An Isoperimetric-Type Inequality for Integrals of Green's Functions

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1. Introduction

Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, with finite volume. Let D^* be the ball centered at the origin and of the same volume as D. Denote by $G_D(w,z)$ and $G_{D^*}(w,z)$ the Green's functions for D and D^*, respectively. The following isoperimetric inequality is now a classical result (see [1, p. 61]):

$$\sup_{w \in D} \int_D \varphi(G_D(w,z))\,dz \leq \int_{D^*} \varphi(G_{D^*}(0,z))\,dz \tag{1.1}$$

for all nonnegative nondecreasing functions φ defined on $[0,\infty)$.

However, for a large class of domains D and functions φ, what determines the finiteness of the quantity on the left-hand side of (1.1) is not the volume of the domain but rather its inner radius. This is true in particular for any simply connected domain in the plane. More precisely, if D is a simply connected domain in the complex plane we let R_D be the radius of the largest disc contained in D (if such a disc exists) and the limit superior of the radii of all discs contained in D otherwise. We call R_D the *inner radius* of D. Assume φ satisfies

$$\int_0^\infty r\varphi(\log(\coth(r)))\,dr < \infty,$$

(e.g., $\varphi(x) = x^p$, $0 < p < \infty$), then by Bañuelos and Carroll [2] we have

$$\sup_{w \in D} \int_D \varphi(G_D(w,z))\,dz < \infty \tag{1.2}$$

if and only if $R_D < \infty$. It is then natural to inquire about the following extremal problem: Amongst all simply connected planar domains D with $R_D = 1$, find those that maximize the left-hand side of (1.2). This problem, which is wide open even for $\varphi(x) = x$, is closely related to an extremal problem for the lowest Dirichlet eigenvalue of D and to the well-known problem in function theory concerning the schlicht Bloch–Landau constant. We refer the reader to [2] for more on this connection. When we restrict ourselves to convex domains, it has been proved by Sperb [9, p. 87] that

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$$\sup_{w \in D} \int_D G_D(w, z)\, dz \le \int_S G_S(0, w)\, dw, \tag{1.3}$$

where

$$S = \{z = x + iy \colon -1 < y < 1\}$$

is the infinite strip of inner radius 1. Sperb's proof, which is based on the maximum principle for the torsion problem $\Delta u = -2$ in D and u vanishing on the boundary, does not seem to apply with other φs. In this paper we prove the followoing theorem.

THEOREM 1. *Let D be a convex domain in the complex plane of inner radius* 1. *Let $\varphi \colon [0, \infty) \to [0, \infty)$ be nondecreasing. Then*

$$\sup_{w \in D} \int_D \varphi(G_D(w, z))\, dz \le \left(\frac{2\pi^2}{\Gamma(1/3)^3} \right)^2 \int_S \varphi(G_S(0, z))\, dz. \tag{1.4}$$

The constant $2\pi^2/\Gamma(1/3)^3$ is approximately 1.0541, and we of course conjecture that the inequality (1.4) holds with this constant replaced by 1.

For simply connected domains D in the plane, integrals involving the Green's function of D can be written in terms of the derivative of the conformal mapping that sends the unit disc U onto D. More precisely, if $F \colon U \to D$ is a conformal mapping with $F(0) = w$, then the conformal invariance of the Green's function gives

$$\begin{aligned} \int_D \varphi(G_D(w, z))\, dz &= \int_U \varphi\left(\frac{1}{2\pi} \log \frac{1}{|z|} \right) |F'(z)|^2\, dz \\ &= \int_0^1 r\varphi\left(\frac{1}{2\pi} \log \frac{1}{r} \right) \left(\int_0^{2\pi} |F'(re^{i\theta})|^2\, d\theta \right) dr. \end{aligned} \tag{1.5}$$

Thus a more general extremal problem, which was first raised in [2], is the following: Define

$$\mathfrak{F} = \{F \text{ univalent in } U \text{ with } R_{F(U)} = 1\}.$$

Is there a $\psi \in \mathfrak{F}$ with the property that, for all $F \in \mathfrak{F}$ and all $r \in (0, 1)$,

$$\int_0^{2\pi} |F'(re^{i\theta})|^2\, d\theta \le \int_0^{2\pi} |\psi'(re^{i\theta})|^2\, d\theta\, ? \tag{1.6}$$

