludes nilmanifolds. Moreover, the construction presented here generalizes the method used by Gordon and Wilson to construct the Heisenberg examples.

Consequently, outside of the nonlocally isometric examples mentioned above, the construction below subsumes all known examples of isospectral manifolds that do not fall under a Sunada set-up.

The strength of this construction is demonstrated by the richness of the properties of resulting new examples. In particular, we use the new construction to produce pairs of isospectral three-step nilmanifolds with the combinations of properties shown in Table I.

The properties listed in Table I are defined as follows. Two cocompact (i.e. $\Gamma\backslash G$ compact), discrete subgroups Γ_1 and Γ_2 of a Lie group G are called *representation equivalent* if the associated quasi-regular representations are

Table I New Examples of Isospectral Manifolds

Pair of 3-Step Isospectral Nilmanifolds	$\forall p$ Same p-form Spectrum	Rep. Equiv. Fundamental Groups	Isomorphic Fundamental Groups	Same Length Spectrum	Same Marked Length Spectrum
I(7 dim)	Yes	Yes	No	No	No
II(5 dim)	Yes	Yes	Yes	Yes	No
III\IV(7\5 dim)	No	No	No	No	No
V(7 dim)	No	No	Yes	Yes	Yes

unitarily equivalent. (See Section 2 for details.) The *length spectrum* of a Riemannian manifold is the set of lengths of closed geodesics, counted with multiplicity. The multiplicity of a length is defined as the number of distinct free homotopy classes of loops in which the length occurs. (Note: other definitions of multiplicity appear in the literature.) The pairs of isospectral manifolds described above have the same lengths of closed geodesics. However, the length spectra often differ in the multiplicities that occur. The *marked length spectrum* takes into account the lengths of the closed geodesics and also records the free homotopy classes in which the geodesics occur.

After establishing notation in Section 2, the new construction for producing pairs of higher-step isospectral nilmanifolds is presented in Section 3. In Section 4, we present the examples described in Table I and compare the quasi-regular representations and the fundamental groups of Examples I through V. We also compare the p-form spectrum, but the calculations are left to an Appendix. The length spectrum and marked length spectrum of these examples will be examined in [Gt4].

Example I is the first example of a pair of *nonisomorphic,* representation equivalent, cocompact, discrete subgroups of a nilpotent Lie group. It is also the first example of a pair of representation equivalent, cocompact, discrete subgroups of a solvable Lie group producing Riemannian manifolds that do not have the same length spectrum. This example has implications in representation theory on nilpotent Lie groups, and motivated [Gt2] and [Gt3].

We prove in the Appendix that the manifolds in Examples III, IV, and V are not isospectral on 1-forms. Outside of the traditional Sunada set-up, no general method is known for comparing the 1-form spectrum of manifolds. The methods illustrated in the Appendix are new, as previously used techniques could not be applied to the higher-step examples. The only previous examples of manifolds that are isospectral on functions but not isospectral on p-forms for all p are the lens spaces and Heisenberg manifolds mentioned above.

Example V is the first example of a pair of isospectral Riemannian manifolds with the same marked length spectrum, but not the same spectrum on 1-forms. This example contrasts with two-step results relating the marked length spectrum and the p-form spectrum [E]. This example will be studied in detail in [Gt4].

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2. Background and Notation

Let G be a simply connected Lie group and let Γ be a cocompact, discrete subgroup of G. A Riemannian metric g is *left-invariant* if the left translations of G are isometries. The left-invariant metric g descends to a Riemannian metric on $\Gamma\backslash G$, which we also denote by g. Note that a left-invariant metric is determined by a choice of orthonormal basis of the Lie algebra $\mathfrak{g}$ of G.

As G is unimodular, the Laplace–Beltrami operator of $(\Gamma\backslash G, g)$ may be written

$$\Delta = -\sum_{i=1}^{n} E_i^2, \tag{2.1}$$

where $\{E_1, \dots, E_n\}$ is an orthonormal basis of the Lie algebra $\mathfrak{g}$ of G.

The Laplace–Beltrami operator acting on smooth p-forms is defined by $\Delta = d\delta + \delta d$. Here δ is the metric adjoint of d. Equivalently

$$\delta = (-1)^{n(p+1)+1} * d*,$$

where $*$ is the Hodge-$*$ operator. We denote the *p-form spectrum* of a Riemannian manifold (M, g) by p-spec(M, g).

The quasi-regular representation ρ_Γ of G on $L^2(\Gamma\backslash G)$ is defined as follows: For all x in G and f in $L^2(\Gamma\backslash G)$,

$$\rho_\Gamma(x)f = f \circ R_x.$$

Here R_x denotes the right action of x on $\Gamma\backslash G$. The quasi-regular representation is known to be unitary.

We say Γ_1 and Γ_2 are *representation equivalent* if ρ_{Γ_1} and ρ_{Γ_2} are unitarily equivalent; that is, Γ_1 and Γ_2 are representation equivalent if there exists a unitary isomorphism $T: L^2(\Gamma_1\backslash G) \to L^2(\Gamma_2\backslash G)$ such that $T(\rho_{\Gamma_1}(x)f) = \rho_{\Gamma_2}(x)Tf$ for every x in G and every f in $L^2(\Gamma_1\backslash G)$.

PROPOSITION 2.2 [GW1]. *Let Γ_1 and Γ_2 be cocompact, discrete subgroups of a simply connected Lie group G. Let g be a left-invariant metric on G. If Γ_1 and Γ_2 are representation equivalent, then*

$$p\text{-spec}(\Gamma_1\backslash G, g) = p\text{-spec}(\Gamma_2\backslash G, g)$$

for $p = 0, 1, \dots, \dim(G)$.

REMARK. Pairs of isospectral manifolds constructed using the traditional Sunada method are of the form $(\Gamma_1\backslash M, g)$ and $(\Gamma_2\backslash M, g)$, where Γ_1 and Γ_2 are representation equivalent, cocompact, discrete subgroups of a group G acting by isometries on a Riemannian manifold (M, g).

For a Lie algebra $\mathfrak{g}$, denote by $\mathfrak{g}^{(1)}$ the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. That is, $\mathfrak{g}^{(1)}$ is the Lie subalgebra of $\mathfrak{g}$ generated by all elements of the form $[X, Y]$ for X, Y in $\mathfrak{g}$. Inductively, define $\mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}]$. A Lie algebra $\mathfrak{g}$ is said to be *k-step nilpotent* if $\mathfrak{g}^{(k)} \equiv 0$ but $\mathfrak{g}^{(k-1)} \not\equiv 0$. A Lie group G is called *k-step nilpotent* if its Lie algebra is.

Let $G^{(k)} = \exp(\mathfrak{g}^{(k)})$ denote the kth derived subgroup of G. We denote the center of G by $Z(G)$ and the center of $\mathfrak{g}$ by $\mathfrak{z}$. Note that if G is k-step nilpotent, then $G^{(k-1)} \subset Z(G)$.

Let exp denote the Lie algebra exponential from $\mathfrak{g}$ to G. The Campbell–Baker–Hausdorff formula gives us the group operation of G in terms of $\mathfrak{g}$. Namely, for $X, Y \in \mathfrak{g}$:

$$\exp(X)\exp(Y) = \exp(X+Y+\tfrac{1}{2}[X,Y]+\tfrac{1}{12}[X,[X,Y]]+\tfrac{1}{12}[Y,[Y,X]]+\cdots),$$

where the remaining terms are higher-order brackets. Note that for two-step nilpotent Lie groups, only the first three terms in the right-hand side are nonzero. For three-step groups, only the first five terms are nonzero. If $\mathfrak{g}$ is nilpotent and G is simply connected, then exp is a diffeomorphism from $\mathfrak{g}$ onto G. Denote its inverse by log.

Let Γ_1 and Γ_2 be cocompact, discrete subgroups of nilpotent Lie groups G_1 and G_2, respectively. Any abstract group isomorphism $\Phi: \Gamma_1 \to \Gamma_2$ extends uniquely to a Lie group isomorphism $\Phi: G_1 \to G_2$.

Let Γ be a cocompact, discrete subgroup of a nilpotent Lie group G with left-invariant metric g. The locally homogeneous space $(\Gamma\backslash G, g)$ is called a *Riemannian nilmanifold.* If G is an abelian Lie group, then Γ is merely a lattice of full rank in G, and in this case $\log \Gamma$ is also a lattice in $\mathfrak{g}$.

Let $\mathfrak{g}_{\mathbf{Q}} = \operatorname{span}_{\mathbf{Q}}\{\log \Gamma\}$. This is a *rational* Lie algebra; that is, there exists a basis of $\mathfrak{g}$ made up of elements of $\log \Gamma$ such that the structure constants are rational. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called *rational* if $\mathfrak{h}$ is spanned by $\mathfrak{h} \cap \mathfrak{g}_{\mathbf{Q}}$. Note that the notion of rational depends on Γ. If $H = \exp(\mathfrak{h})$ is the connected Lie subgroup of G with rational Lie algebra $\mathfrak{h}$, then $\Gamma \cap H$ is a cocompact, discrete subgroup of H. The $\mathfrak{g}^{(k)}$ are always rational Lie subalgebras of $\mathfrak{g}$.

The Kirillov theory of irreducible unitary representations of nilpotent groups gives us a correspondence between irreducible unitary representations of G and elements of $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. In particular, fix $\tau \in \mathfrak{g}^*$. Let $\mathfrak{h}$ be a rational subalgebra of $\mathfrak{g}$ that is maximal with respect to the property that $\tau([\mathfrak{h}, \mathfrak{h}]) \equiv 0$. The subalgebra $\mathfrak{h}$ is called a *polarization* of τ. Let $H = \exp(\mathfrak{h})$ be the connected subgroup of G with Lie algebra $\mathfrak{h}$.

Define a character $\bar{\tau}$ of H by

$$\bar{\tau}(h) = e^{2\pi i \tau(\log(h))} \tag{2.3}$$

for all h in H. Define π_τ to be the irreducible representation of G induced by the representation $\bar{\tau}$ of H. Denote by $\mathcal{H}_\tau$ the representation space of π_τ. Two such irreducible representations π_τ and $\pi_{\tau'}$ are unitarily equivalent if and only if $\tau' = \tau \circ \operatorname{Ad}(x)$ for some x in G. Here $\operatorname{Ad}(x)$ is the adjoint map from $\mathfrak{g}$ to $\mathfrak{g}$.

For τ in $\mathfrak{g}^*$, the *coadjoint orbit* of τ is

$$O(\tau) = \{\tau \circ \operatorname{Ad}(x) : x \in G\}.$$

Hence π_τ and $\pi_{\tau'}$ are unitarily equivalent if and only if τ and τ' lie in the same coadjoint orbit of $\mathfrak{g}^*$.

As G is nilpotent, every irreducible representation of G is unitarily equivalent to π_τ for some $\tau \in \mathfrak{g}^*$, and the quasi-regular representation ρ_Γ is completely reducible. Thus the representation space $L^2(\Gamma\backslash G)$ is unitarily isomorphic to

$$L^2(\Gamma\backslash G) \cong \bigoplus_{\tau\in\mathfrak{J}} m(\tau)\mathcal{H}_\tau$$

for some $\mathfrak{J} \subset \mathfrak{g}^*$. Here $m(\tau)$ denotes the multiplicity of H_τ, and we assume $\mathfrak{J}$ contains at most one element of each coadjoint orbit of $\mathfrak{g}^*$.

A good reference for representation theory on nilpotent Lie groups is [CG].

3. A New Construction of Isospectral Nilmanifolds

Let G be a simply connected, k-step nilpotent Lie group with Lie algebra $\mathfrak{g}$. Define $\bar{G}$ to be the simply connected, $(k-1)$-step nilpotent Lie group $G/G^{(k-1)}$. For Γ a cocompact, discrete subgroup of G, denote by $\bar{\Gamma}$ the image of Γ under the canonical projection from G onto $\bar{G}$. The group $\bar{\Gamma}$ is then a cocompact, discrete subgroup of $\bar{G}$. For a left-invariant metric g on G, we associate a left-invariant metric $\bar{g}$ on $\bar{G}$ by restricting g to an orthogonal complement of $\mathfrak{g}^{(k-1)}$ in $\mathfrak{g}$.

We call the $(k-1)$-step nilmanifold $(\bar{\Gamma}\backslash\bar{G}, \bar{g})$ the *quotient nilmanifold of* $(\Gamma\backslash G, g)$. By using the definition of $\bar{g}$, one easily sees that the projection $(\Gamma\backslash G, g) \to (\bar{\Gamma}\backslash\bar{G}, \bar{g})$ is a Riemannian submersion.

The Lie algebra $\bar{\mathfrak{g}}$ of $\bar{G}$ is just $\mathfrak{g}/\mathfrak{g}^{(k-1)}$. We denote elements of $\bar{\mathfrak{g}}$ by $\bar{U}$, where $\bar{U}$ is the image of U under the canonical projection from $\mathfrak{g}$ onto $\bar{\mathfrak{g}}$.

DEFINITION 3.1. Let G be a simply connected nilpotent Lie group. We say G is *strictly nonsingular* if the following property holds: For every z in $Z(G)$ and every noncentral x in G, there exists an element a in G such that the commutator of x and a is z. That is, $xax^{-1}a^{-1} = z$.

