The Ahlfors Laplacian on a Riemannian Manifold with Boundary

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0. Introduction

In our previous paper [9] we initiated a study of the Ahlfors Laplacian $L = S^*S$, that is, the symmetric and trace-free part of the covariant derivative acting on vector fields on a Riemannian manifold M. Here S is the Ahlfors operator: $SX = 0$ means that the vector field X on M is conformal. The size of SX (in the infinity norm) measures the degree to which X is quasi-conformal (the constant of quasi-conformality). Thus a good understanding of the operators S and L is desirable in connection with studies of quasi-conformal transformations and their geometry.

In this paper we extend some of our earlier results to the case where M has a boundary $\Sigma = \partial M \neq \emptyset$. We investigate the behavior of S on a general hypersurface, thus relating the intrinsic conformal geometry of Σ with that of M. In this way we find geometrically natural boundary conditions for L, giving rise to self-adjoint and elliptic extensions of L up to the boundary. One such condition consists of the elasticity condition investigated by Weyl [16] in dealing with vibrations of an elastic body in the Eulidean space $\mathbb{R}^3$. We are thus able to generalize and sharpen the asymptotic distribution of eigenfrequencies found by Weyl, in a sense finding the "vibrational spectrum" of M. Note that L does not have scalar leading symbol, so that both the spectral asymptotics as well as the boundary conditions are a more delicate matter than, for example, for the Laplacian. Another question we address is that of unique continuation for conformal vector fields given a certain behavior on Σ; this is directly related to the existence of a Poisson kernel for L and for S.

It turns out that the basic formulas of Green's type for S and L are particularly simple. We derive these and show how similar formulas hold for any generalized gradient [14], based on a universal Green's formula for the covariant derivative. Such formulas were in a special case considered by Weyl and also by Yano in [17], where he used them to characterize conformal vector fields and their boundary values.

These general Green's formulas are remarkably simple and could in particular be applied to finding natural boundary conditions of a geometric

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nature for any generalized gradient. Since our emphasis is on elliptic boundary value problems, we extend and clarify Yano's results; also, our proofs and formulations are more conceptual, so that many of our results will carry over easily to other gradients. A particularly interesting prospect is that of considering higher Ahlfors operators, that is, gradients on differential forms.

After establishing notation, we derive the relevant formulas giving the interplay between S near Σ and the intrinsic geometry of Σ. Then, based on the Green's formulas mentioned above, we formulate three basic boundary conditions: Dirichlet, Neumann, and elasticity, and show that they are self-adjoint and elliptic in the strong sense of the usual pseudodifferential calculus; see [5]. Finally, in Section 5 we give the leading asymptotics for the heat semigroup $\exp(-tL)$ for some of the boundary conditions.

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Since this work was completed, a preprint has appeared ("Heat Equation Asymptotics of the Generalized Ahlfors Laplacian on a Manifold with Boundary" by Branson, Gilkey, Ørsted and Pierzchalski) in which other methods are used to calculate more terms in the heat asymptotics for boundary conditions of a similar nature. However, those methods do not apply, for example, to the Dirichlet conditions discussed in the present paper.

1. Preliminaries and Notation

Let (M, g) be a Riemannian manifold (with or without boundary), that is, a C^∞ manifold M equipped with a C^∞ metric tensor $g = (g_{rs})$ which is symmetric and positive definite. Many results extend (but we do not go into this) to the pseudo-Riemannian case, where g is symmetric and nondegenerate. The variable n will always denote the dimension of M. For $p \in M$, T_p and T_p^* denote the tangent and the cotangent space at p, respectively.

The space of all C^∞ vector fields will be denoted by $\mathfrak{X}$. The space of C^∞-p-forms will be denoted by $\mathfrak{D}^p$, and $\mathfrak{M}$ will denote the space of all symmetric C^∞ tensor fields of zero trace with respect to g.

Take ∇, the Levi-Civita connection of the metric g, and extend it naturally to the whole tensor algebra over M. Then, for each $h \in C^\infty(\xi)$ where ξ is an arbitrary tensor bundle over M, ∇h is a section of $C^\infty(T^* \otimes \xi)$ where T^* is the cotangent bundle of M. For example, for a 1-form α we have that

$$(\nabla\alpha)(X, Y) = \nabla_X(\alpha(Y)) - \alpha(\nabla_X Y), \quad X, Y \in \mathfrak{X},$$

while, for a 2-tensor $\varphi \in \mathfrak{M}$,

$$(\nabla\varphi)(X, Y, Z) = \nabla_X(\varphi(Y, Z)) - \varphi(\nabla_X Y, Z) - \varphi(Y, \nabla_X Z), \quad X, Y, Z \in \mathfrak{X}.$$

Extend also g naturally onto arbitrary tensor fields and denote the extended g by the same letter. Consider, for example, the cases of 1-forms and symmetric 2-tensors as follows. If, in coordinates $(x^1, \dots, x^n)$,

$$\alpha = \alpha_r dx^r, \qquad \beta = \beta_s dx^s$$

(summation convention), then

$$g(\alpha, \beta) = g^{rs}\alpha_r \beta_s,$$

where (g^{rs}) is the inverse matrix of $(g_{rs}) = (g(\partial/\partial x^r, \partial/\partial x^s))$. Also, if

$$\varphi = \varphi_{rs} dx^r \otimes dx^s, \qquad \psi = \psi_{tu} dx^t \otimes dx^u,$$

then

$$g(\varphi, \psi) = g^{rt} g^{su} \varphi_{rs} \psi_{tu}.$$

For simplicity we assume that M is orientable. It can then be covered by neighborhoods U with coordinates $(x^1, \ldots, x^n)$ such that each two of them with a nonempty intersection are related by a diffeomorphism with a positive Jacobian. The volume form is defined locally by

$$\mathrm{vol}_{M|U} = G dx^1 \wedge \cdots \wedge dx^n, \tag{1.1}$$

where

$$G = \sqrt{\det(g_{rs})}. \tag{1.2}$$

Now we define the global scalar product of two tensors V, W (e.g. $V, W \in \mathfrak{X}$ or $\mathfrak{D}^1$ or $\mathfrak{M}$) by

$$(V, W) = \int_M g(V, W)\, \mathrm{vol}_M. \tag{1.3}$$

In the case of compact M (with or without boundary), (1.3) is always finite.

An essential role in the theory of quasi-conformal deformations of a Riemannian manifold is played by the Ahlfors differential operator S (of [1; 2; 10; 11; 12]) from the space $\mathfrak{X}$ of all vector fields ($=$ deformations) Z into the space $\mathfrak{M}$ of all symmetric trace-free tensors, defined by

$$SZ(X, Y) = \frac{1}{2}[g(\nabla_X Z, Y) + g(X, \nabla_Y Z)] - \frac{1}{n} \operatorname{div} Z g(X, Y) \tag{1.4}$$

or, equivalently, in a more consistent form by

$$SZ = \frac{1}{2}\mathfrak{L}_Z g - \frac{1}{n} \operatorname{div} Z g, \tag{1.5}$$

where $\mathfrak{L}_Z$ is the Lie derivative in direction Z:

$$\mathfrak{L}_Z g(X, Y) = Zg(X, Y) - g([Z, X], Y) - g(X, [Z, Y]), \quad X, Y, Z \in \mathfrak{X}, \tag{1.6}$$

and $\operatorname{div} Z$ denotes the divergence of Z:

$$\operatorname{div} Z = \operatorname{tr}(X \to \nabla_X Z). \tag{1.7}$$

If α is the 1-form dual to Z in the sense that

$$\alpha(X) = g(Z, X), \quad X \in \mathfrak{X}, \tag{1.8}$$

then

$$\tfrac{1}{2}\mathfrak{L}_Z g = \nabla^s \alpha \quad \text{and} \quad \delta\alpha = -\operatorname{div} Z, \tag{1.9}$$

where ∇^s denotes the symmetrized ∇:

$$(\nabla^s\alpha)(X,Y)=\tfrac{1}{2}((\nabla_X\alpha)(Y)+(\nabla_Y\alpha)(X)), \tag{1.10}$$

and $\delta\alpha$ denotes the codifferential of α. In coordinates,

$$\delta\alpha=-\nabla^i\alpha_i. \tag{1.11}$$

Consequently, by the duality (1.8), S may be realized as an operator acting on 1-forms α:

$$S\alpha=\nabla^s\alpha+\frac{1}{n}\delta\alpha\cdot g. \tag{1.12}$$

In what follows we will use both SZ and $S\alpha$.