An extremal for this problem would also give an extremal for (1.2) and for the schlicht Bloch–Landau constant problem, and also, quite likely, for the extremal problem concerning the lowest Dirichlet eigenvalue for domains of inner radius 1. We again refer the reader to Bañuelos and Carroll [2] for more information on these problems. As before, if we restrict the family $\mathfrak{F}$ to

$$\mathfrak{F}_c = \{F \text{ univalent in } U,\ F(U) \text{ convex},\ R_{F(U)} = 1\},$$

it seems reasonable to conjecture, (see [2] for more motivation on this) that (1.6) should hold with $\psi(0) = 0$ and $\psi'(0) > 0$, where ψ is the map that takes the unit disc onto the strip S. Here we prove the following result, which suggests to us that this conjecture should be true.

THEOREM 2. *Let F be univalent in the unit disc U such that $F(U)$ is convex and $R_{F(U)} = 1$. Then, for all $0 < r \leq 1/2$,*

$$\int_0^{2\pi} |F'(re^{i\theta})|^2 \, d\theta \leq \left(\frac{2\pi^2}{\Gamma(1/3)^3}\right)^2 \int_0^{2\pi} |\psi_S'(re^{i\theta})|^2 \, d\theta, \tag{1.7}$$

where

$$\psi_S(z) = \frac{2}{\pi} \log\left(\frac{1+z}{1-z}\right)$$

is the conformal map from U onto S with $\psi_S(0) = 0$.

Theorems 1 and 2 will be consequences of domain monotonicity (subordination) and the following.

THEOREM 3. *Let F be a conformal mapping of the unit disc U onto a triangle T of inner radius R_T and inner angles $\alpha\pi$, $\beta\pi$, and $\gamma\pi$. Then, for $0 < r < 1$,*

$$\int_0^{2\pi} |F'(re^{i\theta})|^2 \, d\theta \leq C_{\alpha,\beta,\gamma}^2 R_T^2 \int_0^{2\pi} |\psi_S'(re^{i\theta})|^2 \, d\theta, \tag{1.8}$$

where

$$C_{\alpha,\beta,\gamma} = \frac{\pi^2}{(4\Gamma(\alpha)\sin(\alpha\pi/2)\Gamma(\beta)\sin(\beta\pi/2)\Gamma(\gamma)\sin(\gamma\pi/2))}.$$

This inequality is sharp in the sense that $C_{\alpha,\beta,\gamma} \to 1$ as the triangle tends to a strip.

Inequalities for integral means of univalent functions have been extensively studied for many years using, among other methods, extreme point theory and the powerful *-function of Baernstein (see Duren [4] for some of this work). Most of this literature, however, deals with univalent functions F and derivatives F', under the normalization $F'(0) = 1$, and in many of these cases the Koebe function is shown to be the extremal. To the best of our knowledge, integral means of derivatives of univalent functions F under the normalization $R_{F(U)} = 1$ have not been studied before. We know of only one result in the literature which, when properly interpreted, is related to this. First, a univalent function F in the Hardy space H^1 of U is in BMOA if its boundary function on the circle is in BMO, the space of functions of bounded mean oscillation on the circle. It follows from Bañuelos and Øksendal [3] and Bañuelos and Carroll [2] that a univalent function F is in BMOA if and only if $R_{F(U)} < \infty$. Furthermore, there exist universal constants c_1 and c_2 such that

$$c_1 R_{F(U)} \leq \|F\|_{\text{BMOA}} \leq c_2 R_{F(U)}.$$

(See also [8] for more on the BMO connection.) With this notation, the result of Nowak [7] (see also [5]) can be stated as follows.

THEOREM A. *Let F be a univalent function in U with $R_{F(U)} = 1$ and $F(0) = 0$. Then there exists a universal constant A such that*

$$\int_0^{2\pi} \varphi(|F(re^{i\theta})|)\,d\theta \le \int_0^{2\pi} \varphi(A|\psi_S(re^{i\theta})|)\,d\theta, \quad 0 \le r < 1, \tag{1.9}$$

for all nonnegative nondecreasing convex functions φ *on* $[0,\infty)$.

One would hope that, when $F(U)$ is convex, the constant A can be taken to be 1. This is indeed the case in various special situations (such as equilateral triangles), but we have not been able to prove this in general. The proof in [7], via Baernstein's *-function, relies on the decomposition of BMO functions in terms of an L^∞-function plus the Hilbert transform of another L^∞-function. Such a proof provides no information on the size of the constant A.