The nilpotent Lie algebra $\mathfrak{g}$ is *strictly nonsingular* if, for every noncentral X in $\mathfrak{g}$,

$$\mathfrak{z} \subset \operatorname{ad}(X)(\mathfrak{g}).$$

That is, for every X in $\mathfrak{g}-\mathfrak{z}$ and every Z in $\mathfrak{z}$, there exists a vector Y in $\mathfrak{g}$ such that $[X, Y] = Z$.

One easily sees that these notions are equivalent. That is, a nilpotent Lie group is strictly nonsingular if and only if its Lie algebra is strictly nonsingular.

THEOREM 3.2. *Let G be a simply connected, strictly nonsingular nilpotent Lie group with left-invariant metric g. If Γ_1 and Γ_2 are cocompact, discrete subgroups of G such that*

$$\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G) \quad \textit{and} \quad \operatorname{spec}(\bar{\Gamma}_1\backslash\bar{G}, \bar{g}) = \operatorname{spec}(\bar{\Gamma}_2\backslash\bar{G}, \bar{g}),$$

then

$$\operatorname{spec}(\Gamma_1\backslash G, g) = \operatorname{spec}(\Gamma_2\backslash G, g).$$

REMARK. The above construction is a generalization of the construction used by Gordon and Wilson to obtain pairs of isospectral Heisenberg manifolds

[GW2]. If we let the Lie group G be a simply connected, strictly nonsingular, two-step nilpotent Lie group with a 1-dimensional center, then $G = H_n$ for some n, where H_n denotes the $(2n+1)$-dimensional Heisenberg group.

Proof of Theorem 3.2. We use the notation of Section 2.

For $i = 1, 2$, let $\mathfrak{I}_i$ be a subset of $\mathfrak{g}^*$ such that

$$L^2(\Gamma_i\backslash G) \cong \bigoplus_{\tau \in \mathfrak{I}_i} m_i(\tau)\mathcal{H}_\tau.$$

Recall that $m_i(\tau)$ denotes the multiplicity of π_τ in the quasi-regular representation of G on $L^2(\Gamma_i\backslash G)$, and we assume that $\mathfrak{I}_i$ contains at most one element of each coadjoint orbit of $\mathfrak{g}^*$.

We decompose the index set $\mathfrak{I}_i = \mathfrak{I}_i' \cup \mathfrak{I}_i''$ by letting

$$\mathfrak{I}_i' = \{\tau \in \mathfrak{I}_i : \tau(\mathfrak{z}) \equiv 0\} \quad \text{and} \quad \mathfrak{I}_i'' = \{\tau \in \mathfrak{I}_i : \tau(\mathfrak{z}) \not\equiv 0\}.$$

We likewise decompose the representation space $L^2(\Gamma_i\backslash G)$ by letting

$$\mathcal{H}_i' = \bigoplus_{\tau \in \mathfrak{I}_i'} m_i(\tau)\mathcal{H}_\tau \quad \text{and} \quad \mathcal{H}_i'' = \bigoplus_{\tau \in \mathfrak{I}_i''} m_i(\tau)\mathcal{H}_\tau.$$

As representation spaces, $L^2(\Gamma_i\backslash G) = \mathcal{H}_i' \oplus \mathcal{H}_i''$.

Because the Laplace operator acts through the representation, we can decompose the spectrum as $\operatorname{spec}(\Gamma_i\backslash G, g) = \operatorname{spec}'(\Gamma_i\backslash G, g) \cup \operatorname{spec}''(\Gamma_i\backslash G, g)$. Here $\operatorname{spec}'(\Gamma_i\backslash G, g)$ and $\operatorname{spec}''(\Gamma_i\backslash G, g)$ are defined as the spectrum of the Laplacian restricted to acting on $\mathcal{H}_i'$, and $\mathcal{H}_i''$, respectively. The multiplicity of an eigenvalue in $\operatorname{spec}(\Gamma_i\backslash G, g)$ is equal to the sum of its multiplicities in $\operatorname{spec}'(\Gamma_i\backslash G, g)$ and $\operatorname{spec}''(\Gamma_i\backslash G, g)$.

LEMMA 3.3. *The Laplacian of* $(\Gamma_i\backslash G, g)$ *acting on* $\mathcal{H}_i'$ *is precisely the Laplacian of* $(\bar{\Gamma}_i\backslash \bar{G}, \bar{g})$ *acting on* $L^2(\bar{\Gamma}_i\backslash \bar{G})$. *Thus* $\operatorname{spec}(\bar{\Gamma}_i\backslash \bar{G}, \bar{g}) = \operatorname{spec}'(\Gamma_i\backslash G, g)$ *for* $i = 1, 2$.

LEMMA 3.4. *The representations of G on* $\mathcal{H}_1''$ *and* $\mathcal{H}_2''$ *are unitarily equivalent, hence* $\operatorname{spec}''(\Gamma_1\backslash G, g) = \operatorname{spec}''(\Gamma_2\backslash G, g)$.

Theorem 3.2 now follows. □

The proof of Lemma 3.3 is essentially an extension of the first part of the proof used by Gordon and Wilson to construct pairs of isospectral Heisenberg manifolds (see [GW2, Thm. 4.1]). The details are included here for completeness and because of a difference in notation.

Proof of Lemma 3.3. Let $\{Z_1, Z_2, \ldots, Z_T\}$ be an orthonormal basis of $\mathfrak{z} = \mathfrak{g}^{(k-1)}$. Extend it to $\{E_1, E_2, \ldots, E_N, Z_1, Z_2, \ldots, Z_T\}$, an orthonormal basis of $\mathfrak{g}$. By (2.1), the Laplace–Beltrami operator of (G, g) is

$$\Delta = -\sum_{n=1}^{N} E_n{}^2 - \sum_{k=1}^{K} Z_k{}^2.$$

View the functions in $L^2(\Gamma_i\backslash G)$ as left Γ_i-invariant functions of G. The subspace $\mathcal{H}_i'$ then consists of those functions in $L^2(\Gamma_i\backslash G)$ that are independent of the center, which correspond to functions in $L^2(\bar{\Gamma}_i \backslash \bar{G})$ in a natural way. Hence, when we restrict Δ to $\mathcal{H}_i'$, we have

$$\Delta = -\sum_{n=1}^{N} E_n^{\,2},$$

which corresponds to the Laplacian of $(\bar{\Gamma}_i \backslash \bar{G}, \bar{g})$.

The Laplacian of $(\Gamma_i \backslash G, g)$ acting on $\mathcal{H}_i'$ is then precisely the Laplacian of $(\bar{\Gamma}_i \backslash \bar{G}, \bar{g})$ acting on $L^2(\bar{\Gamma}_i \backslash \bar{G})$, so $\mathrm{spec}(\bar{\Gamma}_i \backslash \bar{G}, \bar{g}) = \mathrm{spec}'(\Gamma_i \backslash G, g)$, as desired. □

Before proving Lemma 3.4, we must introduce some of the theory of square integrable representations of nilpotent Lie groups.

DEFINITION 3.5. Let G be a locally compact, unimodular group with center $Z(G)$. We say that an irreducible unitary representation π of G on a Hilbert space $\mathcal{H}$ is *square integrable* if there are nonzero vectors x_1 and x_2 in $\mathcal{H}$ such that

$$\int_{G/Z(G)} |(\pi(s)x_1, x_2)|^2 \, d\bar{\mu}(\bar{s}) < \infty.$$

Here $d\bar{\mu}(\bar{s})$ denotes integration over $G/Z(G)$ with respect to a choice of Haar measure $\bar{\mu}$ on $G/Z(G)$. As the center acts trivially, the integrand may be viewed as a function of $G/Z(G)$.

Let $\mathfrak{z}^\perp$ be the subalgebra of $\mathfrak{g}^*$ defined by $\mathfrak{z}^\perp = \{\mu \in \mathfrak{g}^* : \mu(\mathfrak{z}) \equiv 0\}$. Note that $\mathfrak{z}^\perp \cong (\mathfrak{g}/\mathfrak{z})^*$. For $\tau \in \mathfrak{g}^*$, let b_τ denote the skew-symmetric, bilinear form on $\mathfrak{g}/\mathfrak{z}$ defined by $b_\tau(\bar{X}, \bar{Y}) = \tau([X, Y])$ for all $\bar{X}, \bar{Y}$ in $\mathfrak{g}/\mathfrak{z}$. Here X and Y are any elements of $\mathfrak{g}$ that project onto $\bar{X}$ and $\bar{Y}$, respectively.

THEOREM 3.6 [MW]. *For a linear functional τ in $\mathfrak{g}^*$ with coadjoint orbit $O(\tau)$ and corresponding irreducible unitary representation π_τ, the following three conditions are equivalent:*

(1) *π_τ is square integrable;*
(2) *$O(\tau) = \tau + \mathfrak{z}^\perp$;*
(3) *b_τ is nondegenerate on $\mathfrak{g}/\mathfrak{z}$.*

Proof of Lemma 3.4. Fix $\tau \in \mathfrak{Z}_i''$. Let $Z \in \mathfrak{z}$ be such that $\tau(Z) \neq 0$. By strict nonsingularity, for all noncentral $X \in \mathfrak{g}$ there exists $Y \in \mathfrak{g}$ such that $[X, Y] = Z$. Hence $b_\tau(\bar{X}, \bar{Y}) = \tau([X, Y]) \neq 0$ and so b_τ is nondegenerate. By Theorem 3.6, π_τ is square integrable. Note that b_τ nondegenerate implies that $N = \dim(\mathfrak{g}/\mathfrak{z})$ is even.

Recall from Section 2 that π_τ is independent of the choice of τ in $O(\tau)$. By Theorem 3.6, since π_τ is square integrable, the coadjoint orbit $O(\tau)$ is uniquely determined by the restriction of τ to the center. We may thus assume $\tau \in \mathfrak{z}^*$.

Let α be a volume form on $G/Z(G)$. That is, let α be a fixed, alternating, N-linear form over $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{z}$. Since b_τ is nondegenerate, $b_\tau^{N/2} = b_\tau \wedge \cdots \wedge b_\tau$ is also a volume form on $G/Z(G)$ and hence a scalar multiple of α. Define $P_\alpha(\tau)$ by

$$P_\alpha(\tau)\alpha = b_\tau \wedge \cdots \wedge b_\tau = b_\tau^{N/2}.$$

The polynomial P_α is homogeneous of degree $N/2$ on $\mathfrak{g}^*$ and depends only on the choice of volume form α.

Let $\mathfrak{z}^*$ denote the dual of $\mathfrak{z}$. Let $L = \Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G)$ and let $L^* \subset \mathfrak{z}^*$ be the dual lattice of L. We now use the following occurrence and multiplicity condition, also due to Moore and Wolf.

THEOREM 3.7 [MW]. *Let G be a nilpotent Lie group and Γ a cocompact, discrete subgroup of G. Let $L = \Gamma \cap Z(G)$. Fix a volume form α_Γ on $\mathfrak{g}/\mathfrak{z}$ so that $\bar{\Gamma}\backslash\bar{G}$ has volume 1. Let τ be a nonzero element of $\mathfrak{z}^*$ such that π_τ is square integrable. The representation π_τ occurs in the quasi-regular representation of G on $L^2(\Gamma\backslash G)$ if and only if $\tau \in L^*$. Moreover, its multiplicity $m(\tau)$ is $|P_{\alpha_\Gamma}(\tau)|$.*

By Theorem 3.7, the square integrable representation π_τ occurs in the quasi-regular representation of G on $L^2(\Gamma_i\backslash G)$ if and only if τ is contained in L^*. Thus, for $i = 1, 2$, the coadjoint orbits represented in $\mathfrak{I}_i''$ correspond to the elements of L^*. We may assume that $\mathfrak{I}_1'' = \mathfrak{I}_2''$.

Because the Riemannian metrics of $(\bar{\Gamma}_1\backslash\bar{G}, \bar{g})$ and $(\bar{\Gamma}_2\backslash\bar{G}, \bar{g})$ arise from the same left-invariant metric $\bar{g}$ on $\bar{G}$, we know that the Riemannian volume forms of $(\bar{\Gamma}_1\backslash\bar{G}, \bar{g})$ and $(\bar{\Gamma}_2\backslash\bar{G}, \bar{g})$ arise from the same left-invariant volume form on $\bar{G}$. We will denote by Ω this volume form and its projections onto $(\bar{\Gamma}_1\backslash\bar{G}, \bar{g})$ and $(\bar{\Gamma}_2\backslash\bar{G}, \bar{g})$.

Let α_{Γ_1} and α_{Γ_2} be as in Theorem 3.7. The volume forms α_{Γ_i} are then scalar multiples of Ω. For $i = 1, 2$, let $\alpha_{\Gamma_i} = p_i\Omega$. It follows that

$$\int_{\bar{\Gamma}_1\backslash\bar{G}} \alpha_{\Gamma_1} = 1 = \int_{\bar{\Gamma}_2\backslash\bar{G}} \alpha_{\Gamma_2};$$

$$p_1 \int_{\bar{\Gamma}_1\backslash\bar{G}} \Omega = p_2 \int_{\bar{\Gamma}_2\backslash\bar{G}} \Omega;$$

$$p_1 \operatorname{Vol}(\bar{\Gamma}_1\backslash\bar{G}, \bar{g}) = p_2 \operatorname{Vol}(\bar{\Gamma}_2\backslash\bar{G}, \bar{g}).$$

By hypothesis, $\operatorname{spec}(\bar{\Gamma}_1\backslash\bar{G}, \bar{g}) = \operatorname{spec}(\bar{\Gamma}_2\backslash\bar{G}, \bar{g})$, and the spectrum of the Laplace–Beltrami operator is known to determine the volume of a closed manifold. Thus $\operatorname{Vol}(\bar{\Gamma}_1\backslash\bar{G}, \bar{g}) = \operatorname{Vol}(\bar{\Gamma}_2\backslash\bar{G}, \bar{g})$, which implies $p_1 = p_2$, and so $\alpha_{\Gamma_1} = \alpha_{\Gamma_2}$.