The operator S^*, formally adjoint to S in the sense that the equality

$$(S\alpha,\varphi)=(\alpha,S^*\varphi),\quad \alpha\in\mathfrak{D}^1,\ \varphi\in\mathfrak{M}, \tag{1.13}$$

holds if α or φ is of compact support (not intersecting ∂M if $\partial M\neq\emptyset$), is of divergence type:

$$S^*\varphi=\delta\varphi,\quad \varphi\in\mathfrak{M} \tag{1.14}$$

where, in coordinates, the 1-form $\delta\varphi$ is defined by

$$\delta\varphi=-\nabla^i\varphi_{ij}. \tag{1.15}$$

The operator

$$L=S^*S,$$

called the *Ahlfors Laplacian,* is a formally self-adjoint nonnegative differential operator on M. It is of the form (cf. [9])

$$L=\frac{1}{2}\delta d+\left(1-\frac{1}{n}\right)d\delta-R. \tag{1.16}$$

Here R denotes the Ricci action on 1-forms α:

$$R\alpha=R(Z,\cdot), \tag{1.17}$$

where Z is the field dual to α in the sense of (1.8) by the decomposition formula (1.16). The symbol σ_L is of the form

$$\begin{aligned}\sigma_L(\omega)\alpha&=i(\omega)\epsilon(\omega)\alpha+\left(2-\frac{2}{n}\right)\epsilon(\omega)i(\omega)\alpha\\&=g(\omega,\omega)\alpha+\left(1-\frac{2}{n}\right)(i(\omega)\alpha)\cdot\omega,\quad \alpha,\omega\in\mathfrak{D}^1,\end{aligned}$$

where $i(\omega)$ and $\epsilon(\omega)$ denote (respectively) the interior and exterior product by ω. S may be treated as a generalized gradient in the sense of Stein and Weiss [14]. At each point $p\in M$, the space of all two tensors at p decomposes into three g_p-orthogonal subspaces invariant under the natural action of the orthogonal groups $O(g_p)$: skew-symmetric tensors, symmetric and trace-free tensors, and traces. Accordingly, the covariant derivative of $\alpha \in \mathfrak{D}^1$ decomposes into three pieces:

$$\nabla\alpha = \frac{1}{2} d\alpha + S\alpha - \frac{1}{n}\delta\alpha \cdot g. \tag{1.18}$$

The decomposition formula (1.18) may be briefly interpreted as follows: If we look at α as an infinitesimal deformation of M (the vector field Z dual to α generates a flow of transformations of M), then $d\alpha$ "measures" a rotation, $S\alpha$ an elastic distorsion, and $\delta\alpha$ a stretching caused infinitesimally by α.

2. Behavior on a Hypersurface

Consider the decomposition (1.18). The exterior derivative d is independent of the geometric structure of M. The geometric properties of ∇ are then determined by S and δ or, equivalently, by ∇^s and δ. Thus these two operators determine the behavior both of ∇ and S. Our next aim is to describe their local properties on a hypersurface $\Sigma \subset M$.

By a hypersurface Σ of M we mean an $(n-1)$-dimensional C^∞-submanifold of M. We treat Σ as a Riemannian manifold with the scalar product $\tilde{g}$ induced by g. All objects related to Σ will be denoted by $\tilde{}$; for example, we will use the symbols $\tilde{\nabla}$, $\widetilde{\operatorname{div}}$, and $\tilde{S}$, for the Levi-Civita connection, the divergence, and the Ahlfors operator, respectively, with respect to $\tilde{g}$ on Σ.

Since only local properties of Σ will be investigated in this section, we accept, without loss of generality, some technical assumptions. We may therefore assume that Σ is an imbedded submanifold and that it is contained in a single neighborhood U with coordinates $(x^1, \ldots, x^{n-1}, x^n)$. Moreover, we can choose the coordinates in such a way that the last one, $r = x^n$, denotes the geodesic distance to Σ. The vector fields

$$e_1 = \frac{\partial}{\partial x^1}, \ldots, e_{n-1} = \frac{\partial}{\partial x^{n-1}}$$

are tangent to Σ while

$$N = e_n = \frac{\partial}{\partial r}$$

is orthogonal to it at each point. Moreover, M is a geodesic field in the sense that

$$\nabla_N N = 0 \ \text{ in } U. \tag{2.1}$$

In particular,

$$g(N, N) = 1; \tag{2.2}$$

that is, N is a field of unit vectors.

Take now two arbitrary vector fields tangent to Σ: $X, Y \in \mathfrak{X}(\Sigma)$. By the Gauss and Weingarten formulas (cf. [6, p. 15]) and (2.1), we have that

$$\nabla_X Y = \tilde{\nabla}_X Y + h(X, Y) \cdot N \tag{2.3}$$

and

$$\nabla_X N = -A_N X, \quad g(\nabla_X N, N) = 0, \tag{2.4}$$

at every point of Σ, where h is a symmetric bilinear form (called the *second fundamental* form of Σ) $h: \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(M)$ and A_N is a linear endomorphism $A_N: \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ conjugate to h in the sense that

$$g(A_N X, Y) = h(X, Y), \quad X, Y \in \mathfrak{X}(\Sigma). \tag{2.5}$$

Assume that indices i, j, k, l run over the range $1, \dots, n-1$ while s, t, u, v run over $1, \dots, n$. The summation convention will be used for indices of both kinds. By the assumptions on the coordinates in U, the following relations may be obtained from (2.3), (2.4), and (2.5).

LEMMA 2.1. *At every point of Σ we have:*

$$g(\nabla_{e_i} e_j, e_k) = \tilde{g}(\tilde{\nabla}_{e_i} e_j, e_k), \tag{2.6}$$

$$g(\nabla_{e_i} e_j, e_n) = -g(\nabla_{e_i} e_n, e_j) = -g(\nabla_{e_n} e_i, e_j) = h(e_i, e_j), \tag{2.7}$$

$$g(\nabla_{e_i} e_n, e_n) = g(\nabla_{e_n} e_i, e_n) = g(\nabla_{e_n} e_n, e_i) = g(\nabla_{e_n} e_n, e_n) = 0 \tag{2.8}$$

for $i, j = 1, \dots, n-1$.

Take now an arbitrary vector field Z in U. In the basis $e_1, \dots, e_{n-1}, e_n$,

$$Z = Z^k e_k + Z^n e_n. \tag{2.9}$$

If we let

$$Z^T = Z^k e_k, \qquad Z^N = Z^n e_n, \tag{2.10}$$

then we obtain the decomposition

$$Z = Z^T + Z^N \tag{2.11}$$

in the whole neighborhood U of Σ. Moreover, at every point of Σ, Z^T is tangent while Z^N is orthogonal to Σ.

Extend h to arbitrary vector fields along Σ (not necessarily tangent), setting

$$h(X, Y) = h(X^T, Y^T). \tag{2.12}$$

For such h we have, for example,

$$h(X, N) = 0. \tag{2.13}$$

Let us first observe how the divergence behaves on Σ. By (1.7), the orthogonality relations $e_n \perp e_i$ $(i = 1, \dots, n-1)$ and Lemma 2.1, we obtain the following after direct calculations.

LEMMA 2.2. *For an arbitrary vector field Z in U,*

$$\operatorname{div} Z = \widetilde{\operatorname{div}}\, Z^T - Z^n \operatorname{tr} h + \frac{\partial}{\partial r}(Z^n) \tag{2.14}$$

at each point of Σ. (Note that Z^n is well-defined in the whole neighborhood U of Σ.)

Now observe how ∇^s behaves on Σ. Here again ∇^s denotes the symmetrized ∇; that is, $\nabla^s Z$ is the tensor field defined by

$$\nabla^s Z(X,Y) = \tfrac{1}{2}[g(\nabla_X Z, Y) + g(\nabla_Y Z, X)], \quad X, Y, Z \in \mathfrak{X}(M); \tag{2.15}$$

cf. (1.10).