2. Proofs

In this section we present the proofs of Theorems 1, 2, and 3. We shall first prove Theorem 3 and then show how Theorems 1 and 2 follow from this. We start by computing the inner radius of a triangle in terms of the conformal mapping from the disc to the triangle.

LEMMA 1. *Let* ξ_1, ξ_2, ξ_3 *be three points on* ∂U *and* α, β, γ *three nonnegative numbers such that* $\alpha+\beta+\gamma = 1$. *Let*

$$F'(z) = (z-\xi_1)^{\alpha-1}(z-\xi_2)^{\beta-1}(z-\xi_3)^{\gamma-1} \tag{2.1}$$

be the derivative of a conformal mapping from U *onto a triangle* T *with angles* $\alpha\pi, \beta\pi, \gamma\pi$ *and vertices at* $w_1 = F(\xi_1)$, $w_2 = F(\xi_2)$, *and* $w_3 = F(\xi_3)$, *respectively. Then*

$$R_T = \frac{2}{\pi}\frac{\Gamma(\alpha)\sin(\alpha\pi/2)\Gamma(\beta)\sin(\beta\pi/2)\Gamma(\gamma)\sin(\gamma\pi/2)}{|\xi_2-\xi_3|^\alpha|\xi_1-\xi_3|^\beta|\xi_1-\xi_2|^\gamma}. \tag{2.2}$$

Proof. Let w_0 be the incenter of the triangle T. That is, w_0 is the center of the largest disc contained in T. Then the segments from w_1 to w_0 and from w_2 to w_0 bisect the angles at w_1 and w_2. Thus

$$R_T = \frac{\text{length}(\overline{w_1w_2})}{\cot(\alpha\pi/2)+\cot(\beta\pi/2)}. \tag{2.3}$$

However,

$$\text{length}(\overline{w_1w_2}) = \int_{\xi_1}^{\xi_2} |F'(e^{i\theta})|\,d\theta$$

and we need to compute this quantity. To do this let $w: U \to \mathbb{H}$, where $\mathbb{H}$ is the upper half-space, be defined by

$$w(z) = \frac{\xi_3-\xi_2}{\xi_1-\xi_2}\cdot\frac{\xi_1-z}{\xi_3-z}.$$

Then $w(\xi_1) = 0$, $w(\xi_2) = 1$, and $w(\xi_3) = \infty$. Setting $V(z) = w^{-1}(z)$, we obtain

$$\int_{\xi_1}^{\xi_2} |F'(e^{i\theta})|\,d\theta = \int_0^1 |F'(V(t))||V'(t)|\,dt. \tag{2.4}$$

Let

$$a = \frac{\xi_3 - \xi_2}{\xi_1 - \xi_2}.$$

A simple calculation gives

$$V(z) = \frac{\xi_3 z - a\xi_1}{z - a} \quad \text{and} \quad |V'(t)| = \frac{a|\xi_1 - \xi_3|}{|t-a|^2}.$$

Since

$$|V(t) - \xi_1|^{\alpha-1} = \frac{t^{\alpha-1}|\xi_1 - \xi_3|^{\alpha-1}}{|t-a|^{\alpha-1}},$$

$$|V(t) - \xi_2|^{\beta-1} = \frac{|\xi_2 - \xi_3|^{\beta-1}|t-1|^{\beta-1}}{|t-a|^{\beta-1}},$$

and

$$|V(t) - \xi_3|^{\gamma-1} = \frac{|\xi_3 - \xi_2|^{\gamma-1}|\xi_1 - \xi_3|^{\gamma-1}}{|\xi_1 - \xi_2|^{\gamma-1}|t-a|^{\gamma-1}},$$

we have, after noticing that $1-\alpha+1-\beta+1-\gamma = 2$ and substituting in (2.4), that

$$\begin{aligned}\int_{\xi_1}^{\xi_2} |F'(e^{i\theta})|\, d\theta &= \frac{1}{|\xi_2 - \xi_3|^\alpha |\xi_1 - \xi_3|^\beta |\xi_1 - \xi_2|^\gamma} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)|\xi_2 - \xi_3|^\alpha |\xi_1 - \xi_3|^\beta |\xi_1 - \xi_2|^\gamma}.\end{aligned}$$