As the definition of $P_{\alpha_{\Gamma_i}}$ depends only on the volume form α_{Γ_i}, we must have

$$P_{\alpha_{\Gamma_1}}(\tau) = P_{\alpha_{\Gamma_2}}(\tau)$$

for all τ in $\mathfrak{I}_1'' = \mathfrak{I}_2''$. Hence $m_1(\tau) = m_2(\tau)$ for all τ in $\mathfrak{I}_1'' = \mathfrak{I}_2''$. Thus, the representations of G on $\mathcal{H}_1''$ and $\mathcal{H}_2''$ are unitarily equivalent, and by Proposition 2.2

$$\operatorname{spec}''(\Gamma_1\backslash G, g) = \operatorname{spec}''(\Gamma_2\backslash G, g),$$

as desired.

The proofs of Lemma 3.4 and Theorem 3.2 are now complete. □

COROLLARY 3.8. *Let G be a simply connected, strictly nonsingular nilpotent Lie group. Two cocompact, discrete subgroups Γ_1 and Γ_2 are representation equivalent subgroups of G if and only if $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are representation equivalent subgroups of $\bar{G}$ and $\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G)$.*

Proof. Let Γ_1 and Γ_2 be cocompact, discrete subgroups of G. As in the proof of Theorem 3.2, we decompose the representation spaces of ρ_{Γ_1} and ρ_{Γ_2} as

$$L^2(\Gamma_i\backslash G) = \mathcal{H}_i' \oplus \mathcal{H}_i''$$

for $i = 1, 2$.

If Γ_1 and Γ_2 are representation equivalent, then the square integrable representations occurring in the quasi-regular representations must correspond. We showed in the proof of Lemma 3.4 that the square integrable representations occurring in ρ_{Γ_1} and ρ_{Γ_2} are precisely the irreducible representations appearing in $\mathcal{H}_i''$. Thus Γ_1 and Γ_2 are representation equivalent subgroups of G if and only if the representations of G on $\mathcal{H}_1'$ and $\mathcal{H}_2'$ are unitarily equivalent and the representations of G on $\mathcal{H}_1''$ and $\mathcal{H}_2''$ are unitarily equivalent.

For $i = 1, 2$, the irreducible components of the representation of G on $\mathcal{H}_i'$ correspond to the linear functionals in $\mathfrak{Z}_i$ that are zero on the center of $\mathfrak{g}$. Since these functionals may be viewed as functionals on $\bar{\mathfrak{g}}$, we may likewise view the irreducible components as representations of $\bar{G}$. Hence the representations of G on $\mathcal{H}_1'$ and $\mathcal{H}_2'$ may be viewed as the quasi-regular representations of $\bar{G}$ on $L^2(\bar{\Gamma}_1\backslash\bar{G})$ and $L^2(\bar{\Gamma}_2\backslash\bar{G})$, respectively. Thus Γ_1 and Γ_2 are representation equivalent subgroups of G if and only if $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are representation equivalent subgroups of $\bar{G}$ and the representations of G on $\mathcal{H}_1''$ and $\mathcal{H}_2''$ are unitarily equivalent.

Theorem 3.7 tells us that the irreducible representations occurring in $\mathcal{H}_i''$ correspond to the elements of the dual lattice of $\Gamma_i \cap Z(G)$, so if the representations of G on $\mathcal{H}_1''$ and $\mathcal{H}_2''$ are unitarily equivalent then the duals of $\Gamma_1 \cap Z(G)$ and $\Gamma_2 \cap Z(G)$ must coincide. Hence $\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G)$. The proof of the forward direction is now complete.

The reverse direction follows from Proposition 2.2 applied to the quotient nilmanifolds and Lemma 3.4. □

4. New Examples of Isospectral Nilmanifolds

Using Theorem 3.2, we construct and compare five new pairs of isospectral nilmanifolds. A summary of the properties of these examples was listed in Table I. In this section we compare the quasi-regular representations and the fundamental groups of these examples. The p-form spectra of these examples are also compared, but the calculations are left to the Appendix. The methods used in the Appendix are new, as previously used techniques could not be applied to compare the p-form spectrum of these higher-step examples. The length spectra and marked length spectra of these examples will be studied in [Gt4].

With the exception of the column comparing the representation equivalence of the fundamental groups, all of the properties listed in Table I are geometric invariants. Hence, a "No" in any one of those columns demonstrates that an Example is nontrivial.

Before proceeding, we need the following.

DEFINITION 4.1. Let Φ be a Lie group automorphism of G. Let Γ be a cocompact, discrete subgroup of G. We say Φ is an *almost inner automorphism* if for all elements x of G there exists a_x in G such that $\Phi(x) = a_x x a_x^{-1}$.

THEOREM 4.2 [GW1]. *Let G be a nilpotent Lie group and let Γ be a cocompact, discrete subgroup of G. If Φ is an almost inner automorphism of G then Γ and $\Phi(\Gamma)$ are representation equivalent subgroups of G.*

Example I

Consider the simply connected, strictly nonsingular, three-step nilpotent Lie group G with Lie algebra

$$\mathfrak{g} = \operatorname{span}_{\mathbf{R}}\{X_1, X_2, Y_1, Y_2, Z_1, Z_2, W\}$$

and Lie brackets

$$[X_1, Y_1] = [X_2, Y_2] = Z_1,$$
$$[X_1, Y_2] = Z_2,$$
$$[X_1, Z_1] = [X_2, Z_2] = [Y_1, Y_2] = W,$$

and all other basis brackets zero.

Let Γ_1 be the cocompact, discrete subgroup of G generated by

$$\{\exp(2X_1), \exp(2X_2), \exp(Y_1), \exp(Y_2), \exp(Z_1), \exp(Z_2), \exp(W)\}.$$

Let Γ_2 be the cocompact, discrete subgroup of G generated by

$$\{\exp(2X_1), \exp(2X_2), \exp(Y_1), \exp(Y_2 + \tfrac{1}{2}Z_2), \exp(Z_1), \exp(Z_2), \exp(W)\}.$$

Note that $\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G) = \{\exp(jW) : j \in \mathbf{Z}\}$.

Now $\bar{\Gamma}_2 = \Phi(\bar{\Gamma}_1)$, where Φ is the almost inner automorphism of $\bar{G}$ given on the Lie algebra level by

$$\bar{X}_1 \to \bar{X}_1,$$
$$\bar{X}_2 \to \bar{X}_2,$$
$$\bar{Y}_1 \to \bar{Y}_1,$$
$$\bar{Y}_2 \to \bar{Y}_2 + \tfrac{1}{2}\bar{Z}_2,$$
$$\bar{Z}_1 \to \bar{Z}_1,$$
$$\bar{Z}_2 \to \bar{Z}_2.$$

By Theorem 4.2, $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are representation equivalent subgroups of $\bar{G}$. By Corollary 3.8, Γ_1 and Γ_2 are representation equivalent subgroups of G. By Proposition 2.2, for any choice of left-invariant metric g of G, we have $p\text{-spec}(\Gamma_1\backslash G, g) = p\text{-spec}(\Gamma_2\backslash G, g)$ for $p = 0, 1, \ldots, 7$.

PROPOSITION 4.3 [Gt2]. *The subgroups Γ_1 and Γ_2 are not isomorphic as groups.*

REMARK. The author previously established the representation equivalence of Γ_1 and Γ_2 in [Gt2] by using a direct calculation. This example was presented in [Gt2] as the first example of a pair of nonisomorphic, representation equivalent subgroups of a solvable Lie group. Note that a nilpotent Lie group is necessarily solvable. Also, this example was presented in [Gt3] as the first example of a pair of representation equivalent subgroups of a solvable Lie group producing nilmanifolds with unequal length spectra. Contrast this example with what must happen in the two-step case.

DEFINITION 4.4. Let G be a two-step nilpotent Lie group and let Γ be a cocompact, discrete subgroup of G. We call the automorphism Φ of G a Γ-*equivalence* if for all γ in Γ there exist a_γ in G and γ'_γ in $\Gamma \cap G^{(1)}$ such that $\Phi(\gamma) = a_\gamma \gamma a_\gamma^{-1} \gamma'_\gamma$.

THEOREM 4.5 [Gt2; Gt3]. *Let G be a two-step nilpotent Lie group. Let Γ_1 and Γ_2 be cocompact, discrete subgroups of G. The subgroups Γ_1 and Γ_2 are representation equivalent if and only if there exists Φ, a Γ_1-equivalence of G, such that $\Phi(\Gamma_1) = \Gamma_2$. Thus, if Γ_1 and Γ_2 are representation equivalent then they are necessarily isomorphic. In addition, if Γ_1 and Γ_2 are representation equivalent then $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$ have the same length spectrum for any choice of left-invariant metric g of G.*

Example II

Consider the simply connected, strictly nonsingular, three-step nilpotent Lie group G with Lie algebra

$$\mathfrak{g} = \text{span}_{\mathbf{R}}\{X_1, Y_1, Y_2, Z, W\}$$

and Lie brackets

$$[X_1, Y_1] = Z,$$
$$[X_1, Z] = [Y_1, Y_2] = W,$$

and all other basis brackets zero.

Let Γ_1 be the cocompact, discrete subgroup of G generated by

$$\{\exp(2X_1), \exp(Y_1), \exp(Y_2), \exp(Z), \exp(W)\}.$$

Let Γ_2 be the cocompact, discrete subgroup of G generated by

$$\{\exp(2X_1), \exp(Y_1 + \tfrac{1}{2}Z), \exp(Y_2), \exp(Z), \exp(W)\}.$$

Note that $\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G) = \{\exp(jW) : j \in \mathbf{Z}\}$.

Now $\bar{\Gamma}_2 = \Phi(\bar{\Gamma}_1)$, where Φ is the inner automorphism of $\bar{G}$ given on the Lie algebra level by

$$\begin{aligned}\bar{X}_1 &\to \bar{X}_1,\\ \bar{Y}_1 &\to \bar{Y}_1 + \tfrac{1}{2}\bar{Z},\\ \bar{Y}_2 &\to \bar{Y}_2,\\ \bar{Z} &\to \bar{Z}.\end{aligned}$$

Note that an inner automorphism is necessarily almost inner.

By Theorem 4.2, $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are representation equivalent subgroups of $\bar{G}$. By Corollary 3.8, Γ_1 and Γ_2 are representation equivalent subgroups of G. By Proposition 2.2, $p\text{-spec}(\Gamma_1\backslash G, g) = p\text{-spec}(\Gamma_2\backslash G, g)$ for $p = 0, 1, \ldots, 5$, for any choice of left-invariant metric g of G.

Here Γ_1 and Γ_2 are isomorphic. Indeed, a simple calculation shows that the isomorphism Ψ given on the Lie algebra level by

$$\begin{aligned}X_1 &\to X_1 + \tfrac{1}{2}Y_2,\\ Y_1 &\to Y_1 + \tfrac{1}{2}Z,\\ Y_2 &\to Y_2,\\ Z &\to Z,\\ W &\to W,\end{aligned}$$

is an isomorphism of G such that $\Psi(\Gamma_1) = \Gamma_2$. Note, however, that Ψ is not almost inner, as $\exp(X_1 + \frac{1}{2}Y_2)$ and $\exp(X_1)$ are not conjugate.

Proposition 4.6. *No isomorphism between Γ_1 and Γ_2 will project to a $\bar{\Gamma}_1$-equivalence of $\bar{G}$.*

Remark. Example I illustrates that in the higher-step case, the representation equivalence of Γ_1 and Γ_2 and the isomorphism class of Γ_1 and Γ_2 need not be related. Example II shows that, in contrast to Theorem 4.5, even in the case where Γ_1 and Γ_2 are isomorphic, knowing the isomorphisms between Γ_1 and Γ_2 is not enough to use Corollary 3.8 to establish whether or not Γ_1 and Γ_2 are representation equivalent.

Proof of Proposition 4.6. Let Ψ be an isomorphism from Γ_1 to Γ_2. Extend it to the Lie group isomorphism $\Psi: G \to G$ such that $\Psi(\Gamma_1) = \Gamma_2$.

On the Lie algebra level, any such isomorphism must preserve the following ideals of $\mathfrak{g}$:

(1) $\mathfrak{g}^{(2)} = \text{span}_{\mathbf{R}}\{W\}$;
(2) $\mathfrak{g}^{(1)} = \text{span}_{\mathbf{R}}\{Z, W\}$;
(3) $\mathfrak{U} = \text{span}_{\mathbf{R}}\{Y_2, Z, W\}$; and
(4) $\mathfrak{C} = \text{span}_{\mathbf{R}}\{Y_1, Y_2, Z, W\}$, the centralizer of $\mathfrak{g}^{(1)}$ in $\mathfrak{g}$.