Equations (2.15) and (2.9) together with Lemma 2.1 establish the following lemma.

LEMMA 2.3. *For an arbitrary vector field Z in U, at each point of Σ we have:*

$$(\nabla^s Z)(e_i, e_j) = (\tilde{\nabla}^s Z^T)(e_i, e_j) - Z^n h(e_i, e_j), \tag{2.16}$$

$$\begin{aligned}(\nabla^s Z)(e_i, e_n) &= \tfrac{1}{2}[e_i(Z^n) + e_n(Z^k) g(e_n, e_i)] \\ &= \tfrac{1}{2}[e_i(Z^n) + g([N, Z], e_i)],\end{aligned} \tag{2.17}$$

$$(\nabla^s Z)(e_n, e_n) = \frac{\partial}{\partial r} Z^n g(e_n, e_n) \tag{2.18}$$

for $i, j = 1, \ldots, n-1$.

REMARK 2.3′. In (2.18) $g(e_n, e_n) = 1$, but we have retained it in order to stress the tensorial character of $\nabla^s Z$.

COROLLARY 2.1. *For an arbitrary vector field Z in U,*

$$\begin{aligned} SZ(e_i, e_j) &= \tilde{S}Z^T(e_i, e_j) + \frac{1}{n(n-1)} \widetilde{\operatorname{div}}\, Z^T g_{ij} \\ &\quad - Z^n h(e_i, e_j) + \frac{1}{n} Z^n \operatorname{tr} h \cdot g_{ij} - \frac{1}{n}\frac{\partial}{\partial r} Z^n g_{ij}, \end{aligned} \tag{2.19}$$

$$\begin{aligned} SZ(e_i, e_n) &= \tfrac{1}{2}[e_i(Z^n) + e_n(Z^k) g_{ki}] \\ &= \tfrac{1}{2}[e_i(Z^n) + g([N, Z], e_i)], \end{aligned} \tag{2.20}$$

$$SZ(e_n, e_n) = \left(1 - \frac{1}{n}\right)\frac{\partial}{\partial r} Z^n - \frac{1}{n}\widetilde{\operatorname{div}}\, Z^T + \frac{1}{n} Z^n \operatorname{tr} h \tag{2.21}$$

at each point of Σ.

Proof. The formulas (2.19)–(2.21) are a simple consequence of Lemmas 2.2 and 2.3. □

Let us next investigate the behavior of SZ on Σ under different assumptions on Z itself. The results obtained will be applied in formulating self-adjoint and elliptic boundary conditions (Section 4).

First, observe that the restricting of a conformal field to hypersurfaces preserves conformality. This may be stated more precisely as follows.

LEMMA 2.4. *If Z is a conformal vector field on M (i.e., $SZ = 0$ on M) and if Z is tangent to Σ (i.e., $Z^N = 0$ on Σ), then the vector field $\tilde{Z} \in \mathfrak{X}(\Sigma)$, the restriction of Z to Σ, is a conformal vector field on Σ:*

$$\tilde{S}\tilde{Z} = 0 \tag{2.22}$$

at each point of Z.

Proof. Since $Z^N = 0$, by (2.19) and (2.21) we obtain

$$\tilde{S}\tilde{Z}(e_i, e_j) = SZ(e_i, e_j) + \frac{1}{n-1} SZ(e_n, e_n) g_{ij} \tag{2.23}$$

at each point of Σ. The assumption $SZ = 0$ now implies (2.22). □

LEMMA 2.5. *If Z_1, Z_2 are arbitrary vector fields on M such that*

$$Z_1 = Z_2 = 0 \ \ on\ \Sigma, \tag{2.24}$$

then

$$SZ_1(Z_2, N) = 0 \tag{2.25}$$

at each point of Z.

Proof. This is a simple consequence of SZ_1 being a tensor field. □

LEMMA 2.6. *If Z_1, Z_2 are vector fields on M such that*

$$Z_1^N = Z_2^N = 0 \quad and \quad (\nabla_N Z_1)^T = 0 \quad on\ \Sigma, \tag{2.26}$$

then

$$SZ_1(Z_2, N) = \tfrac{1}{2} h(Z_1, Z_2) \tag{2.27}$$

at each point of Σ.

Proof. Since $g(Z_2, N) = 0$, by (1.4) we obtain that

$$SZ_1(Z_2, N) = \tfrac{1}{2}[g(\nabla_{Z_2} Z_1, N) + g(Z_2, \nabla_N Z_1)] \tag{2.28}$$

at each point of Σ. Using the assumption and the Gauss formula (2.3) yields the assertion (2.27). □

REMARK 2.6′. If we replace the conditions (2.26) by

$$Z_1^N = Z_2^N = 0 \quad \text{and} \quad [N, Z_1]^T = 0 \quad \text{on } M, \tag{2.26′}$$

then we obtain that

$$SZ_1(Z_2, N) = 0 \ \text{ on } \Sigma \tag{2.27′}$$

instead of (2.27).

LEMMA 2.7. *If Z_1, Z_2 are arbitrary vector fields on M such that*

$$Z_1^T = Z_2^T = 0 \quad and \quad \operatorname{div} Z_1 = 0 \quad on\ \Sigma, \tag{2.29}$$

then

$$SZ_1(Z_2, N) = Z_1^n Z_2^n \operatorname{tr} h \tag{2.30}$$

at each point of Σ.

Proof. By assumptions (2.29) and formula (2.14), we obtain that

$$\frac{\partial}{\partial r} Z_1^n = Z_1^n \operatorname{tr} h \tag{2.31}$$

at each point of Σ. Using formula (2.21) then yields (2.30). □

REMARK 2.7′. If we replace the conditions (2.29) by

$$Z_1^T = Z_2^T = 0 \quad \text{and} \quad \operatorname{div} Z_1 = -\frac{n}{n-1} Z^n \operatorname{tr} h \quad \text{on } \Sigma, \tag{2.29′}$$

we obtain that

$$SZ_1(Z_2, N) = 0 \tag{2.30′}$$

on Σ instead of (2.30).

3. Green's Formulas

In this section we are going to derive Green-type formulas for the operators S, S^* and $L = S^*S$. Some of them were used implicitly by mathematicians considering conformal Killing vector fields on manifolds with boundary (cf. e.g. Yano [17] and references mentioned there). We decided to formulate them explicitly because our formulation extends automatically to all generalized gradients in the sense of Stein and Weiss [14], that is, differential operators as projections of ∇ onto irreducible pieces of arbitrary tensor (or spin) bundles over M.

This last point is important enough that we want to stress it in the beginning of this section: For each Stein–Weiss gradient corresponding to a connection ∇, there is a Green-type formula as follows: Let ξ be an arbitrary tensor (spinor) bundle over M. Then $\nabla: C^\infty(\xi) \to C^\infty(T^* \otimes \xi)$. At each point $p \in M$, the orthogonal group $O(g_p)$ acts naturally on $T_p^* \otimes \xi_p$. Denote by $F_p^1, \dots, F_p^\mu$ all the irreducible invariant subspaces of $T_p^* \otimes \xi_p$, and by π_ν: $C^\infty(T^* \otimes \xi) \to C^\infty(F^\nu)$ ($\nu = 1, \dots, \mu$) the natural orthogonal projections. Consider the first-order differential operators

$$P_\nu = \pi_\nu \circ \nabla, \quad \nu = 1, \dots, \mu,$$

and denote by P_ν^* the operator formally adjoint to P_ν. Then our Green's formula is given in the following theorem.

THEOREM 3.1. *If $\alpha \in C^\infty(\xi)$ and $\beta \in C^\infty(F^\nu)$, then*

$$(P_\nu \alpha, \beta) - (\alpha, P_\nu^* \beta) = \int_{\partial M} *i(\alpha)\beta$$

where $i(\alpha)\beta$ denotes contraction of α and β to a 1*-form via the metric.*

Here the contraction between α and β is the pairing between ξ and itself.

Theorems 3.2 and 3.3 are just special cases of this general result; however, in order to illustrate the geometry of our special situation, we give the specialization in a precise way. Hence they are also seen as consequences of the Stokes theorem. We give the proof of Theorem 3.1 at the end of this section.