Thus

$$\begin{aligned} R_T &= \frac{\Gamma(\alpha)\Gamma(\beta)}{[\cot(\alpha\pi/2) + \cot(\beta\pi/2)]\Gamma(\alpha+\beta)|\xi_2 - \xi_3|^\alpha |\xi_1 - \xi_3|^\beta |\xi_1 - \xi_2|^\gamma} \\ &= \frac{\sin(\alpha\pi/2)\Gamma(\alpha)\sin(\beta\pi/2)\Gamma(\beta)}{\sin((\alpha+\beta)\pi/2)\Gamma(\alpha+\beta)|\xi_2 - \xi_3|^\alpha |\xi_1 - \xi_3|^\beta |\xi_1 - \xi_2|^\gamma} \\ &= \frac{2}{\pi} \frac{\Gamma(\alpha)\sin(\alpha\pi/2)\Gamma(\beta)\sin(\beta\pi/2)\Gamma(\gamma)\sin(\gamma\pi/2)}{|\xi_2 - \xi_3|^\alpha |\xi_1 - \xi_3|^\beta |\xi_1 - \xi_2|^\gamma},\end{aligned}$$

where the last equality follows from the fact that $\alpha+\beta+\gamma = 1$ and the identity

$$\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin(\pi t)}.$$

This completes the proof of the lemma. □

REMARK. In [10], Szegö presents a similar computation for the inner radius of T in terms of the conformal mapping from the upper half-space to T. His formula is a little cleaner.

Next, for any $\xi \in \partial U$, we denote by

$$K_\xi(re^{i\theta}) = \frac{1-r^2}{|re^{i\theta} - \xi|^2}$$

the Poisson kernel for the unit disc.

LEMMA 2. *Let ξ_1 and ξ_2 be any two points on the unit circle. Then, for any* $0<r<1$,

$$\int_0^{2\pi} K_{\xi_1}(re^{i\theta})K_{\xi_2}(re^{i\theta})\,d\theta \le \frac{4}{|\xi_1-\xi_2|^2}\int_0^{2\pi} K_{-1}(re^{i\theta})K_1(re^{i\theta})\,d\theta. \tag{2.5}$$

Proof. Let $\xi_1 = e^{i\theta_1}$ and $\xi_2 = e^{i\theta_2}$. By the semigroup property of the Poisson kernel,

$$\frac{1}{2\pi}\int_0^{2\pi} K_{\xi_1}(re^{i\theta})K_{\xi_2}(re^{i\theta})\,d\theta = \frac{1-r^4}{|r^2e^{i\theta_2}-e^{i\theta_1}|^2} \tag{2.6}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} K_1(re^{i\theta})K_{-1}(re^{i\theta})\,d\theta = \frac{1-r^2}{1+r^2}. \tag{2.7}$$

Thus (2.5) is equivalent to

$$\frac{(1+r^2)^2}{4}\frac{|e^{i\theta_1}-e^{i\theta_2}|^2}{|r^2e^{i\theta_2}-e^{i\theta_1}|^2} \le 1. \tag{2.8}$$

However,

$$\begin{aligned}\frac{(1+r^2)^2}{4}\frac{|e^{i\theta_1}-e^{i\theta_2}|^2}{|r^2e^{i\theta_2}-e^{i\theta_1}|^2} &= \frac{2(1+r^2)^2(1-\cos(\theta_1-\theta_2))}{4(1+r^4-2r^2\cos(\theta_1-\theta_2))}\\ &= \frac{(1+r^2)^2(1-\cos(\theta_1-\theta_2))}{2[(1-r^2)^2+2r^2(1-\cos(\theta_1-\theta_2))]}.\end{aligned}$$

Since $1-\cos(\theta_1-\theta_2)\le 2$, we have

$$(1-r^2)^2(1-\cos(\theta_1-\theta_2)) \le 2(1-r^2)^2.$$

Thus,

$$(1+r^2)^2(1-\cos(\theta_1-\theta_2)) \le 2[(1-r^2)^2+2r^2(1-\cos(\theta_1-\theta_2))],$$

which by (2.8) proves the lemma. □

REMARK. We notice that without the factor $4/|\xi_1-\xi_2|^2$, the inequality (2.5) is in fact reversed. Also, it is easy to check, using the conformal invariance of the Poisson kernel, that (2.5) is equivalent to

$$\int_0^{2\pi} K_{-1}(\tau(re^{i\theta}))K_1(\tau(re^{i\theta}))\,d\theta \le \int_0^{2\pi} K_{-1}(re^{i\theta})K_1(re^{i\theta})\,d\theta \tag{2.9}$$

for all Möbius transformations τ of the unit disc. The inequality (2.9) also has an interesting application to conditioned Brownian motion. Multiplying both sides of (2.9) by r and integrating in r from 0 to 1, we have that the lifetime of Brownian motion in U conditioned to go from ξ_1 to ξ_2 is maximized when the points ξ_1 and ξ_2 are diametrically opposite, a result first proved in [6] by a different argument.