To see (3), note that $\text{ad}(U)(\mathfrak{g}) \subset \mathfrak{g}^{(2)}$ if and only if $U \in \mathfrak{U}$.

Note that the generators of Γ_1 and Γ_2 presented above are canonical in the sense that every element of Γ_1 may be expressed uniquely as

$$\exp(2n_1X_1)\exp(m_1Y_1)\exp(m_2Y_2)\exp(kZ)\exp(jW)$$

for integers n_1, m_1, m_2, k, j, and similarly for Γ_2.

Because $\Psi(\Gamma_1)=\Gamma_2$, generators of Γ_1 must go to generators of Γ_2, and these generators must be expressible in terms of the canonical generators of Γ_2 given above. Combining this fact with properties (1) through (4), we obtain:

$$\Psi_*(W)=\pm W \quad \text{by (1).}$$
$$\Psi_*(Z)=\pm Z+h_0W \quad \text{using (2).}$$
$$\Psi_*(Y_2)=\pm Y_2 \bmod \mathfrak{g}^{(1)} \quad \text{using (3).}$$
$$\Psi_*(Y_1)=\pm(Y_1+\tfrac{1}{2}Z)+h_1Y_2+h_2Z \bmod \mathfrak{g}^{(2)} \quad \text{using (4).}$$
$$\Psi_*(X_1)=\pm X_1+\tfrac{1}{2}h_3Y_1+\tfrac{1}{2}h_4Y_2 \bmod \mathfrak{g}^{(1)}.$$

Here h_0, h_1, h_2, h_3, and h_4 are integers.

Finally, we use the fact that Ψ_* is a Lie algebra isomorphism. By examining the W coefficient of $\Psi_*([X_1,Y_1])=[\Psi_*(X_1),\Psi_*(Y_1)]$, we have the equation

$$h_0=\pm\tfrac{1}{2}+\tfrac{1}{2}h_1h_3\pm h_2\pm\tfrac{1}{2}h_4.$$

Thus either $h_3\neq 0$ or $h_4\neq 0$.

As $\bar{Y}_1$ and $\bar{Y}_2$ are not in $[\bar{X}_1,\bar{\mathfrak{g}}]$ and not in $\bar{\mathfrak{g}}^{(1)}$, we see that the projection of Ψ cannot possibly be a $\bar{\Gamma}_1$-equivalence. □

Example III

Consider again the 7-dimensional Lie group G presented in Example I. We again let Γ_1 be the cocompact, discrete subgroup of G generated by

$$\{\exp(2X_1),\exp(2X_2),\exp(Y_1),\exp(Y_2),\exp(Z_1),\exp(Z_2),\exp(W)\},$$

and let Γ_2 be the cocompact, discrete subgroup of G generated by

$$\{\exp(X_1),\exp(X_2),\exp(2Y_1),\exp(2Y_2),\exp(Z_1),\exp(Z_2),\exp(W)\}.$$

Note that $\Gamma_1\cap Z(G)=\Gamma_2\cap Z(G)=\{\exp(jW): j\in\mathbf{Z}\}$.

Let g be the left-invariant metric on G defined by letting

$$\{X_1,X_2,Y_1,Y_2,Z_1,Z_2,W\}$$

be an orthonormal basis of $\mathfrak{g}$.

Now $\bar{\Gamma}_2=\Phi(\bar{\Gamma}_1)$, where Φ is the automorphism of $\bar{G}$ given on the Lie algebra level by

$$\bar{X}_1\to\bar{Y}_2,$$
$$\bar{X}_2\to\bar{Y}_1,$$
$$\bar{Y}_1\to\bar{X}_2,$$
$$\bar{Y}_2\to\bar{X}_1,$$
$$\bar{Z}_1\to-\bar{Z}_1,$$
$$\bar{Z}_2\to-\bar{Z}_2.$$

The automorphism Φ is also an isometry of $(\bar{G}, \bar{g})$, and an isometry must preserve the spectrum. Thus $\text{spec}(\bar{\Gamma}_1\backslash\bar{G}, \bar{g}) = \text{spec}(\bar{\Gamma}_2\backslash\bar{G}, \bar{g})$. By Theorem 3.2, $\text{spec}(\Gamma_1\backslash G, g) = \text{spec}(\Gamma_2\backslash G, g)$.

PROPOSITION 4.7. *The manifolds* $(\Gamma_1\backslash G, g)$ *and* $(\Gamma_2\backslash G, g)$ *are not isospectral on* 1*-forms.*

The proof of Proposition 4.7 is left to the Appendix. Note that Proposition 4.7 and Proposition 2.2 together imply that Γ_1 and Γ_2 are not representation equivalent subgroups of G.

PROPOSITION 4.8. *The subgroups* Γ_1 *and* Γ_2 *are not isomorphic as groups.*

Proof. If there existed a group isomorphism between Γ_1 and Γ_2, it would extend to a Lie group automorphism Ψ of G such that $\Psi(\Gamma_1) = \Gamma_2$.

Each of the following ideals of $\mathfrak{g}$ must be preserved by the Lie algebra automorphism Ψ_*:

(1) $\mathfrak{g}^{(2)} = \mathfrak{z} = \text{span}_{\mathbf{R}}\{W\}$;
(2) $\mathfrak{g}^{(1)} = \text{span}_{\mathbf{R}}\{Z_1, Z_2, W\}$;
(3) $\mathfrak{C} = \text{span}_{\mathbf{R}}\{Y_1, Y_2, Z_1, Z_2, W\}$, the centralizer of $\mathfrak{g}^{(1)}$ in $\mathfrak{g}$; and
(4) $\mathfrak{U} = \text{span}_{\mathbf{R}}\{X_2, Y_1, Z_1, Z_2, W\}$.

To see (4), note that the image of $\text{ad}(U)$ has dimension < 3 if and only if $U \in \mathfrak{U}$.

Now $\Psi(\Gamma_1) = \Gamma_2$. Consequently, generators of Γ_1 must go to generators of Γ_2, and these generators must be expressible in terms of the canonical generators of Γ_2 given above. Combining this fact with properties (1) through (4), we obtain:

$$\Psi_*(Y_1) = \pm 2Y_1 \bmod \mathfrak{g}^{(1)} \qquad \text{by (3) and (4).}$$
$$\Psi_*(Y_2) = \pm 2Y_2 \bmod \text{span}_{\mathbf{R}}\{Y_1, Z_1, Z_2, W\} \qquad \text{by (3).}$$
$$\Psi_*(W) = \pm W \qquad \text{by (1).}$$

But then $\Psi_*([Y_1, Y_2]) \neq [\Psi_*(Y_1), \Psi_*(Y_2)]$. □

Example IV

Consider again the 5-dimensional Lie group G presented in Example II. We again let Γ_1 be the cocompact, discrete subgroup of G generated by

$$\{\exp(2X_1), \exp(Y_1), \exp(Y_2), \exp(Z), \exp(W)\},$$

and let Γ_2 be the cocompact, discrete subgroup of G generated by

$$\{\exp(X_1), \exp(2Y_1), \exp(Y_2), \exp(Z), \exp(W)\}.$$

Note that $\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G) = \{\exp(jW) : j \in \mathbf{Z}\}$.

Let g be the left-invariant metric on G defined by letting

$$\{X_1, Y_1, Y_2, Z, W\}$$

be an orthonormal basis of $\mathfrak{g}$.

Now $\bar{\Gamma}_2 = \Phi(\bar{\Gamma}_1)$, where Φ is the automorphism of $\bar{G}$ given on the Lie algebra level by

$$\begin{aligned}\bar{X}_1 &\to \bar{Y}_1,\\ \bar{Y}_1 &\to \bar{X}_1,\\ \bar{Y}_2 &\to \bar{Y}_2,\\ \bar{Z} &\to -\bar{Z}.\end{aligned}$$

The automorphism Φ is clearly an isometry of $(\bar{G}, \bar{g})$, and an isometry preserves the spectrum. Thus $\operatorname{spec}(\bar{\Gamma}_1\backslash\bar{G}, \bar{g}) = \operatorname{spec}(\bar{\Gamma}_2\backslash\bar{G}, \bar{g})$. By Theorem 3.2, $\operatorname{spec}(\Gamma_1\backslash G, g) = \operatorname{spec}(\Gamma_2\backslash G, g)$.

PROPOSITION 4.9. *The manifolds* $(\Gamma_1\backslash G, g)$ *and* $(\Gamma_2\backslash G, g)$ *are not isospectral on* 1-*forms.*

The proof of Proposition 4.9 is left to the Appendix.

PROPOSITION 4.10. *The subgroups* Γ_1 *and* Γ_2 *are not isomorphic as groups.*

REMARK. The combination of properties exhibited by Examples III and IV are similar to properties exhibited by pairs of isospectral Heisenberg manifolds constructed by Gordon and Wilson [GW2; G2].

Proof of Proposition 4.10. If there existed a group isomorphism between Γ_1 and Γ_2, it would extend to a Lie group automorphism Ψ of G such that $\Psi(\Gamma_1) = \Gamma_2$.

As before, generators of Γ_1 must go to generators of Γ_2, and these generators must be expressible in terms of the canonical generators of Γ_2 given above. Combining this with properties (1) through (4) from the proof of Proposition 4.6, we have:

$$\Psi_*(Y_1) = \pm 2Y_1 \bmod \operatorname{span}_{\mathbb{R}}\{Y_2, Z, W\}.$$

$$\Psi_*(Y_2) = \pm Y_2 \bmod \mathfrak{g}^{(1)}.$$

$$\Psi_*(W) = \pm W.$$

But then $\Psi_*([Y_1, Y_2]) \neq [\Psi_*(Y_1), \Psi_*(Y_2)]$. □

Example V

Consider again the 7-dimensional Lie group G presented in Example I. We fix a left-invariant metric on G by letting $\{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$ be an orthonormal basis of $\mathfrak{g}$, where

$$\begin{aligned}E_1 &= X_1 - \tfrac{1}{2}X_2 - \tfrac{1}{4}Y_2,\\ E_2 &= X_2 - \tfrac{1}{4}Y_1,\\ E_3 &= Y_1,\\ E_4 &= Y_1 + Y_2,\end{aligned}$$

$$E_5 = Z_1,$$
$$E_6 = \tfrac{1}{2}Z_1 + Z_2,$$
$$E_7 = W.$$

Let Φ be the automorphism of G defined on the Lie algebra level by

$$\begin{aligned}
X_1 &\to -X_1 + X_2 + \tfrac{1}{4}Y_1 + \tfrac{1}{2}Y_2,\\
X_2 &\to X_2 - \tfrac{1}{2}Y_1 + \tfrac{1}{4}Z_1,\\
Y_1 &\to -Y_1,\\
Y_2 &\to 2Y_1 + Y_2 + Z_2,\\
Z_1 &\to Z_1 + \tfrac{1}{2}W,\\
Z_2 &\to -Z_1 - Z_2 + \tfrac{1}{4}W,\\
W &\to -W.
\end{aligned}$$

A straightforward calculation shows that $\Phi_*([U, V]) = [\Phi_*(U), \Phi_*(V)]$ for all U, V in $\mathfrak{g}$. Thus Φ is indeed a Lie group automorphism.

Let Γ_1 be the cocompact, discrete subgroup of G generated by

$$\{\exp(2X_1), \exp(2X_2), \exp(Y_1), \exp(Y_2), \exp(Z_1), \exp(Z_2), \exp(W)\},$$

and let $\Gamma_2 = \Phi(\Gamma_1)$. Note that $\Gamma_1 \cap Z(G) = \Gamma_2 \cap Z(G) = \{\exp(jW) : j \in \mathbf{Z}\}$.

Let $\bar{\Phi}$ be the projection of Φ onto $\bar{G}$. Then $\bar{\Phi}$ factors as $\bar{\Phi} = \Psi_1 \circ \Psi_2$, where Ψ_1 is the automorphism of $\bar{G}$ given on the Lie algebra level by

$$\begin{aligned}
\bar{X}_1 &\to -\bar{X}_1 + \bar{X}_2 + \tfrac{1}{4}\bar{Y}_1 + \tfrac{1}{2}\bar{Y}_2,\\
\bar{X}_2 &\to \bar{X}_2 - \tfrac{1}{2}\bar{Y}_1,\\
\bar{Y}_1 &\to -\bar{Y}_1,\\
\bar{Y}_2 &\to 2\bar{Y}_1 + \bar{Y}_2,\\
\bar{Z}_1 &\to \bar{Z}_1,\\
\bar{Z}_2 &\to -\bar{Z}_1 - \bar{Z}_2,
\end{aligned}$$

and where Ψ_2 is the automorphism of $\bar{G}$ given on the Lie algebra level by

$$\begin{aligned}
\bar{X}_1 &\to \bar{X}_1,\\
\bar{X}_2 &\to \bar{X}_2 + \tfrac{1}{4}\bar{Z}_1,\\
\bar{Y}_1 &\to \bar{Y}_1,\\
\bar{Y}_2 &\to \bar{Y}_2 - \bar{Z}_1 - \bar{Z}_2,\\
\bar{Z}_1 &\to \bar{Z}_1,\\
\bar{Z}_2 &\to \bar{Z}_2.
\end{aligned}$$

By rewriting Ψ_1 in terms of the orthonormal basis $\{\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4, \bar{E}_5, \bar{E}_6\}$ of $\bar{\mathfrak{g}}$, one easily sees that $\Psi_1(\bar{E}_i) = \pm E_i$ for $i = 1, \dots, 6$. Thus the automorphism Ψ_1 is also an isometry of $\bar{G}$ and must preserve the spectrum. A simple calculation shows that Ψ_2 is an almost inner automorphism of $\bar{G}$, which by Theorem 4.2 also preserves the spectrum. Thus $\operatorname{spec}(\bar{\Gamma}_1\backslash\bar{G}, \bar{g}) = \operatorname{spec}(\bar{\Gamma}_2\backslash\bar{G}, \bar{g})$. By Theorem 3.2, $\operatorname{spec}(\Gamma_1\backslash G, g) = \operatorname{spec}(\Gamma_2\backslash G, g)$.