Let again M be an orientable compact Riemannian manifold with a smooth boundary ∂M consisting of a finite number of $(n-1)$-dimensional manifolds. M may then be covered by a finite number of neighborhoods with coordinates $(x^1, \dots, x^n)$ which are of two kinds: First-kind coordinate systems map open sets in $M \setminus \partial M$ onto open sets in $\mathbb{R}^n$ and cover an open set V_I; second-kind coordinate systems cover an open neighborhood V_B of ∂M in M ($V_B \cup V_I = M$). For each boundary point p there exist coordinates $(x^1, \dots, x^n = r)$ in a neighborhood of p, $|x_i| < 1$ and $0 \le x_n \le 1$, such that (like the coordinate system in a neighborhood U of Σ in Section 2) r measures the geodesic distance of points of V_B to the boundary ∂M. The last coordinates of a point of V_B in two overlapping systems are then the same. V_B therefore has the form of a product of ∂M with a half-open interval $0 \le r < 1$. The vector field $N = \partial/\partial r$ is then well-defined in the whole neighborhood V_B.

For a given manifold M with boundary ∂M, it is possible to construct its "double" M', which is a compact manifold consisting of two copies of M, suitably oriented, glued together smoothly at the boundary ∂M. Coordinate systems of the second kind make the gluing process possible: two copies of such coordinates glue together in a coordinate system of the first kind. ∂M may therefore be treated as a hypersurface Σ of M', and all results of Section 2 apply to ∂M.

Recall that M is an orientable manifold. This means that coordinates of both kinds may be chosen in such a way that each two overlapping systems are related by a diffeomorphism with a positive Jacobian. The volume form vol_M can then be locally expressed by (1.3). If $(x^1, \dots, x^n)$ are coordinates of the second kind in U, then the local formula

$$\mathrm{vol}_{\partial M}|_{U \cap \partial M} = (-1)^{n-1} G\, dx^1 \wedge \cdots \wedge dx^{n-1} \tag{3.1}$$

defines a global $(n-1)$-form on ∂M.

We will use the following Stokes formula in this section:

$$\int_M d\omega = \int_{\partial M} \omega, \tag{3.2}$$

where M and ∂M are oriented according to (1.3) and (3.1), respectively. In particular, for an arbitrary 1-form α on M we obtain

$$\int_M d * a = \int_{\partial M} *\alpha, \tag{3.3}$$

where $*\colon \mathfrak{D}^p \to \mathfrak{D}^{n-p}$ is the Hodge star isomorphism defined by

$$\omega \wedge *\mu = g(\omega, \mu)\, \mathrm{vol}_M, \quad \omega, \mu \in \mathfrak{D}^p. \tag{3.4}$$

In a coordinate system $(x^1, \dots, x^n)$ in U, let

$$\alpha = \alpha_s dx^s. \tag{3.5}$$

By (1.3) and (3.4) one can calculate that

$$*\alpha = G\alpha_s g^{st}(-1)^{t-1} dx^1 \wedge \cdots \wedge \widehat{dx^t} \wedge \cdots \wedge dx^n, \tag{3.6}$$

where $g^{st}g_{tn} = \delta_n^s$ and where $\hat{}$ over a factor means that this factor is missing from the product.

Since

$$\delta\alpha = -* d* \tag{3.7}$$

on 1-forms α (cf. [15, p. 220]), applying $*$ to both sides of (3.7) and using the fact that $*f = f \operatorname{vol}_M$ for $f \in C^\infty(M)$ yields

$$\delta\alpha \operatorname{vol}_M = -**(d*\alpha). \tag{3.8}$$

Since

$$**|_{\mathfrak{D}^n} = \mathrm{id}|_{\mathfrak{D}^n}, \tag{3.9}$$

we find

$$d*\alpha = -\delta\alpha \operatorname{vol}_M, \tag{3.10}$$

and by (3.3) we obtain

$$\int_M \delta\alpha \operatorname{vol}_M = -\int_{\partial M} *\alpha. \tag{3.11}$$

Consequently, by (3.1) and (3.6) we see that

$$\int_M \delta\alpha \operatorname{vol}_M = -\int_{\partial M} i(dr)\alpha \operatorname{vol}_{\partial M} = -\int_{\partial M} \alpha(N) \operatorname{vol}_{\partial M}, \tag{3.12}$$

where, for any p-form β, $i(\beta)$ denotes the inner product by β, and where N is the unit normal field along ∂M defined in the beginning of this section.

Now we are ready to prove Green-type formulas for S, S^* and $L = S^*S$. These formulas generalize those obtained by Ahlfors in [1] in the case where M is the Euclidean ball or by the second-named author in [10] in the case of an arbitrary domain in the Euclidean space in $\mathbb{R}^n$.

THEOREM 3.2. *For arbitrary 1-form $\alpha \in \mathfrak{D}^1$ and $\varphi \in \mathfrak{M}$,*

$$(S\alpha, \varphi) - (\alpha, S^*\varphi) = \int_{\partial M} *i(\alpha)\varphi; \tag{3.13}$$

by duality, for an arbitrary vector field Z,

$$(SZ, \varphi) - (Z, S^*\varphi) = \int_{\partial M} \varphi(Z, N) \operatorname{vol}_{\partial M} \tag{3.14}$$

(here $(Z, S^*\varphi) = (S^*\varphi)(Z)$).

Proof. By the Stokes theorem, it is enough to show that if we express the left-hand side of (3.13) in the integral form (1.3) then the integrand is equal to $d(*i(\alpha)\varphi)$. By (3.9) and (3.8), we have that

$$d(*i(\alpha)\varphi) = **(d*i(\alpha)\varphi) = *(*d*i(\alpha)\varphi) = -*\delta i(\alpha)\varphi. \tag{3.15}$$

In local coordinates,

$$\delta(i(\alpha)\varphi) = -\nabla^u g^{st}\alpha_t\varphi_{su} = -g^{st}(\nabla^u\alpha_t)\varphi_{su} - g^{st}\alpha_t\nabla^u\varphi_{su}. \tag{3.16}$$

Consequently,

$$-*\,\delta(i(\alpha)\varphi) = [g^{st}(\nabla^u\alpha_t)\varphi_{su} + g^{st}\alpha_t\nabla^u\varphi_{su}]\,\mathrm{vol}_M. \tag{3.17}$$

On the other hand, using (1.12), the symmetry of φ, and the fact that $\operatorname{tr}\varphi = 0$, we obtain

$$\begin{aligned} g(S\alpha, \varphi) &= S\alpha_{us}g^{ut}g^{sv}\varphi_{tv} = \tfrac{1}{2}(\nabla_u\alpha_s + \nabla_s\alpha_u)g^{ut}g^{sv}\varphi_{tv} \\ &= g^{su}(\nabla^t\alpha_s)\varphi_{tu}. \end{aligned} \tag{3.18}$$

After re-ordering indices, (3.18) is equal to the first summand in the brackets of (3.17). Similarly, by (1.15), we get the other summand in (3.17):

$$-g(\alpha, S^*\varphi) = -\alpha_t g^{ts}(S^*\varphi)_s = \alpha_t g^{ts}\nabla^u\varphi_{us}. \tag{3.19}$$

By (3.17)–(3.19) we have

$$d(*i(\alpha)\varphi) = (g(S\alpha, \varphi) - g(\alpha, S^*\varphi))\,\mathrm{vol}_M. \tag{3.20}$$

Now the Stokes formula (3.3) completes the proof. □

REMARK 3.2′. By changing signs in (3.13) or (3.14), we obtain Green's formulas for S^*.

Combining (3.13) or (3.14) with the analogous formula for S^*, we now obtain directly the following formula for $L = S^*S$.