We are now ready for the proof of Theorem 3. We need to show that

$$\int_0^{2\pi} |F'(re^{i\theta})|^2\,d\theta \le C^2_{\alpha,\beta,\gamma}R_T^2\int_0^{2\pi} |\psi_S'(re^{i\theta})|^2\,d\theta, \tag{2.10}$$

where

$$|F'(z)|^2 = |z-\xi_1|^{2(\alpha-1)}|z-\xi_2|^{2(\beta-1)}|z-\xi_3|^{2(\gamma-1)}$$

and R_T is as in (2.2). Since $K_{\xi_j}(z) = (1-|z|^2)/|z-\xi_j|^2$ and $\alpha+\beta+\gamma=1$, we see that

$$|F'(z)|^2 = \frac{1}{(1-|z|^2)^{(1-\alpha)+(1-\beta)+(1-\gamma)}}[K_{\xi_1}(z)^{1-\alpha}K_{\xi_2}(z)^{1-\beta}K_{\xi_3}(z)^{1-\gamma}]$$
$$= \frac{1}{(1-|z|^2)^2}K_{\xi_1}(z)^{1-\alpha}K_{\xi_2}(z)^{1-\beta}K_{\xi_3}(z)^{1-\gamma}. \tag{2.11}$$

Also,

$$\psi_S'(z) = \frac{4}{\pi}\frac{1}{(1-z^2)}$$

and

$$|\psi_S'(z)|^2 = \frac{16}{\pi^2}\frac{1}{|1-z^2|^2}$$
$$= \frac{16}{\pi^2}\frac{1}{(1-|z|^2)^2}K_1(z)K_{-1}(z). \tag{2.12}$$

Substituting (2.11) and (2.12) in (2.10), we see that the inequality (1.8) is equivalent to

$$\int_0^{2\pi} K_{\xi_1}(re^{i\theta})^{1-\alpha}K_{\xi_2}(re^{i\theta})^{1-\beta}K_{\xi_3}(re^{i\theta})^{1-\gamma}\,d\theta$$
$$\le \frac{16}{\pi^2}C^2_{\alpha,\beta,\gamma}R_T^2\int_0^{2\pi}K_1(re^{i\theta})K_{-1}(re^{i\theta})\,d\theta$$
$$= \frac{4}{|\xi_2-\xi_3|^{2\alpha}|\xi_1-\xi_3|^{2\beta}|\xi_1-\xi_2|^{2\gamma}}\int_0^{2\pi}K_1(re^{i\theta})K_{-1}(e^{i\theta})\,d\theta \tag{2.13}$$

for any three points ξ_1, ξ_2, ξ_3 on the unit circle and any three nonnegative integers α, β, γ with $\alpha+\beta+\gamma=1$.

We now write the left-hand side of (2.13) as

$$\int_0^{2\pi}(K_{\xi_2}(re^{i\theta})K_{\xi_3}(re^{i\theta}))^{\alpha}(K_{\xi_1}(re^{i\theta})K_{\xi_3}(re^{i\theta}))^{\beta}(K_{\xi_1}(re^{i\theta})K_{\xi_2}(re^{i\theta}))^{\gamma}\,d\theta. \tag{2.14}$$

Since $\alpha+\beta+\gamma=1$, we can apply Hölders inequality to conclude that the quantity in (2.14) is

$$\le\left(\int_0^{2\pi}K_{\xi_2}(re^{i\theta})K_{\xi_3}(re^{i\theta})\,d\theta\right)^{\alpha}\left(\int_0^{2\pi}K_{\xi_1}(re^{i\theta})K_{\xi_3}(re^{i\theta})\,d\theta\right)^{\beta}$$
$$\times\left(\int_0^{2\pi}K_{\xi_1}(re^{i\theta})K_{\xi_2}(re^{i\theta})\,d\theta\right)^{\gamma}$$
$$\le \frac{4}{|\xi_2-\xi_3|^{2\alpha}|\xi_1-\xi_3|^{2\beta}|\xi_1-\xi_2|^{2\gamma}}\int_0^{2\pi}K(re^{i\theta})K_{-1}(re^{i\theta})\,d\theta,$$

where the last inequality follows from Lemma 2. This completes the proof of (2.13) and hence of Theorem 3. □