PROPOSITION 4.11. *The manifolds $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$ are not isospectral on 1-forms.*

The proof of Proposition 4.11 is left to the Appendix.

REMARK. We will show in [Gt4] that the automorphism Φ marks the length spectrum of these examples. This is the first example of a pair of isospectral manifolds with the same marked length spectrum but not the same spectrum on 1-forms.

Appendix: Comparing the p-Form Spectrum of Nilmanifolds

In this Appendix, we show that the pairs of isospectral manifolds in Examples III, IV, and V are not isospectral on 1-forms.

Recall that on smooth p-forms, the Laplace–Beltrami operator is defined as

$$\Delta = d\delta + \delta d.$$

Here δ is the metric adjoint of d. Equivalently, $\delta = (-1)^{n(p+1)+1} * d*$, where $*$ is the Hodge-$*$ operator. Let $E^p(M)$ denote the exterior algebra of smooth differential p-forms on M. Then, for $f \in C^\infty(M)$ and $\tau \in E^p(M)$, Gordon and Wilson [GW1] showed that

$$\Delta(f\tau) = (\Delta f)\tau + f(\Delta\tau) - 2\nabla_{\mathrm{grad}\, f}\tau. \tag{A.1}$$

For G a simply connected Lie group with cocompact, discrete subgroup Γ, view $E^p(\Gamma\backslash G)$ as

$$E^p(\Gamma\backslash G) = C^\infty(\Gamma\backslash G) \otimes \Lambda^p(\mathfrak{g}^*).$$

Here elements of $\Lambda^p(\mathfrak{g}^*)$ are viewed as left-invariant p-forms of G and also as elements of $E^p(\Gamma\backslash G)$.

PROPOSITION 4.7. *The nilmanifolds $(\Gamma_1\backslash G, g)$ and $(\Gamma_2\backslash G, g)$ as presented in Example III are not isospectral on 1-forms.*

Outline of Proof. In Step 1 we decompose 1-spec$(\Gamma_i\backslash G, g)$ into four components:

$$\text{1-spec}^{\mathrm{I}}(\Gamma_i\backslash G, g) \cup \text{1-spec}^{\mathrm{II}}(\Gamma_i\backslash G, g) \cup \text{1-spec}^{\mathrm{III}}(\Gamma_i\backslash G, g) \cup \text{1-spec}^{\mathrm{IV}}(\Gamma_i\backslash G, g).$$

The multiplicity of an eigenvalue in 1-spec$(\Gamma_i\backslash G, g)$ is the sum of its multiplicities in each of the four components. In Step 2, using representation theory, we show that

$$\text{1-spec}^{\mathrm{IV}}(\Gamma_1\backslash G, g) = \text{1-spec}^{\mathrm{IV}}(\Gamma_2\backslash G, g)$$

and

$$\text{1-spec}^{\mathrm{III}}(\Gamma_1\backslash G, g) = \text{1-spec}^{\mathrm{III}}(\Gamma_2\backslash G, g).$$

In Step 3 we show that the (complex) multiplicity of every eigenvalue in 1-spec$^{\mathrm{II}}(\Gamma_i\backslash G, g)$ is congruent to 0 modulo 4. Finally, in Step 4 we show that the eigenvalue π^2+1 does not occur in 1-spec$^{\mathrm{I}}(\Gamma_1\backslash G, g)$ but does occur with (complex) multiplicity 2 in 1-spec$^{\mathrm{I}}(\Gamma_2\backslash G, g)$.

Proof of Proposition 4.7.

Step 1: Using the notation of Section 2, for $i=1,2$ let $\mathfrak{I}_i$ be a subset of $\mathfrak{g}^*$ such that

$$L^2(\Gamma_i\backslash G) \cong \bigoplus_{\tau\in\mathfrak{I}_i} m_i(\tau)\mathcal{H}_\tau.$$

Let $\mathfrak{I}_i = \mathfrak{I}_i^{\mathrm{I}} \cup \mathfrak{I}_i^{\mathrm{II}} \cup \mathfrak{I}_i^{\mathrm{III}} \cup \mathfrak{I}_i^{\mathrm{IV}}$ where

$$\begin{aligned}
\mathfrak{I}_i^{\mathrm{I}} &= \{\tau\in\mathfrak{I}_i\colon \tau(Z_1)=\tau(Z_2)=\tau(W)=0\},\\
\mathfrak{I}_i^{\mathrm{II}} &= \{\tau\in\mathfrak{I}_i\colon \tau(Z_2)\neq 0,\ \tau(Z_1)=\tau(W)=0\},\\
\mathfrak{I}_i^{\mathrm{III}} &= \{\tau\in\mathfrak{I}_i\colon \tau(Z_1)\neq 0,\ \tau(W)=0\},\\
\mathfrak{I}_i^{\mathrm{IV}} &= \{\tau\in\mathfrak{I}_i\colon \tau(W)\neq 0\}.
\end{aligned}$$

Let

$$\mathcal{H}_i^{\mathrm{I}} = \bigoplus_{\tau\in\mathfrak{I}_i^{\mathrm{I}}} m_i(\tau)\mathcal{H}_\tau, \qquad \mathcal{H}_i^{\mathrm{II}} = \bigoplus_{\tau\in\mathfrak{I}_i^{\mathrm{II}}} m_i(\tau)\mathcal{H}_\tau,$$

$$\mathcal{H}_i^{\mathrm{III}} = \bigoplus_{\tau\in\mathfrak{I}_i^{\mathrm{III}}} m_i(\tau)\mathcal{H}_\tau, \qquad \mathcal{H}_i^{\mathrm{IV}} = \bigoplus_{\tau\in\mathfrak{I}_i^{\mathrm{IV}}} m_i(\hat{\tau})\mathcal{H}_\tau.$$

As representation spaces,

$$L^2(\Gamma_i\backslash G) = \mathcal{H}_i^{\mathrm{I}} \oplus \mathcal{H}_i^{\mathrm{II}} \oplus \mathcal{H}_i^{\mathrm{III}} \oplus \mathcal{H}_i^{\mathrm{IV}}.$$

Decompose 1-spec$(\Gamma_i\backslash G, g)$ as

$$\text{1-spec}^{\mathrm{I}}(\Gamma_i\backslash G, g) \cup \text{1-spec}^{\mathrm{II}}(\Gamma_i\backslash G, g) \cup \text{1-spec}^{\mathrm{III}}(\Gamma_i\backslash G, g) \cup \text{1-spec}^{\mathrm{IV}}(\Gamma_i\backslash G, g),$$

where 1-spec$^{\mathrm{I}}(\Gamma_i\backslash G, g)$ is defined as the spectrum of the Laplacian acting on $\mathcal{H}_i^{\mathrm{I}} \otimes \Lambda^1(\mathfrak{g}^*)$. Define 1-spec$^{\mathrm{II}}(\Gamma_i\backslash G, g)$, 1-spec$^{\mathrm{III}}(\Gamma_i\backslash G, g)$, and 1-spec$^{\mathrm{IV}}(\Gamma_i\backslash G, g)$ similarly. Because the Laplace operator acting on functions acts through the representation, the multiplicity of an eigenvalue in 1-spec$(\Gamma_i\backslash G, g)$ is equal to the sum of its multiplicities in each of the four components.

Step 2: By Lemma 3.4, the representations of G on $\mathcal{H}_1^{\mathrm{IV}}$ and $\mathcal{H}_2^{\mathrm{IV}}$ are unitarily equivalent and

$$\text{1-spec}^{\mathrm{IV}}(\Gamma_1\backslash G, g) = \text{1-spec}^{\mathrm{IV}}(\Gamma_2\backslash G, g).$$

We will now show that the representations of G on $\mathcal{H}_1^{\mathrm{III}}$ and $\mathcal{H}_2^{\mathrm{III}}$ are unitarily equivalent.

The irreducible representations of G corresponding to elements of $\mathfrak{g}^*$ that are zero on the center may be viewed as irreducible representations of $\bar{G} = G/Z(G)$. It is easy to see that such representations of G are unitarily equivalent if and only if the corresponding representations of $\bar{G}$ are unitarily equivalent.

For all $\tau \in \mathfrak{Z}_i^{\mathrm{III}}$, $\tau(\mathfrak{z}) = 0$. We may thus use the following proposition to calculate the representations of G on $\mathcal{H}_1^{\mathrm{III}}$ and $\mathcal{H}_2^{\mathrm{III}}$.

PROPOSITION A.2 (cf. [P2]). *Let N be a simply connected, two-step nilpotent Lie group with Γ a cocompact, discrete subgroup of N. Let $\tau \in \mathfrak{N}^*$. Define $\mathfrak{N}_\tau = \{Y \in \mathfrak{N} : \tau([Y, \mathfrak{N}]) \equiv 0\}$. Then the irreducible representation π_τ appears in the quasi-regular representation of N on $L^2(\Gamma \backslash N)$ if and only if $\tau(\log \Gamma \cap \mathfrak{N}_\tau) \subset \mathbf{Z}$. The multiplicity of π_τ is 1 if $\tau(\mathfrak{N}^{(1)}) \equiv 0$. Define a nondegenerate, skew-symmetric bilinear form on $\mathfrak{N}/\mathfrak{N}_\tau$ by $B_\tau(U, V) = \tau([U, V])$ for all $U, V \in \mathfrak{N}/\mathfrak{N}_\tau$. Then, if $\tau(\mathfrak{N}^{(1)}) \not\equiv 0$, the multiplicity of π_τ is equal to $\sqrt{\det B_\tau}$, where the determinant is calculated with respect to any basis of $\mathcal{L}_\tau = \log \Gamma / (\log \Gamma \cap \mathfrak{N}_\tau)$.*

REMARK. The occurrence condition above actually follows directly from a more general occurrence and multiplicity theorem due independently to Richardson [R] and Howe [H].

Let $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \zeta_1, \zeta_2, \omega\}$ be the dual basis to the orthonormal basis $\{X_1, X_2, Y_1, Y_2, Z_1, Z_2, W\}$ of $\mathfrak{g}$. If $\tau \in \mathfrak{Z}_i^{\mathrm{III}}$, then

$$\tau = A_1\alpha_1 + A_2\alpha_2 + B_1\beta_1 + B_2\beta_2 + C_1\zeta_1 + C_2\zeta_2$$

for some $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathbf{R}$ with $C_1 \neq 0$.

Now $\bar{\mathfrak{g}}_\tau = \mathrm{span}_{\mathbf{R}}\{\bar{Z}_1, \bar{Z}_2\}$. Hence

$$\begin{aligned} \log \bar{\Gamma}_1 \cap \bar{\mathfrak{g}}_\tau &= \log \bar{\Gamma}_2 \cap \bar{\mathfrak{g}}_\tau \\ &= \log(\exp(\mathbf{Z}\bar{Z}_1)\exp(\mathbf{Z}\bar{Z}_2)) \\ &= \mathrm{span}_{\mathbf{Z}}\{\bar{Z}_1, \bar{Z}_2\}. \end{aligned}$$

Hence $\tau(\log \bar{\Gamma}_i \cap \bar{\mathfrak{g}}_\tau) \subset \mathbf{Z}$ if and only if $C_1 \in \mathbf{Z}$ and $C_2 \in \mathbf{Z}$. By Proposition A.2, we see that $\tau \in \mathfrak{Z}_i^{\mathrm{III}}$ if and only if $C_1 \in \mathbf{Z}$ and $C_2 \in \mathbf{Z}$. Moreover, distinct values of C_1 and C_2 determine distinct coadjoint orbits of $\mathfrak{g}^*$. Since these conditions are the same for both $\mathfrak{Z}_1^{\mathrm{III}}$ and $\mathfrak{Z}_2^{\mathrm{III}}$, we may assume $\mathfrak{Z}_1^{\mathrm{III}} = \mathfrak{Z}_2^{\mathrm{III}}$.

We now calculate multiplicities. A basis for $\mathcal{L}_{1_\tau} = \log \bar{\Gamma}_1 / (\log \bar{\Gamma}_1 \cap \bar{\mathfrak{g}}_\tau)$ is $\{2\bar{X}_1, 2\bar{X}_2, \bar{Y}_1, \bar{Y}_2\}$. A basis for $\mathcal{L}_{2_\tau} = \log \bar{\Gamma}_2 / (\log \bar{\Gamma}_2 \cap \bar{\mathfrak{g}}_\tau)$ is $\{\bar{X}_1, \bar{X}_2, 2\bar{Y}_1, 2\bar{Y}_2\}$. In both cases, $\sqrt{\det B_\tau} = 4C_1^2$. Thus for $\tau \in \mathfrak{Z}_1^{\mathrm{III}} = \mathfrak{Z}_2^{\mathrm{III}}$, the multiplicities $m_1(\tau)$ and $m_2(\tau)$ are equal. Hence the representations of $\bar{G}$ are unitarily equivalent, so the representations of G on $\mathcal{H}_1^{\mathrm{III}}$ and $\mathcal{H}_2^{\mathrm{III}}$ are unitarily equivalent.