THEOREM 3.3. *For arbitrary* 1*-forms* $\alpha_1, \alpha_2 \in \mathfrak{D}^1$,

$$(S^*S\alpha_1, \alpha_2) - (\alpha_1, S^*S\alpha_2) = \int_{\partial M} *(i(\alpha_1)S\alpha_2 - i(\alpha_2)S\alpha_1); \tag{3.21}$$

by duality, for arbitrary vector fields $Z_1, Z_2 \in \mathfrak{X}(M)$ *we have*

$$(S^*SZ_1, Z_2) - (Z_1, S^*SZ_2) = \int_{\partial M} [SZ_2(Z_1, N) - SZ_1(Z_2, N)]\,\mathrm{vol}_{\partial M}. \tag{3.22}$$

It is worth noticing that the reasoning in the proof of Theorem 3.1 applies *mutatis mutandis* to each orthogonal summand of $\nabla\alpha$ in (1.18) and, finally, to the $\nabla\alpha$ itself. We can therefore formally obtain Green-type formulas for d, δ, or ∇ by replacing S in (3.13) with d, δ, or ∇ and suitably correcting the range for φ. Analogously, we can derive Green-type formulas for $d^*d = \delta d$, $\delta^*\delta = d\delta$, or $\nabla^*\nabla$ from (3.21).

The same theorems may also be formulated for ∇ considered as an operator acting on arbitrary tensors (spinors), so the Green-type theorems may be proved analogously for each of the generalized gradients in the sense of Stein and Weiss. Recalling the notation in Theorem 3.1, we state these more precisely. Let ξ be an arbitrary tensor (spinor) bundle over M. Then

$\nabla: C^\infty(\xi) \to C^\infty(T^* \otimes \xi)$. At each point $p \in M$, the orthogonal group $O(g_p)$ acts naturally on $T_p^* \otimes \xi_p$. Denote by $F_p^1, \ldots, F_p^\mu$ all the irreducible invariant subspaces of $T_p^* \otimes \xi_p$, and by $\pi_\nu: C^\infty(T^* \otimes \xi) \to C^\infty(F^\nu)$ ($\nu = 1, \ldots, \mu$) the natural orthogonal projections. Consider the first-order differential operators

$$P_\nu = \pi_\nu \circ \nabla, \quad \nu = 1, \ldots, \mu,$$

and denote by P_ν^* the operator formally adjoint to P_ν.

THEOREM 3.4. *If $\alpha \in C^\infty(\xi)$ and $\beta \in C^\infty(F^\nu)$, then*

$$(P_\nu \alpha, \beta) - (\alpha, P_\nu^* \beta) = \int_{\partial M} *i(\alpha)\beta, \tag{3.23}$$

where $i(\alpha)\beta$ denotes contraction of α and β to a vector via the metric.

If $\alpha_1, \alpha_2 \in C^\infty(\xi)$, then

$$(P_\nu^* P_\nu \alpha_1, \alpha_2) - (\alpha_1, P_\nu^* P_\nu \alpha_2) = \int_{\partial M} *(i(\alpha_1) P_\nu \alpha_2 - i(\alpha_2) P_\nu \alpha_1). \tag{3.24}$$

Proof. In local coordinates and corresponding multi-indices I, we differentiate

$$\nabla^i(\alpha^I \beta_{Ii}) = (\nabla^i \alpha^I)\beta_{Ii} + \alpha^I \nabla^i \beta_{Ii}$$

and integrate over M, obtaining the universal formula

$$(\nabla\alpha, \beta) - (\alpha, \nabla^* \beta) = \int_{\partial M} *i(\alpha)\beta, \tag{3.25}$$

from which (3.23) (i.e. Theorem 3.1) follows by applying the orthogonal projection P_ν. Now (3.24) is a direct consequence of setting $\beta = P_\nu \alpha_2$. □

4. Elliptic Boundary Value Problems

For S and L, several boundary conditions are geometrically natural. We are interested in self-adjoint and elliptic extensions of L up to the boundary, and also in the extent to which these boundary data determine the solutions to either $SX = 0$ or $LX = 0$ inside M.

In his paper on elasticity, Weyl [16] used the boundary condition of vanishing divergence and vanishing tangential part of the vector field at Σ. In our general setting these are again very natural, and together with Dirichlet and Neumann conditions turn out to be self-adjoint and elliptic in the sense of [5].

Let us now formulate these boundary conditions more precisely. Toward this end we define the following subspaces of $\mathfrak{X}$:

$$\mathfrak{X}_D = \{Z \in \mathfrak{X} \mid Z_p = 0 \text{ for all } p \in \partial M\} \tag{4.1}$$

for the boundary problem of the Dirichlet type (D). Also, define

$$\mathfrak{X}_N = \{Z \in \mathfrak{X} \mid Z_p^N = 0 \text{ and } (\nabla_N Z)_p^T = 0 \text{ for all } p \in \partial M\} \tag{4.2}$$

for the boundary problem of the Neumann type (N). In coordinates, the condition (N) reads (cf. (2.4)):

$$Z^n = 0 \quad \text{and} \quad \frac{\partial}{\partial r} Z^k = (A_N Z)^k \ \ (k = 1, \ldots, n-1) \quad \text{on } \partial M. \tag{4.2$'$}$$

Finally, define

$$\mathfrak{X}_E = \{ Z \in \mathfrak{X} \mid Z_p^T = 0 \text{ and } \operatorname{div} Z_p = 0 \text{ for all } p \in \partial M \} \tag{4.3}$$

for the boundary problem of the theory of elasticity (E). In coordinates, we obtain

$$Z^1 = \cdots = Z^{n-1} = 0 \quad \text{and} \quad \frac{\partial}{\partial r} Z^n = -Z^n \operatorname{tr} h \tag{4.3$'$}$$

by (2.14).

We will use the symbol $\mathfrak{X}_B$ when we do not distinguish any of the subspaces (4.1)–(4.3) so that $\mathfrak{X}_B$ may denote any of them.

The boundary conditions of type D, N, or E are self-adjoint in the following sense.

THEOREM 4.1. *If $Z_1, Z_2 \in \mathfrak{X}_B$ then*

$$(LZ_1, Z_2) = (Z_1, LZ_2). \tag{4.4}$$

Proof. By (3.22), it is enough to show that

$$SZ_2(Z_1, N) - SZ_1(Z_2, N) = 0. \tag{4.5}$$

This is a consequence of Lemmas 2.5, 2.6, and 2.7, respectively, and of the symmetry of h. □

Now we turn to the question of the elliptic properties of these boundary conditions. We follow the definitions of [5, esp. Chap. 1.9], together with the special geometric nature of S.

We consider a general operator with leading term $L_0 = ad\delta + b\delta d$, $a, b > 0$. This has leading symbol

$$p(\xi) = a\epsilon(\xi)\, i(\xi) + b i(\xi)\, \epsilon(\xi), \tag{4.6}$$

where ξ is a cotangent vector and i (resp. ϵ) denotes the interior (resp. exterior) product:

$$i(\xi)v = v(\xi, \cdot), \qquad \epsilon(\xi)v = \xi \wedge v.$$

Here we use the metric to identify tangent and cotangent vectors, so that $i(\xi)^* = \epsilon(\xi)$. Also, these operators act on alternating forms, so the v in the preceding formulas stands for (in our case) either a 1-form, a 2-form, or a 0-form. The ordinary differential equation governing the ellipticity at the boundary is in this case the constant coefficient

$$p(\zeta, D_r)\varphi = \lambda\varphi. \tag{4.7}$$

Here we again introduce normal coordinates near Σ: $(y, r) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $r \geq 0$, where the r-coordinate is the normal distance to the boundary. Then ζ is a cotangent vector to Σ, and $D_r = -i\partial/\partial r$. We need to solve (4.7) with $0 \neq \lambda$ a complex number, not in the positive real half-axis, such that

$$\varphi(r) \to 0 \quad \text{for } r \to \infty. \tag{4.8}$$

We recall for the convenience of the reader the notion of elliptic boundary value problem, following [5, pp. 70–72]. Let (as in our case) P be an elliptic formally self-adjoint second-order partial differential operator with leading symbol p, acting on the sections of some vector bundle V over M. At the boundary Σ we consider the bundle of Cauchy data $W = W_0 \oplus W_1 \cong V \oplus V$, where we may restrict the value and the normal derivative of a section of V to Σ to obtain a section of W. A boundary operator

$$B: C^\infty(W) \to C^\infty(W')$$

is a tangential differential operator over Σ going from the sections of W to the sections of some auxiliary graded vector bundle over Σ: $W_0' \oplus W_1'$. The graded leading symbol of B is denoted $\sigma(B)$, and is at each point y and covector ζ a 2×2 matrix. We say that (P, B) is *elliptic* with respect to $\mathbb{C} \backslash \mathbb{R}_+$ if $\det(p(x, \xi) - \lambda) \neq 0$ on the interior for all $(\xi, \lambda) \in T^*M \times (\mathbb{C} \backslash \mathbb{R}_+) \backslash \{(0, 0)\}$, and if on the boundary there always exists a unique solution to the ordinary differential equation (4.7) with boundary condition (4.8) such that $\sigma(B)(y, \zeta)\gamma(f) = f'$ for any prescribed $f' \in W'$. Here γ is the restriction described above.