We next show how Theorems 1 and 2 follow from Theorem 3. First note that if D is convex and $R_D = 1$, then either $D \subseteq S$, where S is the strip, or $D \subseteq T$, where T is a triangle and $R_T = 1$. This can be seen in the following way: If D contains a largest disc, then that disc is tangent to the boundary of D somewhere. Divide the disc into quarters so that this tangent point is on the boundary of two quarters. There must be another tangent point in each of the opposite quarters or else one could move the center of the disc slightly and replace this disc with a larger one. The extreme places for these additional tangent points both yield strips, and the others yield a triangle inside which the domain D lies. Convexity allows the choice of any tangent point in the opposite quarters. If the domain does not contain a largest disc, then by convexity it must be contained in a strip of the same inner radius.

If the domain is contained in a strip, then the domain monotonicity of the Green's function gives (1.4) with $2\pi^2/\Gamma(1/3)^3$ replaced by 1. If the domain is contained in a triangle, then again domain monotonicity gives

$$\sup_{w\in D}\int_D \varphi(G_D(w,z))\,dz \le \sup_{w\in T}\int_T \varphi(G_T(w,z))\,dz \tag{2.15}$$

and

$$\int_T \varphi(G_T(w,z))\,dz = \int_0^1 r\varphi\left(\frac{1}{2\pi}\log\frac{1}{r}\right)\left(\int_0^{2\pi}|F'(re^{i\theta})|^2\,d\theta\right)dr,$$

where F is the conformal mapping from U onto T with $F(0) = w$. By Theorem 3,

$$\int_0^{2\pi}|F'(re^{i\theta})|^2\,d\theta \le C^2_{\alpha,\beta,\gamma}\int_0^{2\pi}|\psi_S'(re^{i\theta})|^2\,d\theta \tag{2.16}$$

and hence to finish the proof of Theorem 1 we need only show that

$$C_{\alpha,\beta,\gamma} \le \frac{2\pi^2}{\Gamma(1/3)^3}, \tag{2.17}$$

for any positive α, β, γ with $\alpha+\beta+\gamma = 1$. To verify this, we set

$$g(\alpha,\beta) = \Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)\Gamma(\beta)\sin\left(\frac{\beta\pi}{2}\right)\Gamma(1-\alpha-\beta)\sin\left(\frac{(1-\alpha-\beta)\pi}{2}\right).$$

By differentiating $\log g(\alpha,\beta)$ we notice that, at the local extrema,

$$\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \cot\left(\frac{\alpha\pi}{2}\right) = \frac{\Gamma'(\beta)}{\Gamma(\beta)} + \cot\left(\frac{\beta\pi}{2}\right). \tag{2.18}$$

Since $\Gamma'(s)/\Gamma(s)$ and $\cot(s\pi/2)$ are both increasing functions on $(0,1]$, (2.18) is only satisfied when $\alpha = \beta = \gamma = 1/3$. It is then easy to see, by direct calculation, that this gives a minimum for g. Thus,

$$C_{\alpha,\beta,\gamma} \leq \frac{\pi^2}{4\Gamma(1/3)^3(\sin(\pi/6))^3} = \frac{2\pi^2}{\Gamma(1/3)^3},$$

which is (2.17).

Next we recall that if $D_1 \subset D_2$, where D_1 and D_2 are simply connected in the plane, then F_1 is subordinate to F_2, where F_1 and F_2 are the conformal mappings from U onto D_1 and D_2 (respectively) with $F_1(0) = F_2(0)$ (see [4, Chap. 6]). If this is the case then ([4, p. 194]),

$$\int_0^{2\pi} |F_1'(re^{i\theta})|^2 \, d\theta \leq \int_0^{2\pi} |F_2'(re^{i\theta})|^2 \, d\theta, \quad 0 < r \leq 1/2. \tag{2.19}$$

Now, if F is as in Theorem 2 then $F(U) \subseteq S$ or $F(U) \subseteq T$, where T is a triangle with $R_T = 1$. Theorem 2 follows from this, (2.19), and Theorem 3.

□

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