By Proposition 2.2,

$$1\text{-spec}^{\mathrm{III}}(\Gamma_1 \backslash G, g) = 1\text{-spec}^{\mathrm{III}}(\Gamma_2 \backslash G, g),$$

as desired.

Step 3: We now show that for any eigenvalue in $1\text{-spec}^{\mathrm{II}}(\Gamma_i \backslash G, g)$, its multiplicity in $1\text{-spec}^{\mathrm{II}}(\Gamma_i \backslash G, g)$ is always congruent to 0 modulo 4. We first compute the multiplicity of the irreducible representations occurring here, using the same technique as in Step 2.

For all $\tau \in \mathfrak{Z}_i^{\mathrm{II}}$, $\tau = A_1\alpha_1 + A_2\alpha_2 + B_1\beta_1 + B_2\beta_2 + C_2\zeta_2$ for some $A_1, A_2, B_1, B_2, C_2 \in \mathbf{R}$ with $C_2 \neq 0$. We may again use Proposition A.2.

Now $\bar{\mathfrak{g}}_\tau = \text{span}_{\mathbf{R}}\{\bar{X}_2, \bar{Y}_1, \bar{Z}_1, \bar{Z}_2\}$. Hence

$$\begin{aligned}\log \bar{\Gamma}_1 \cap \bar{\mathfrak{g}}_\tau &= \log(\exp(2\mathbf{Z}\bar{X}_2)\exp(\mathbf{Z}\bar{Y}_1)\exp(\mathbf{Z}\bar{Z}_1)\exp(\mathbf{Z}\bar{Z}_2)) \\ &= \text{span}_{\mathbf{Z}}\{2\bar{X}_2, \bar{Y}_1, \bar{Z}_1, \bar{Z}_2\},\end{aligned}$$

and $\tau(\log \bar{\Gamma}_1 \cap \bar{\mathfrak{g}}_\tau) \subset \mathbf{Z}$ if and only if $A_2 \in \frac{1}{2}\mathbf{Z}$, $B_1 \in \mathbf{Z}$, and $C_2 \in \mathbf{Z}$. However,

$$\begin{aligned}\log \bar{\Gamma}_2 \cap \bar{\mathfrak{g}}_\tau &= \log(\exp(\mathbf{Z}\bar{X}_2)\exp(2\mathbf{Z}\bar{Y}_1)\exp(\mathbf{Z}\bar{Z}_1)\exp(\mathbf{Z}\bar{Z}_2)) \\ &= \text{span}_{\mathbf{Z}}\{\bar{X}_2, 2\bar{Y}_1, \bar{Z}_1, \bar{Z}_2\},\end{aligned}$$

so $\tau(\log \bar{\Gamma}_2 \cap \bar{\mathfrak{g}}_\tau) \subset \mathbf{Z}$ if and only if $A_2 \in \mathbf{Z}$, $B_1 \in \frac{1}{2}\mathbf{Z}$, and $C_2 \in \mathbf{Z}$.

We now calculate $m_i(\tau)$. A basis for $\mathcal{L}_{1_\tau} = \log \bar{\Gamma}_1/(\log \bar{\Gamma}_1 \cap \bar{\mathfrak{g}}_\tau)$ is $\{2\bar{X}_1, \bar{Y}_2\}$. A basis for $\mathcal{L}_{2_\tau} = \log \bar{\Gamma}_2/(\log \bar{\Gamma}_2 \cap \bar{\mathfrak{g}}_\tau)$ is $\{\bar{X}_1, 2\bar{Y}_2\}$. In both cases, $\sqrt{\det B_\tau} = 2|C_2|$.

Thus, if τ is in $\mathfrak{J}_i^{\text{II}}$, the irreducible representation π_τ occurs in the representation of G on $\mathcal{H}_i^{\text{II}}$ with multiplicity $2|C_2|$. Since the integer $C_2 \neq 0$, the multiplicity must be even. Hence, any eigenvalue of Δ acting on $\mathcal{H}_i^{\text{II}} \otimes \Lambda^1(\mathfrak{g}^*)$ must occur in 1-spec$^{\text{II}}(\Gamma_i \backslash G, g)$ with multiplicity congruent to 0 modulo 2.

We now use the following.

Proposition A.3 [G2]. *Let G be a simply connected Lie group with left-invariant metric g, and let Γ_1 and Γ_2 be cocompact, discrete subgroups of G. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be invariant subspaces of ρ_{Γ_1} and ρ_{Γ_2}, respectively. Denote by p-spec$'(\Gamma_i \backslash G, g)$ the spectrum of Δ restricted to acting on $\mathcal{H}_i \otimes \Lambda^p(\mathfrak{g}^*)$. If there exists an automorphism Φ of G such that*

(1) *Φ is also an isometry of (G, g) and*

(2) *ρ_{Γ_1} restricted to $\mathcal{H}_1$ and $\rho_{\Gamma_2} \circ \Phi$ restricted to $\mathcal{H}_2$ are unitarily equivalent,*

then

$$p\text{-spec}'(\Gamma_1 \backslash G, g) = p\text{-spec}'(\Gamma_2 \backslash G, g).$$

Here $\rho_{\Gamma_2} \circ \Phi$ is defined by $((\rho_{\Gamma_2} \circ \Phi)(x))f = f \circ R_{\Phi(x)}$ for x in G and f in $L^2(\Gamma_1 \backslash G)$.

Let Φ be the Lie group automorphism of G defined on the Lie algebra level by

$$\begin{aligned} X_1 &\to -X_1, \\ X_2 &\to X_2, \\ Y_1 &\to -Y_1, \\ Y_2 &\to Y_2, \\ Z_1 &\to Z_1, \\ Z_2 &\to -Z_2, \\ W &\to -W. \end{aligned}$$

The automorphism Φ is also an isometry of (G, g). Note that $\tau \circ \Phi_* = -A_1\alpha_1 + A_2\alpha_2 - B_1\beta_1 + B_2\beta_2 - C_2\zeta_2$. Clearly, if τ satisfies Condition (*) or (**), then so does $\tau \circ \Phi_*$. A straightforward calculation shows that since

$C_2 \neq 0$, the functionals τ and $\tau \circ \Phi_*$ are not in the same coadjoint orbit of $\mathfrak{g}^*$, so π_τ and $\pi_{\tau \circ \Phi_*}$ are not unitarily equivalent. Thus if π_τ occurs in $\mathcal{H}_i^{II}$ with multiplicity $2|C_2|$ then so does $\pi_{\tau \circ \Phi_*}$, also with multiplicity $2|C_2|$.

Note that by (2.3), $\pi_{\tau \circ \Phi_*} = \pi_\tau \circ \Phi$. Using Proposition A.3, any eigenvalue of Δ acting on $\mathcal{H}_\tau \otimes \Lambda^1(\mathfrak{g}^*)$ must also occur as an eigenvalue of Δ acting on $\mathcal{H}_{\tau \circ \Phi_*} \otimes \Lambda^1(\mathfrak{g}^*)$. Moreover, each of the representation spaces $\mathcal{H}_\tau$ and $\mathcal{H}_{\tau \circ \Phi_*}$ occurs in $\mathcal{H}_i^{II}$ with multiplicity $2|C_2|$. Consequently, the multiplicity of any eigenvalue in $\text{spec}^{II}(\Gamma_i \backslash G, g)$ is a multiple of $4|C_2|$, which is clearly congruent to 0 modulo 4, as desired.

Step 4: We now show that the eigenvalue $\pi^2 + 1$ does not occur in $1\text{-spec}^{I}(\Gamma_1 \backslash G, g)$ but does occur with multiplicity 2 in $1\text{-spec}^{I}(\Gamma_2 \backslash G, g)$.

For $\tau \in \mathfrak{J}_1^I$ or $\tau \in \mathfrak{J}_2^I$, $\tau(\mathfrak{g}^{(1)}) \equiv 0$. We again use Proposition A.2 to calculate the irreducible representations occurring here. We write $\tau = A_1\alpha_1 + A_2\alpha_2 + B_1\beta_1 + B_2\beta_2$ for some $A_1, A_2, B_1, B_2 \in \mathbf{R}$.

Now $\bar{\mathfrak{g}}_\tau = \bar{\mathfrak{g}}$, so $\tau(\log \bar{\Gamma}_i \cap \bar{\mathfrak{g}}_\tau) \subset \mathbf{Z}$ if and only if $\tau(\log \bar{\Gamma}_i) \subset \mathbf{Z}$. Thus, $\tau \in \mathfrak{J}_1^I$ if and only if

$$A_1, A_2 \in \tfrac{1}{2}\mathbf{Z} \quad \text{and} \quad B_1, B_2 \in \mathbf{Z}, \tag{$*$}$$

and $\tau \in \mathfrak{J}_2^I$ if and only if

$$A_1, A_2 \in \mathbf{Z} \quad \text{and} \quad B_1, B_2 \in \tfrac{1}{2}\mathbf{Z}. \tag{$**$}$$

Let $\mathcal{H}_\tau$ be the associated representation space of π_τ. Then $\mathcal{H}_\tau$ may be viewed as the 1-dimensional subspace of $L^2(\Gamma_i \backslash G)$ generated by

$$\begin{aligned} &F_\tau(\exp(x_1X_1)\exp(x_2X_2)\exp(y_1Y_1)\exp(y_2Y_2)\exp(z_1Z_1)\exp(z_2Z_2)\exp(wW)) \\ &\quad = \exp\{2\pi i\tau(x_1X_1 + x_2X_2 + y_1Y_1 + y_2Y_2)\} \end{aligned}$$

(see Section 2). That is, $\mathcal{H}_\tau = \mathbf{C}F_\tau$.

We now calculate Δ acting on $\mathcal{H}_\tau \otimes \Lambda^1(\mathfrak{g}^*)$. Note that if we let $\mathbf{C}\Lambda^1(\mathfrak{g}^*)$ denote $\Lambda^1(\mathfrak{g}^*)$ with complex coefficients, then

$$\mathcal{H}_\tau \otimes \Lambda^1(\mathfrak{g}^*) = F_\tau \otimes \mathbf{C}\Lambda^1(\mathfrak{g}^*).$$

Let $F_\tau \otimes \mu \in F_\tau \otimes \mathbf{C}\Lambda^1(\mathfrak{g}^*)$. Then

$$\mu = a_1\alpha_1 + a_2\alpha_2 + b_1\beta_1 + b_2\beta_2 + z_1\zeta_1 + z_2\zeta_2 + w\omega$$

for some $a_1, a_2, b_1, b_2, z_1, z_2, w \in \mathbf{C}$. Since F_τ is independent of z_1, z_2, and w we have

$$\begin{aligned} \Delta F_\tau &= -X_1^2F_\tau - X_2^2F_\tau - Y_1^2F_\tau - Y_2^2F_\tau \\ &= 4\pi^2(A_1^2 + A_2^2 + B_1^2 + B_2^2)F_\tau \\ &= 4\pi^2S^2F_\tau, \end{aligned}$$

where $S^2 = A_1^2 + A_2^2 + B_1^2 + B_2^2$.

Let $*$ denote the Hodge-$*$ operator. One easily sees that $d * \mu = 0$ for all $\mu \in \mathfrak{g}^*$. Hence $\delta\mu = \pm * d * \mu = 0$ for all $\mu \in \mathfrak{g}^*$. Consequently $\Delta = \delta d$ on $\Lambda^1(\mathfrak{g}^*)$.

For $\mu \in \Lambda^1(\mathfrak{g}^*)$, $d\mu(U,V) = -\mu([U,V])$ for all $U, V \in \mathfrak{g}$. Using this fact together with the definition of δ as the metric adjoint of d, one easily computes that $\Delta\alpha_1 = \Delta\alpha_2 = \Delta\beta_1 = \Delta\beta_2 = 0$, $\Delta\zeta_1 = 2\zeta_1$, $\Delta\zeta_2 = \zeta_2$, and $\Delta\omega = 3\omega$.

To calculate $\Delta(F_\tau \otimes \mu)$, it remains to calculate

$$\nabla_{\operatorname{grad} F_\tau}\mu = 2\pi i F_\tau(A_1\nabla_{X_1}\mu + A_2\nabla_{X_2}\mu + B_1\nabla_{Y_1}\mu + B_2\nabla_{Y_2}\mu).$$

For Lie algebras with a left-invariant metric, the covariant derivatives can be calculated via the following equation:

$$\langle \nabla_U V, U'\rangle = \tfrac{1}{2}\langle [U', U], V\rangle + \tfrac{1}{2}\langle [U', V], U\rangle + \tfrac{1}{2}\langle [U, V], U'\rangle,$$

for U, V, U' left-invariant vector fields of $\mathfrak{g}$. A simple calculation shows that $\nabla_U(V^b) = (\nabla_U V)^b$, where V^b denotes the dual of V in $\mathfrak{g}^*$ with respect to our choice of orthonormal basis; that is, $V^b(U) = \langle V, U\rangle$ for all U in $\mathfrak{g}^*$. We thus obtain Table II.