As usual for constant coefficient equations, we set $\varphi(r) = \varphi_0 e^{i\mu r}$ with φ_0 a fixed vector and μ complex. Then, with $\xi = (\zeta, \mu)$, $\zeta \neq 0$, (4.7) becomes

$$[a\epsilon(\xi) i(\xi) + b i(\xi) \epsilon(\xi)] \varphi_0 = \lambda \varphi_0. \tag{4.9}$$

In case φ_0 is proportional to (ζ, μ), this gives

$$a(|\zeta|^2 + \mu^2) = \lambda; \tag{4.10}$$

that is, μ is not real. Choose the unique root μ of (4.10) with positive imaginary part. Then our solution is $\varphi(r) = \varphi_0 e^{i\mu r}$, which is determined uniquely by φ_0. Similarly, in case φ_0 is orthogonal to (ζ, μ), we have

$$b(|\zeta|^2 + \mu^2) = \lambda_0,$$

where the solution satisfying (4.8), $\varphi(r) = \varphi_0 e^{i\mu r}$, is determined by φ_0.

Now the question is to what extent the boundary data determine the φ_0. This is clear for the Dirichlet condition, which simply evaluates $\varphi(0)$.

For the elasticity condition, we first consider the form of the boundary operator B:

$$B: W_0 \oplus W_1 \to W_0' \oplus W_1',$$

where $W_0 \cong \mathbb{R}^n$ and $W_1 \cong \mathbb{R}^n$ represent the values of a field and its r-derivative at Σ. Take $W_0' \cong \mathbb{R}^{n-1}$ as the tangential part and $W_1' \cong \mathbb{R}$ as the divergence of the field at Σ, respectively. Then, from (4.3′) and the definition of the grading, the graded symbol of B in this case becomes

$$\sigma_g(B) = \begin{pmatrix} P_{n-1} & 0 \\ 0 & P_n' \end{pmatrix}$$

acting as $W_0 \oplus W_1$. Here P_{n-1} and P_n' denote projection on the first $n-1$ coordinates and the last coordinate, respectively. Clearly, the value of

$$\begin{pmatrix} P_{n-1} & 0 \\ 0 & P_n' \end{pmatrix} \begin{pmatrix} \varphi_0 \\ i\mu\varphi_0 \end{pmatrix}$$

determines φ_0, given $\mu \neq 0$.

For the Neumann condition, we arrive in a similar way at the boundary operator B, this time with graded symbol

$$\sigma_g(B) = \begin{pmatrix} P_n' & 0 \\ 0 & P_{n-1} \end{pmatrix}.$$

Again, φ_0 will be uniquely determined given $\mu \neq 0$ from the value of this matrix applied to $(\varphi_0, i\mu\varphi_0)$. Thus we have established the following theorem.

THEOREM 4.2. *With respect to either of the three boundary conditions D, N, or E, the Ahlfors Laplacian L is self-adjoint and elliptic. In particular, L has a complete orthonormal system of eigenfields $Z_1, Z_2, \dots$ for each boundary condition: $LZ_k = \lambda_k Z_k$, with Z_k of class C^∞ and satisfying the boundary condition in question. Here the eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ grow exponentially.*

In the rest of this section, we study the problem of uniqueness of fields on M satisfying either $SX = 0$ or $LX = 0$ with boundary data given via D, N, or E.

Let us first look at a special case, namely $M \subseteq \mathbb{R}^3$, a bounded Euclidean domain. Here a conformal vector field has the general form

$$X(x) = v + \lambda x + Ax + (w,x)x + x^2 w, \tag{4.11}$$

where $v, w \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$, A is skew-symmetric, and $x^2 = (x,x)$ is the scalar product. (In fact, (4.11) is also the formula in $\mathbb{R}^n$.) The zero set for this is $\{x \mid X(x) = 0\}$, which entails three quadratic equations. Two of these describe, for $w \neq 0$, hyperbolic paraboloids; the third describes an ellipsoid. The intersection of three such quadrics is generically eight isolated points, and never a hypersurface. For isometries we have $w = 0$ and hence a linear system of equations. In any case we see that if a conformal vector field vanishes on the boundary of M, then it is identically zero in M. A similar result holds in $\mathbb{R}^n$ by the same reasoning.

Thus an interesting problem would be to characterize the zero sets of conformal vector fields on manifolds—of course, for isometrics these are totally geodesic—and we suspect that these are very special sets. What we address in the following is the slightly weaker problem: Given a conformal vector field X on M (i.e. $SX = 0$) satisfying boundary conditions D, N, or E, when is X identically zero in M? Also, we consider the same problem for the weaker equation $S^*SX = 0$.

THEOREM 4.3. *If M and Σ are real analytic, $SZ = 0$ on M, and $Z \in \mathfrak{X}_D$, then $Z = 0$ on M.*

Proof. Take a point $p \in \Sigma$ and an analytic coordinate chart around p. In this chart $SZ = 0$ is a first-order system of differential equations with the initial condition $Z = 0$ on Σ. By the Cauchy–Kowalewska theorem (of [7, p. 36]) it follows that this initial value problem has a unique analytic solution in the small; that is, there exists a neighborhood V of p such that $Z = 0$ in V. By [8, Lemma 2], $Z = 0$ everywhere on M. □

THEOREM 4.4. *If M and Σ are real analytic, $SZ = 0$ on M, and $Z \in \mathfrak{X}_N$, then $Z = 0$ on M unless h, the second fundamental form of Σ, is degenerate everywhere.*

Proof. The assumption $Z \in \mathfrak{X}_N$ means that

$$Z^n = 0 \quad \text{and} \quad (\nabla_N Z)^T = 0. \tag{4.12}$$

By (1.4) and the orthogonality $e_i \perp e_n$, we obtain

$$SZ(e_i, e_n) = \tfrac{1}{2}[g(\nabla_{e_i} Z, e_n) + g(e_i, \nabla_{e_i} Z)].$$

Now, by (4.12) and (2.7) we have that

$$SZ(e_i, e_n) = \tfrac{1}{2} h(Z, e_i).$$

Since $SZ = 0$ it follows that

$$h(Z, e_i) = 0, \quad i = 1, \ldots, n = 1$$

on Σ. If now h is nondegenerate in a single point then it is nondegenerate in a neighborhood (contained in Σ) of this point, so $Z = 0$ in this neighborhood. Applying Theorem 4.3, we obtain our assertion. □

REMARK 4.4′. An analogous uniqueness theorem may be formulated if we replace the condition $Z \in \mathfrak{X}_N$ by $Z \in \mathfrak{X}_{N'}$, that is, (4.12) by

$$Z^n = 0 \quad \text{and} \quad [N, Z]^T = 0. \tag{4.12′}$$

THEOREM 4.5. *If M and Σ are real analytic, $SZ = 0$ on M, and $Z \in \mathfrak{X}_E$, then $Z = 0$ on M unless Σ is totally geodesic.*

Proof. The assumption $Z \in \mathfrak{X}_E$ means that

$$Z^T = 0 \quad \text{and} \quad \operatorname{div} Z = 0 \tag{4.13}$$

on Σ. Since $\operatorname{div} Z = 0$, by (2.19) we have

$$SZ(e_i,e_j)=Z^n h(e_i,e_j).$$

By the assumption $SZ=0$, it follows that at each part of Σ

$$Z^n h(e_i,e_j)=0. \tag{4.14}$$

If now $h\neq 0$ in a single point, then $h=0$ in some neighborhood (contained in Σ) of this point and so $Z^n=0$ in this neighborhood; that is, $Z=0$ there. Applying Theorem (4.3) we obtain that $Z=0$ everywhere on M. ❑

REMARK 4.5′. If we replace the assumption $Z\in\mathfrak{X}_E$ by $Z\in\mathfrak{X}_{E'}$, that is, (4.13) by

$$Z^T=0 \quad\text{and}\quad \operatorname{div} Z=-\frac{n}{n-1}Z^n\operatorname{tr}h, \tag{4.13′}$$

then we have

$$Z^n\left(h(e_i,e_j)-\frac{1}{n-1}\operatorname{tr}h\,g_{ij}\right)=0 \tag{4.14′}$$

instead of (4.13), and so $Z=0$ on M unless

$$h(e_i,e_j)-\frac{1}{n-1}\operatorname{tr}h\,g_{ij}=0,\quad i,j=1,\dots,n-1,$$

everywhere on Σ.