Table II

$\nabla_U \mu$	α_1	α_2	β_1	β_2	ζ_1	ζ_2	ω
X_1	0	0	$\frac{1}{2}\zeta_1$	$\frac{1}{2}\zeta_2$	$-\frac{1}{2}\beta_1+\frac{1}{2}\omega$	$-\frac{1}{2}\beta_2$	$-\frac{1}{2}\zeta_1$
X_2	0	0	0	$\frac{1}{2}\zeta_1$	$-\frac{1}{2}\beta_2$	$\frac{1}{2}\omega$	$-\frac{1}{2}\zeta_2$
Y_1	$-\frac{1}{2}\zeta_1$	0	0	$\frac{1}{2}\omega$	$\frac{1}{2}\alpha_1$	0	$-\frac{1}{2}\beta_2$
Y_2	$-\frac{1}{2}\zeta_2$	$-\frac{1}{2}\zeta_1$	$-\frac{1}{2}\omega$	0	$\frac{1}{2}\alpha_2$	$\frac{1}{2}\alpha_1$	$\frac{1}{2}\beta_1$

Using (A.1) and the information from Table II, a straightforward calculation shows that if we let E_τ equal

$$\begin{pmatrix} 4\pi^2S^2 & 0 & 0 & 0 & -2\pi iB_1 & -2\pi iB_2 & 0 \\ 0 & 4\pi^2S^2 & 0 & 0 & -2\pi iB_2 & 0 & 0 \\ 0 & 0 & 4\pi^2S^2 & 0 & 2\pi iA_1 & 0 & -2\pi iB_2 \\ 0 & 0 & 0 & 4\pi^2S^2 & 2\pi iA_2 & 2\pi iA_1 & 2\pi iB_1 \\ 2\pi iB_1 & 2\pi iB_2 & -2\pi iA_1 & -2\pi iA_2 & 4\pi^2S^2+2 & 0 & 2\pi iA_1 \\ 2\pi iB_2 & 0 & 0 & -2\pi iA_1 & 0 & 4\pi^2S^2+1 & 2\pi iA_2 \\ 0 & 0 & 2\pi iB_2 & -2\pi iB_1 & -2\pi iA_1 & -2\pi iA_2 & 4\pi^2S^2+3 \end{pmatrix}$$

then $\Delta(F_\tau \otimes \mu) = \lambda(F_\tau \otimes \mu)$ if and only if λ is an eigenvalue of the matrix E_τ.

We now calculate necessary conditions on $\tau = A_1\alpha_1 + A_2\alpha_2 + B_1\beta_1 + B_2\beta_2$ for π^2+1 to be an eigenvalue of E_τ. Since $\tau \in \mathfrak{Z}_1^1$ or $\tau \in \mathfrak{Z}_2^1$, we know $A_1, A_2, B_1, B_2 \in \mathbf{Q}$. If $\det(E_\tau - (\pi^2+1)I_7) = 0$, then π is the root of a polynomial with rational coefficients. However, π is transcendental. Thus, the coefficients of the powers of π must be zero.

A straightforward calculation shows that π^{14} is the highest power of π occurring in the polynomial, and the coefficient of π^{14} is equal to $(4S^2-1)^7$. Thus, if π^2+1 is an eigenvalue of E_τ then $S^2=\frac{1}{4}$. Recall that

$$S^2 = A_1^2+A_2^2+B_1^2+B_2^2.$$

For τ in $\mathfrak{Z}_1^{\mathrm{I}}$, $S^2=\frac{1}{4}$ if and only if (see (*)) $\tau=\pm\frac{1}{2}\alpha_1$ or $\tau=\pm\frac{1}{2}\alpha_2$. For τ in $\mathfrak{Z}_2^{\mathrm{I}}$, $S^2=\frac{1}{4}$ if and only if (see (**)) $\tau=\pm\frac{1}{2}\beta_1$ or $\tau=\pm\frac{1}{2}\beta_2$.

For $\tau=\pm\frac{1}{2}\alpha_1$, $\tau=\pm\frac{1}{2}\alpha_2$, or $\tau=\pm\frac{1}{2}\beta_2$, a simple calculation shows that $\det(E_\tau-(\pi^2+1)I_7)\neq 0$. Thus, the eigenvalue π^2+1 does not arise from the Laplacian acting on $\mathcal{H}_{\pm\frac{1}{2}\alpha_1}\otimes\Lambda^1(\mathfrak{g}^*)$, $\mathcal{H}_{\pm\frac{1}{2}\alpha_2}\otimes\Lambda^1(\mathfrak{g}^*)$, or $\mathcal{H}_{\pm\frac{1}{2}\beta_2}\otimes\Lambda^1(\mathfrak{g}^*)$, and

$$\pi^2+1\notin \text{1-spec}^{\mathrm{I}}(\Gamma_1\backslash G, g).$$

However, for $\tau=\pm\frac{1}{2}\beta_1$, $\det(E_\tau-(\pi^2+1)I_7)=0$. Thus π^2+1 is an eigenvalue for the Laplacian acting on $\mathcal{H}_{\pm\frac{1}{2}\beta_1}\otimes\Lambda^1(\mathfrak{g}^*)$. Indeed, the eigenspace of π^2+1 in $\mathcal{H}^{\mathrm{I}}\otimes\Lambda^1(\mathfrak{g}^*)$ is

$$\operatorname{span}_{\mathbf{C}}\{F_{\frac{1}{2}\beta_1}\otimes\zeta_2, F_{-\frac{1}{2}\beta_1}\otimes\zeta_2\},$$

which has dimension 2. Thus $\pi^2+1\in\text{1-spec}^{\mathrm{I}}(\Gamma_2\backslash G, g)$ with multiplicity 2, as desired.

The proof of Proposition 4.7 is now complete. □

PROPOSITION 4.9. *The nilmanifolds* $(\Gamma_1\backslash G, g)$ *and* $(\Gamma_2\backslash G, g)$ *as presented in Example IV are not isospectral on* 1*-forms.*

Proof. Again using the notation of Section 2, for $i=1,2$ let $\mathfrak{Z}_i$ be a subset of $\mathfrak{g}^*$ such that

$$L^2(\Gamma_i\backslash G)\cong\bigoplus_{\tau\in\mathfrak{Z}_i} m_i(\tau)\mathcal{H}_\tau.$$

Let $\mathfrak{Z}_i=\mathfrak{Z}_i'\cup\mathfrak{Z}_i''\cup\mathfrak{Z}_i'''$, where

$$\begin{aligned}\mathfrak{Z}_i' &= \{\tau\in\mathfrak{Z}_i : \tau(\mathfrak{g}^{(1)})\equiv 0\},\\ \mathfrak{Z}_i'' &= \{\tau\in\mathfrak{Z}_i : \tau(\mathfrak{g}^{(2)})\equiv 0,\ \tau(\mathfrak{g}^{(1)})\not\equiv 0\},\\ \mathfrak{Z}_i''' &= \{\tau\in\mathfrak{Z}_i : \tau(\mathfrak{g}^{(2)})\not\equiv 0\}.\end{aligned}$$

As in the proof of Proposition 4.7, decompose the representation spaces and spectrum accordingly.

By Lemma 3.4, the representations of G on $\mathcal{H}_1'''$ and $\mathcal{H}_2'''$ are unitarily equivalent and

$$\text{1-spec}'''(\Gamma_1\backslash G, g)=\text{1-spec}'''(\Gamma_2\backslash G, g).$$

A calculation almost identical to Step 2 in the proof of Proposition 4.7 shows that the representations of G on $\mathcal{H}_1''$ and $\mathcal{H}_2''$ are unitarily equivalent. Thus

$$\text{1-spec}''(\Gamma_1\backslash G, g)=\text{1-spec}''(\Gamma_2\backslash G, g).$$

It remains to show that $\text{1-spec}'(\Gamma_1\backslash G, g)\neq\text{1-spec}'(\Gamma_2\backslash G, g)$. The proof of this corresponds to Step 4 in the proof of Proposition 4.7.

Let $\{\alpha_1, \beta_1, \beta_2, \gamma, \omega\}$ be the dual to the orthonormal basis $\{X_1, Y_1, Y_2, Z, W\}$ given in Example IV. For $\tau \in \mathfrak{Z}_i'$, $\tau = A_1\alpha_1 + B_1\beta_1 + B_2\beta_2$ for some $A_1, B_1, B_2 \in \mathbf{R}$. We again use Proposition A.2.

Now $\bar{\mathfrak{g}}_\tau = \bar{\mathfrak{g}}$. Hence $\tau(\log \bar{\Gamma}_i \cap \bar{\mathfrak{g}}_\tau) \subset \mathbf{Z}$ if and only if $\tau(\log \bar{\Gamma}_i) \subset \mathbf{Z}$. Thus, $\tau \in \mathfrak{Z}_1^{\mathrm{I}}$ if and only if

$$A_1 \in \tfrac{1}{2}\mathbf{Z} \quad \text{and} \quad B_1, B_2 \in \mathbf{Z}, \tag{*}$$

and $\tau \in \mathfrak{Z}_2^{\mathrm{I}}$ if and only if

$$A_1, B_2 \in \mathbf{Z} \quad \text{and} \quad B_1 \in \tfrac{1}{2}\mathbf{Z}. \tag{**}$$

Let $\mathcal{H}_\tau$ be the associated representation space of π_τ. As in the proof of Proposition 4.7, $\mathcal{H}_\tau \otimes \Lambda^1(\mathfrak{g}^*) = F_\tau \otimes \mathbf{C}\Lambda^1(\mathfrak{g}^*)$, where

$$\begin{aligned} &F_\tau(\exp(x_1X_1)\exp(y_1Y_1)\exp(y_2Y_2)\exp(zZ)\exp(wW)) \\ &\quad = \exp\{2\pi i\tau(x_1X_1 + y_1Y_1 + y_2Y_2)\}. \end{aligned}$$

Let $F_\tau \otimes \mu \in F_\tau \otimes \mathbf{C}\Lambda^1(\mathfrak{g}^*)$ with $\mu = a_1\alpha_1 + b_1\beta_1 + b_2\beta_2 + z\zeta + w\omega$ for some $a_1, b_1, b_2, z, w \in \mathbf{C}$. Then $\Delta F_\tau = 4\pi^2S^2F_\tau$, where $S^2 = A_1^2 + B_1^2 + B_2^2$. Also, $\Delta\alpha_1 = \Delta\beta_1 = \Delta\beta_2 = 0$, $\Delta\zeta = \zeta$, and $\Delta\omega = 2\omega$. We thus obtain Table III.

Table III

$\nabla_U\mu$	α_1	β_1	β_2	ζ	ω
X_1	0	$\frac{1}{2}\zeta$	0	$-\frac{1}{2}\beta_1 + \frac{1}{2}\omega$	$-\frac{1}{2}\zeta$
Y_1	$-\frac{1}{2}\zeta$	0	$\frac{1}{2}\omega$	$\frac{1}{2}\alpha_1$	$-\frac{1}{2}\beta_2$
Y_2	0	$-\frac{1}{2}\omega$	0	0	$\frac{1}{2}\beta_1$

Using (A.1) and the information from Table III, if we let

$$E_\tau = \begin{bmatrix} 4\pi^2S^2 & 0 & 0 & -2\pi iB_1 & 0 \\ 0 & 4\pi^2S^2 & 0 & 2\pi iA_1 & -2\pi iB_2 \\ 0 & 0 & 4\pi^2S^2 & 0 & 2\pi iB_1 \\ 2\pi iB_1 & -2\pi iA_1 & 0 & 4\pi^2S^2+1 & 2\pi iA_1 \\ 0 & 2\pi iB_2 & -2\pi iB_1 & -2\pi iA_1 & 4\pi^2S^2+2 \end{bmatrix}$$

then $\Delta(F_\tau \otimes \mu) = \lambda(F_\tau \otimes \mu)$ if and only if λ is an eigenvalue of the matrix E_τ.

We calculate necessary conditions on $\tau = A_1\alpha_1 + B_1\beta_1 + B_2\beta_2$ for $\pi^2 + 1$ to be an eigenvalue of E_τ. We assumed $\tau \in \mathfrak{Z}_1'$ or $\tau \in \mathfrak{Z}_2'$, so we know that $A_1, B_1, B_2 \in \mathbf{Q}$. If $\det(E_\tau - (\pi^2 + 1)I_5) = 0$, then π is the root of a polynomial with rational coefficients. However, π is transcendental. Thus, the coefficients of the powers of π must be zero.

A simple calculation shows that π^{10} is the highest power of π occurring in the polynomial, and the coefficient of π^{10} is equal to $(4S^2-1)^5$. Thus, if π^2+1 is an eigenvalue of E_τ then $S^2 = \frac{1}{4}$. Recall that $S^2 = A_1^2 + B_1^2 + B_2^2$.

For τ in $\mathfrak{J}_1'$, $S^2=\frac{1}{4}$ if and only if (see (*)) $\tau=\pm\frac{1}{2}\alpha_1$. For τ in $\mathfrak{J}_2'$, $S^2=\frac{1}{4}$ if and only if (see (**)) $\tau=\pm\frac{1}{2}\beta_1$.