It is worth noting that the uniqueness of the conformal extension may also be obtained under some assumptions on the geometry of M and ∂M (cf. [17, Prop. 3.7 & 3.8, p. 124]).

Consider now a more general question of uniqueness for the equation $S^*SZ=0$. Since by (3.14)

$$(S^*SZ,Z)-(SZ,SZ)=-\int_{\partial M}SZ(Z,N)\operatorname{vol}_{\partial M}, \tag{4.15}$$

we obtain the following lemma.

LEMMA 4.1. *If $S^*SZ=0$ on M, then*

$$(SZ,SZ)=\begin{cases}0 & \text{if } Z\in\mathfrak{X}_D, & \text{(i)}\\ -\frac{1}{2}\int_{\partial M}h(Z,Z)\operatorname{vol}_{\partial M} & \text{if } Z\in\mathfrak{X}_N, & \text{(ii)}\\ \int_{\partial M}Z^n\operatorname{tr}h\operatorname{vol}_{\partial M} & \text{if } Z\in\mathfrak{X}_E. & \text{(iii)}\end{cases}$$

Proof. Formulas (i)–(iii) are obtained by applying (4.15) and Lemmas 2.5, 2.6, and 2.7, respectively. □

REMARK 4.1′. If in (ii) we replace $Z\in\mathfrak{X}_N$ by $Z\in\mathfrak{X}_{N'}$, that is, the boundary conditions (4.12) by (4.12′); or if in (iii) we replace $Z\in\mathfrak{X}_E$ by $Z\in\mathfrak{X}_{E'}$, that is, the boundary conditions (4.13) by (4.13′), then in these two cases we also have that $S^*SZ=0$ on M implies $(SZ,SZ)=0$.

We finish this section with the remark that when L is any of the self-adjoint elliptic extensions assumed to be without zero modes, then we may infer the existence of a Poisson kernel for L. In this case, there is a kernel $P(x,y)$ such that

$$X(x)=\int_{\partial M}P(x,y)X'(y)\,dy$$

gives the unique solution to $LX=0$ with boundary data X' (considered as an element in a complement to the boundary condition subspace $W_0'\oplus W_1'\subseteq W_0\oplus W_1$; see above).

5. The Heat Kernel for L and its Asymptotics

In this section, L will be one of the self-adjoint elliptic extensions of the Ahlfors Laplacian on M with boundary Σ. We consider the heat semigroup $\exp(-tL)$, which is a kernel operator with smooth kernel

$$H(t,x,y)=\sum_{k=1}^{\infty}e^{-t\lambda_k}Z_k(x)\otimes Z_k(y)^*, \tag{5.1}$$

where we use notation as in Theorem 4.2. Equation (5.1) is infinitely smoothing, and its trace is

$$\operatorname{tr}\exp(-tL)=\sum_{k=1}^{\infty}e^{-t\lambda_k}=\int_M H(t,x,x)\,dx. \tag{5.2}$$

It follows from general principles (see e.g. [5]) that (5.2) has an asymptotic expansion

$$\operatorname{tr}\exp(-tL)\sim\sum_{i=0}^{\infty}a_i t^{(i-n)/2},\quad t\downarrow 0, \tag{5.3}$$

where the coefficients a_i are integrals of local expressions in the jets of the symbol of L. It follows from the invariant theory in [5] that the first two terms are given by

$$a_0=\alpha_0\cdot\operatorname{vol}(M),\qquad a_1=\alpha_1\cdot\operatorname{vol}(\Sigma),$$

where α_0 and α_1 are universal constants depending only on the dimension (and L).

The leading term can be found by calculating the kernel for the Euclidean case: Let $L_0=ad\delta+b\delta d$ in $\mathbb{R}^n$ with $a,b>0$. Using the Fourier transformation, the heat kernel is

$$H_0(t,x)=\int_{\mathbb{R}^n}\operatorname{tr}e^{-t\lambda(y)}\,dy. \tag{5.4}$$

When $a\neq b$, $\lambda(y)$ is not diagonal, but we may easily diagonalize it. With P_y denoting the orthogonal projection on y, we have

$$\lambda(y)=(y,y)(aP_y+b(I-P_y)),$$

so we obtain the trace

$$\operatorname{tr}e^{-t\lambda(y)}=e^{-a(y,y)t}+(n-1)e^{-b(y,y)t}.$$

Taking the Fourier transformation yields the following.

PROPOSITION 5.1. *In Euclidean space* $\mathbb{R}^n$, *the heat kernel has the pointwise trace*

$$\operatorname{tr} H_0(t,y) = (2\pi)^{-n}\int_{\mathbb{R}^n}(e^{-a(y,y)t}+(n-1)e^{-b(y,y)t})e^{i(x,y)}\,dy$$

with coincidence value

$$\operatorname{tr} H_0(t,0) = (2\pi)^{-n}\cdot\pi^{n/2}\cdot(a^{-n/2}+(n-1)b^{-n/2})\cdot t^{-n/2}. \tag{5.5}$$

Thus the coefficient to $t^{-n/2}$ in (5.5) gives α_0.

To calculate α_1 we look at a model case, namely the cube $M=[0,\pi]^n\subseteq\mathbb{R}^n$. Although the boundary of M is not smooth, its singularities will only contribute to the higher-order terms beyond the first two. We proceed in the usual way: Since L_0 is constant-coefficient, we look for trigonometric solutions of the form $\psi = ve^{i(x,\xi)}$ for a fixed $\xi\in\mathbb{R}^n$. Here

$$\begin{aligned} L_0\psi &= [a\epsilon(\xi)i(\xi)+bi(\xi)\epsilon(\xi)]\psi \\ &= \lambda(\xi)\psi, \end{aligned}$$

so these provide eigenfunctions with eigenvalues given by those of $\lambda(\xi)$. The problem now is to find the ξs satisfying the boundary conditions. This will mean (just as in the well-known case of the Laplacian on functions) restricting ξ to some integral lattice in $\mathbb{R}^n$.

We first find the eigenfunctions satisfying the elasticity conditions (E). Consider

$$\begin{aligned} \psi = (&v_1\cos x_1\xi_1\sin x_2\xi_2\sin x_3\xi_3\ldots, \\ &v_2\sin x_1\xi_1\cos x_2\xi_2\sin x_3\xi_3\ldots, \\ &v_3\sin x_1\xi_1\sin x_2\xi_2\cos x_3\xi_3\ldots, \\ &\vdots \\ &v_n\sin x_1\xi_1\quad\ldots\quad\cos x_n\xi_n). \end{aligned} \tag{5.6}$$

Note that if ξ is integral (i.e., all $\xi_i\in\mathbb{Z}$) then (5.6) is normal to the boundary, has zero divergence there, and is a basis of all such fields. Now, using $L_0 = b\Delta+(a-b)d\delta$ with Δ the Laplacian, one calculates the action of L_0 on $\psi=\psi_{v,\xi}$ again to be

$$L_0\psi = \psi_{\lambda(\xi)v,\xi}. \tag{5.7}$$

(This is perhaps slightly surprising, since (5.6) is in a complicated way a superposition of the original trigonometric solutions.) The redundancy in the parameter ξ is such that we get all solutions once by taking

$$v=\xi,\qquad \xi_i>0; \tag{5.8a}$$

$$v\perp\xi,\qquad \xi_i\ge 0\quad\text{(no two } \xi_i \text{ are zero)}. \tag{5.8b}$$

Collecting these facts, we may now compute the trace in (5.2) in this case to be

$$\sum_{\xi}e^{-a(\xi,\xi)t}+\sum_{\xi}m(\xi)e^{-b(\xi,\xi)t}, \tag{5.9}$$

where the sums are over (5.9a) and (5.9b), respectively, and where the multiplicity is $m(\xi) = n-1$ if all $\xi_i \geq 0$ and $m(\xi) = 1$ otherwise. Recall that, from the heat equation on the circle, we know that

$$\sum_{k=-\infty}^{\infty} e^{-tk^2} \sim (4\pi t)^{-1/2} \cdot 2\pi, \quad t \downarrow 0, \tag{5.10}$$

from which we can actually derive the full asymptotic behavior of (5.9) (but we need only the first two terms): this is (note the agreement with (5.5))

$$\begin{aligned}
&\left((4\pi a t)^{-1/2} \cdot 2\pi \cdot \frac{1}{2} - \frac{1}{2}\right)^n \\
&\quad + (n-1) \cdot \left[\left((4\pi b t)^{-1/2} \cdot 2\pi \cdot \frac{1}{2} - \frac{1}{2}\right)^n + \frac{n}{n-1}\left((4\pi b t)^{1/2} \cdot 2\pi \cdot \frac{1}{2} - \frac{1}{2}\right)^{n-1}\right] \\
&\quad = (4\pi t)^{-n/2} \cdot \mathrm{vol}(M) \cdot [a^{-n/2} + (n-1) b^{-n/2}] \\
&\quad + (4\pi t)^{-(n-1)/2} \cdot \mathrm{vol}(\partial M) \cdot \left[-\frac{1}{4} a^{-(n-1)/2} - \frac{n-3}{4} b^{-(n-1)/2}\right] \\
&\quad + \cdots.
\end{aligned} \tag{5.11}$$

We can deal with the Neumann condition in the same way. In (5.5), simply change all cosines to sines and vice versa; then, again for the ξ integral, this is tangential to the boundary and the normal derivative vanishes at the boundary. Also, it is again an eigenvector as in (5.7), with v either parallel to ξ or orthogonal to ξ. This time the redundancy is taken into account by

$$v = \xi, \qquad \xi_i \geq 0 \quad \text{(not all } \xi_i \text{ are zero);} \tag{5.12a}$$

$$v \perp \xi, \qquad \xi_i \geq 0 \quad \text{(not all } \xi_i \text{ are zero).} \tag{5.12b}$$

Using (5.9), (5.10), and (5.12), the asymptotics are therefore

$$\begin{aligned}
&\left((4\pi a t)^{-1/2} \cdot 2\pi \cdot \frac{1}{2} - \frac{1}{2}\right)^n + (n-1)\left((4\pi b t)^{-1/2} \cdot 2\pi \cdot \frac{1}{2} + \frac{1}{2}\right)^n - n \\
&\quad = (4\pi t)^{-n/2} \cdot \mathrm{vol}(M) \cdot [a^{-n/2} + (n-1) b^{-n/2}] \\
&\quad + \frac{1}{4}(4\pi t)^{-(n-1)/2} \cdot \mathrm{vol}(\delta M) \cdot [a^{-(n-1)/2} + (n-1) b^{-(n-1)/2}] \\
&\quad + \cdots.
\end{aligned} \tag{5.13}$$

Finally, in the Dirichlet case we can take all the trigonometric functions in (5.6) as sines, and the redundancy is now removed by taking all $\xi_i > 0$. This gives the asymptotics

$$\begin{aligned}
&\left((4\pi a t)^{-1/2} \cdot 2\pi \cdot \frac{1}{2} - \frac{1}{2}\right)^n + (n-1)\left((4\pi b t)^{-1/2} \cdot 2\pi \cdot \frac{1}{2} - \frac{1}{2}\right)^n \\
&\quad = (4\pi t)^{-n/2} \cdot \mathrm{vol}(M) \cdot [a^{-n/2} + (n-1) b^{-n/2}] \\
&\quad - \frac{1}{4}(4\pi t)^{-(n-1)/2} \cdot \mathrm{vol}(\partial M) \cdot [a^{-(n-1)/2} + (n-1) b^{-(n-1)/2}] \\
&\quad + \cdots.
\end{aligned} \tag{5.14}$$

Collecting the information above—especially (5.11), (5.13), and (5.14)—establishes the following result.

THEOREM 5.1. *Let L be the self-adjoint elliptic extension of the Ahlfors Laplacian with boundary conditions D, N, or E on the manifold M. Then the small-time asymptotics of the heat kernel have the first two terms as follows.*

Case D:

$$\operatorname{tr}\exp(-tL) \sim (4\pi t)^{-n/2}\cdot \operatorname{vol}(M)\cdot[a^{-n/2}+(n-1)b^{-n/2}] \\ -\tfrac{1}{4}(4\pi t)^{-(n-1)/2}\cdot \operatorname{vol}(\partial M)\cdot[a^{-(n-1)/2}+(n-1)b^{-(n-1)/2}].$$

Case N:

$$\operatorname{tr}\exp(-tL) \sim (4\pi t)^{-n/2}\cdot \operatorname{vol}(M)\cdot[a^{-n/2}+(n-1)b^{-n/2}] \\ +\tfrac{1}{4}(4\pi t)^{-(n-1)/2}\cdot \operatorname{vol}(\partial M)\cdot[a^{-(n-1)/2}+(n-3)b^{-(n-1)/2}].$$

Case E:

$$\operatorname{tr}\exp(-tL) \sim (4\pi t)^{-n/2}\cdot \operatorname{vol}(M)\cdot[a^{-n/2}+(n-1)b^{-n/2}] \\ -\tfrac{1}{4}(4\pi t)^{-(n-1)/2}\cdot \operatorname{vol}(\partial M)\cdot[a^{-(n-1)/2}+(n-3)b^{-(n-1)/2}].$$

Here a, b are the values in the Ahlfors Laplacian: $a=(n-1)/n$ and $b=\frac{1}{2}$. (Actually, in Case E they could be arbitrary positive; L would still be self-adjoint and elliptic.)

Note that the terms agree with the pattern established in [3], even though these authors consider only operators with metric leading symbol. From Theorem 5.1 we may, via the Tauberian theorem, deduce the asymptotic distribution of eigenvalues, thus generalizing and sharpening the main theorem in [16].

COROLLARY 5.1. *Let $N(\lambda)$ denote the number of eigenvalues for L less than λ. Then for our three cases the first terms in the asymptotic expansion for $N(\lambda)$ are as follows.*

Case D:

$$N(\lambda) \sim \frac{(4\pi)^{-n/2}}{\Gamma(n/2+1)}\cdot \operatorname{vol}(M)\cdot[a^{-n/2}+(n-1)b^{-n/2}]\cdot\lambda^{n/2}.$$

Case N:

$$N(\lambda) \sim \frac{(4\pi)^{-n/2}}{\Gamma(n/2+1)}\cdot \operatorname{vol}(M)\cdot[a^{-n/2}+(n-1)b^{-n/2}]\cdot\lambda^{n/2}.$$

Case E:

$$N(\lambda) \sim \frac{(4\pi)^{-n/2}}{\Gamma(n/2+1)}\cdot \operatorname{vol}(M)\cdot[a^{-n/2}+(n-1)b^{-n/2}]\cdot\lambda^{n/2}.$$

When $n=3$, for the coefficient $(4\pi)^{-3/2}/\Gamma(3/2+1)$ we obtain the value given by Weyl, $\pi^{-2}/6$.

We conclude by remarking that, with an elliptic L as above, we may use (as in [9]) the semigroup $\exp(-tL)$ to smoothen vector fields into quasi-conformal fields and eventually in the limit $t\to\infty$ to the fields in the kernel of L. Via Sobolev estimates as in [9], one may control the constant of quasi-conformality. We think this idea will be important in the study of 1-parameter families of quasi-conformal transformations, and perhaps also in the theory of moduli of Riemannian manifolds, one of our long-term motivations.

An interesting topic of further study would involve calculating the higher terms coming from the boundary in the asymptotic expansion (5.3) as well as interpreting these geometrically. See [3] and [9].

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