For $\tau=\pm\frac{1}{2}\alpha_1$, the determinant of $(E_\tau-(\pi^2+1)I_5)$ is 0. The eigenspace of π^2+1 in $\mathcal{H}'\otimes\Lambda^1(\mathfrak{g}^*)$ is

$$\text{span}_{\mathbf{C}}\{F_{\frac{1}{2}\alpha_1}\otimes(\pi i\beta_1+\zeta+\pi i\omega),\, F_{-\frac{1}{2}\alpha_1}\otimes(\pi i\beta_1-\zeta+\pi i\omega)\},$$

which has dimension 2. Thus $\pi^2+1\in 1\text{-spec}'(\Gamma_1\backslash G,g)$ with multiplicity 2. However, for $\tau=\pm\frac{1}{2}\beta_1$, $\det(E_\tau-(\pi^2+1)I_5)\neq 0$. Thus π^2+1 cannot occur in $1\text{-spec}'(\Gamma_2\backslash G,g)$ and

$$1\text{-spec}'(\Gamma_1\backslash G,g)\neq 1\text{-spec}'(\Gamma_2\backslash G,g),$$

as desired.

The proof of Proposition 4.9 is now complete. □

PROPOSITION 4.11. *The nilmanifolds $(\Gamma_1\backslash G,g)$ and $(\Gamma_2\backslash G,g)$ as presented in Example V are not isospectral on 1-forms.*

Proof. Using the notation of Section 2, for $i=1,2$ let $\mathfrak{J}_i$ be a subset of $\mathfrak{g}^*$ such that

$$L^2(\Gamma_i\backslash G)\cong\bigoplus_{\tau\in\mathfrak{J}_i} m_i(\tau)\mathcal{H}_\tau.$$

Let $\mathfrak{J}_i=\mathfrak{J}_i^{\text{I}}\cup\mathfrak{J}_i^{\text{II}}\cup\mathfrak{J}_i^{\text{III}}\cup\mathfrak{J}_i^{\text{IV}}$ be defined as in the proof of Proposition 4.7, and decompose the representation spaces and spectrum accordingly.

By Lemma 3.4, the representations of G on $\mathcal{H}_1^{\text{IV}}$ and $\mathcal{H}_2^{\text{IV}}$ are unitarily equivalent and

$$1\text{-spec}^{\text{IV}}(\Gamma_1\backslash G,g)=1\text{-spec}^{\text{IV}}(\Gamma_2\backslash G,g).$$

A calculation almost identical to that in Step 2 in the proof of Proposition 4.7 shows that the representations of G on $\mathcal{H}_1^{\text{III}}$ and $\mathcal{H}_2^{\text{III}}$ are unitarily equivalent. Thus

$$1\text{-spec}^{\text{III}}(\Gamma_1\backslash G,g)=1\text{-spec}^{\text{III}}(\Gamma_2\backslash G,g).$$

A calculation almost identical to that in Step 3 in the proof of Proposition 4.7 shows that the representations of G on $\mathcal{H}_1^{\text{II}}$ and $\mathcal{H}_2^{\text{II}}$ are unitarily equivalent. Thus

$$1\text{-spec}^{\text{II}}(\Gamma_1\backslash G,g)=1\text{-spec}^{\text{II}}(\Gamma_2\backslash G,g).$$

It remains to show that $1\text{-spec}^{\text{I}}(\Gamma_1\backslash G,g)\neq 1\text{-spec}^{\text{I}}(\Gamma_1\backslash G,g)$.

For $\tau\in\mathfrak{J}_1^{\text{I}}$ or $\tau\in\mathfrak{J}_2^{\text{I}}$, $\tau(\mathfrak{g}^{(1)})\equiv 0$. We again use Proposition A.2 to calculate the irreducible representations occurring here. Let $\{\epsilon_1,\dots,\epsilon_7\}$ be the dual to the orthonormal basis $\{E_1,\dots,E_7\}$ given in Example V. Then $\tau=A_1\epsilon_1+A_2\epsilon_2+A_3\epsilon_3+A_4\epsilon_4$ for some $A_1,\dots,A_4\in\mathbf{R}$.

Now $\bar{\mathfrak{g}}_\tau=\bar{\mathfrak{g}}$. Hence $\tau(\log\bar{\Gamma}_i\cap\bar{\mathfrak{g}}_\tau)\subset\mathbf{Z}$ if and only if $\tau(\log\bar{\Gamma}_i)\subset\mathbf{Z}$. Thus, $\tau\in\mathfrak{J}_1^{\text{I}}$ if and only if

$$2A_1+A_2-\tfrac{1}{4}A_3+\tfrac{1}{2}A_4\in\mathbf{Z},\quad 2A_2+\tfrac{1}{2}A_3\in\mathbf{Z},\quad\text{and}\quad A_3,A_4\in\mathbf{Z},\qquad (*)$$

and $\tau \in \mathfrak{J}_2^{\mathrm{I}}$ if and only if

$$2A_1+A_2+\tfrac{1}{4}A_3+\tfrac{1}{2}A_4 \in \mathbf{Z}, \quad 2A_2+\tfrac{1}{2}A_3 \in \mathbf{Z}, \quad \text{and} \quad A_3, A_4 \in \mathbf{Z}. \qquad (**)$$

Note that the only distinction between the two conditions is in the sign of $\frac{1}{4}A_3$.

Let $\mathcal{H}_\tau$ be the associated representation space of π_τ. As in the proof of Proposition 4.7, $\mathcal{H}_\tau \otimes \Lambda^1(\mathfrak{g}^*) = F_\tau \otimes \mathbf{C}\Lambda^1(\mathfrak{g}^*)$, where

$$\begin{aligned} &F_\tau(\exp(e_1E_1)\exp(e_2E_2)\exp(e_3E_3)\exp(e_4E_4)\exp(e_5E_5)\exp(e_6E_6)\exp(e_7E_7)) \\ &\quad = \exp\{2\pi i\tau(e_1E_1+e_2E_2+e_3E_3+e_4E_4)\}. \end{aligned}$$

Let $F_\tau \otimes \mu \in F_\tau \otimes \mathbf{C}\Lambda^1(\mathfrak{g}^*)$ with

$$\mu = a_1\epsilon_1+a_2\epsilon_2+a_3\epsilon_3+a_4\epsilon_4+a_5\epsilon_5+a_6\epsilon_6+a_7\epsilon_7$$

for some $a_i \in \mathbf{C}$, $i=1,\ldots,7$. As in the proof of Proposition 4.7, $\Delta F_\tau = 4\pi^2S^2F_\tau$, where $S^2 = A_1^2+A_2^2+A_3^2+A_4^2$. Also, $\Delta\epsilon_1 = \Delta\epsilon_2 = \Delta\epsilon_3 = \Delta\epsilon_4 = 0$, $\Delta\epsilon_5 = 2\epsilon_5$, $\Delta\epsilon_6 = \epsilon_6+\frac{1}{4}\epsilon_7$, and

$$\Delta\epsilon_7 = \frac{1}{4}\epsilon_6+\left(3+\frac{1}{256}+\frac{3}{16}\right)\epsilon_7.$$

We thus obtain Table IV.

Table IV

$\nabla_U\mu$	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7
E_1	0	$-\frac{1}{32}\epsilon_7$	$\frac{1}{2}\epsilon_5+\frac{1}{8}\epsilon_7$	$\frac{1}{2}\epsilon_6+\frac{1}{8}\epsilon_7$	$-\frac{1}{2}\epsilon_3+\frac{1}{2}\epsilon_7$	$-\frac{1}{2}\epsilon_4$	$\frac{1}{32}\epsilon_2-\frac{1}{8}\epsilon_3-\frac{1}{8}\epsilon_4-\frac{1}{2}\epsilon_5$
E_2	$\frac{1}{32}\epsilon_7$	0	0	$\frac{1}{2}\epsilon_5-\frac{1}{8}\epsilon_7$	$-\frac{1}{2}\epsilon_4$	$\frac{1}{2}\epsilon_7$	$-\frac{1}{32}\epsilon_1+\frac{1}{8}\epsilon_4-\frac{1}{2}\epsilon_6$
E_3	$-\frac{1}{2}\epsilon_5-\frac{1}{8}\epsilon_7$	0	0	$\frac{1}{2}\epsilon_7$	$\frac{1}{2}\epsilon_1$	0	$\frac{1}{8}\epsilon_1-\frac{1}{2}\epsilon_4$
E_4	$-\frac{1}{2}\epsilon_6-\frac{1}{8}\epsilon_7$	$-\frac{1}{2}\epsilon_5+\frac{1}{8}\epsilon_7$	$-\frac{1}{2}\epsilon_7$	0	$\frac{1}{2}\epsilon_2$	$\frac{1}{2}\epsilon_1$	$\frac{1}{8}\epsilon_1-\frac{1}{8}\epsilon_2+\frac{1}{2}\epsilon_3$

Using (A.1) and the information from Table IV, a straightforward calculation shows that if we let E_τ be the skew-Hermitian matrix defined by

$$\begin{pmatrix} 4\pi^2S^2 & 0 & 0 & 0 & 2\pi iA_3 & 2\pi iA_4 & \pi i(-\frac{1}{8}A_2+\frac{1}{2}A_3+\frac{1}{2}A_4) \\ & 4\pi^2S^2 & 0 & 0 & 2\pi iA_4 & 0 & \pi i(\frac{1}{8}A_1-\frac{1}{2}A_4) \\ & & 4\pi^2S^2 & 0 & -2\pi iA_1 & 0 & \pi i(-\frac{1}{2}A_1+2A_4) \\ & & & 4\pi^2S^2 & -2\pi iA_2 & -2\pi iA_1 & \pi i(-\frac{1}{2}A_1+\frac{1}{2}A_2-2A_3) \\ & & & & 4\pi^2S^2+2 & 0 & -2\pi iA_1 \\ & & & & & 4\pi^2S^2+1 & -2\pi iA_2+\frac{1}{4} \\ & & & & & & 4\pi^2S^2+817/256 \end{pmatrix}$$

then $\Delta(F_\tau \otimes \mu) = \lambda(F_\tau \otimes \mu)$ if and only if λ is an eigenvalue of the matrix E_τ.

Let

$$\lambda = \frac{17}{4}\pi^2+1+\sqrt{\frac{17}{4}\pi^2+1}.$$

We now calculate necessary conditions on $\tau = A_1\epsilon_1+A_2\epsilon_2+A_3\epsilon_3+A_4\epsilon_4$ for λ to be an eigenvalue of E_τ. Since $\tau\in\mathfrak{J}_1^{\mathrm{I}}$ or $\tau\in\mathfrak{J}_2^{\mathrm{I}}$, we know that $A_1, A_2, A_3, A_4\in \mathbf{Q}$. By a computation using any computer symbolic manipulation package (such as Maple or Mathematica), if $\det(E_\tau-\lambda I_7)=0$ then $x=\sqrt{(17/4)\pi^2+1}$ is the root of a polynomial with rational coefficients. Because x is transcendental, the coefficients of the powers of x must be zero.

A straightforward calculation shows that the leading coefficient is $((16/17)S^2-1)^7$. Thus, if λ is an eigenvalue of E_τ then $S^2=17/16$. Recall that $S^2=A_1^2+A_2^2+A_3^2+A_4^2$.

For τ in $\mathfrak{J}_1^{\mathrm{I}}$, $S^2=17/16$ if and only if (see (*)) $\tau=\pm(\frac{1}{4}\epsilon_2+\epsilon_3)$ or $\tau=\pm\frac{1}{4}\epsilon_1\pm\epsilon_4$. For τ in $\mathfrak{J}_2^{\mathrm{I}}$, $S^2=17/16$ if and only if (see (**)) $\tau=\pm(\frac{1}{4}\epsilon_2-\epsilon_3)$ or $\tau=\pm\frac{1}{4}\epsilon_1\pm\epsilon_4$. Note that the only difference is in the sign of ϵ_3.

For $\tau=\pm(\frac{1}{4}\epsilon_2+\epsilon_3)$ or $\tau=\pm\frac{1}{4}\epsilon_1\pm\epsilon_4$, a calculation using Maple or Mathematica shows that $\det(E_\tau-\lambda I_7)\neq 0$. Thus, $(17/4)\pi^2+1+\sqrt{(17/4)\pi^2+1}\notin$ 1-spec$^{\mathrm{I}}(\Gamma_1\backslash G, g)$. However, for $\tau=\pm(\frac{1}{4}\epsilon_2-\epsilon_3)$, $\det(E_\tau-\lambda I_7)=0$. Thus $(17/4)\pi^2+1+\sqrt{(17/4)\pi^2+1}$ is an eigenvalue for the Laplacian acting on $\mathcal{H}_{\pm(\frac{1}{4}\epsilon_2-\epsilon_3)}\otimes\Lambda^1(\mathfrak{g}^*)$, and $(17/4)\pi^2+1+\sqrt{(17/4)\pi^2+1}\in$ 1-spec$^{\mathrm{I}}(\Gamma_2\backslash G, g)$. Consequently,

$$\text{1-spec}^{\mathrm{I}}(\Gamma_1\backslash G, g)\neq\text{1-spec}^{\mathrm{I}}(\Gamma_2\backslash G, g),$$

as desired.

The proof of Proposition 4.11 is now complete. □

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Department of Mathematics
Texas Tech University
Lubbock, TX 79409-1042

gornet@math.ttu.edu