The Pluri-Complex Green Function and a Covering Mapping

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1. Introduction

Myrberg [12] proved that, if M is a hyperbolic Riemann surface with the Green function $g^M(\cdot, p)$ with pole at $p \in M$ and if $\pi: E \to M$ is a covering mapping from the unit disk $E = \{\lambda \in \mathbf{C}; |\lambda| < 1\}$ in $\mathbf{C}$ to M, then

$$g^M(q, p) = \sum_{j \geq 0} \log \left| \frac{1 - \bar{a} b_j}{b_j - a} \right|,$$

where $a \in \pi^{-1}(p)$ and $\{b_0, b_1, \dots\} = \pi^{-1}(q)$.

In any complex manifold M, we can define the pluri-complex Green function $G_p^M(\cdot)$ with pole at $p \in M$ in such a manner that if M is a hyperbolic Riemann surface, the negative of $G_p^M(\cdot)$ is nothing other than the Green function on M with pole at p [9; 10; 11; 2; 3; 6; 8]. Since $b \mapsto \log|(1 - \bar{a}b)/(b - a)|$ is the Green function on E with pole at $a \in E$, Myrberg's theorem is rewritten as follows:

$$G_p^M(q) = \sum_{j \geq 0} G_a^E(b_j),$$

where $a \in \pi^{-1}(p)$ and $\{b_0, b_1, \dots\} = \pi^{-1}(q)$.

In this paper we shall show the following.

THEOREM A. *Let $\pi: N \to M$ be a covering mapping from a complex manifold N to another one M. For $p, q \in M$, let $a \in \pi^{-1}(p)$ and $\{b_0, b_1, \dots\} = \pi^{-1}(q)$. Then*

$$G_p^M(q) \geq \sum_{j \geq 0} G_a^N(b_j).$$

When the covering is regular, we have the following.

THEOREM B. *Let $\pi: N \to M$ be a regular covering mapping from a complex manifold N to another one M. For $p, q \in M$, let $\{a = a_0, a_1, \dots\} = \pi^{-1}(p)$ and $\{b = b_0, b_1, \dots\} = \pi^{-1}(q)$. Then*

$$G_p^M(q) \geq \sum_{j \geq 0} G_a^N(b_j) = \sum_{j \geq 0} G_{a_j}^N(b).$$

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With respect to the invariant pseudometric $P^M(X)$ induced from the pluri-complex Green function $G_p^M(q)$ [1; 2; 4; 6; 8], we obtain the following.

THEOREM C. *Let $\pi: N \to M$ be a regular covering mapping from a complex manifold N to another one M. For $p \in M$, let $\{a_0, a_1, \ldots\} = \pi^{-1}(p)$. Then*

$$P^N(X) \geq P^M(\pi_* X) \geq C(p) P^N(X)$$

for all holomorphic tangent vectors $X \in T_{a_0}N$ at $a_0 \in N$, where

$$C(p) = \prod_{j \geq 1} \exp G_{a_0}^N(a_j) = \prod_{j \geq 1} \exp G_{a_j}^N(a_0).$$

The function $C(p)$ is $[0,1]$-valued and does not depend on the choice of $a_0 \in \pi^{-1}(p)$.

In Section 4, we quote Poletzkiǐ's interpretation of the pluri-complex Green function using holomorphic mappings from the unit disk E, and deduce some consequences from it. In the final section we give an example that illustrates Theorem C and an application of Theorem C to the estimate of the invariant metric induced from the pluri-complex Green function of a non-convex Thullen domain.

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2. The Pluri-Complex Green Function and the Invariant Pseudometric

Let p be a point of a complex manifold M. Denote by $PS^M(p)$ the family of all $[-\infty, 0)$-valued plurisubharmonic functions f on M such that

$$f(q) - \log\|z(q) - z(p)\| \leq O(1)$$

as $q \to p$ for some holomorphic coordinate z around p and the Euclidean norm $\|\cdot\|$ on $\mathbf{C}^m$, $m = \dim M$. The *pluri-complex Green function $G_p^M(\cdot)$ on M with pole at p* is, by definition, given by $G_p^M(q) = \sup\{f(q); f \in PS^M(p)\}$, $q \in M$ (cf. [9; 10; 11; 2; 3; 6; 8]). The following properties are fundamental.

(2.1) (Decreasing property). If $\Phi \in \mathrm{Hol}(N, M)$ is a holomorphic mapping between complex manifolds N and M, then

$$G_{\Phi(a)}^M(\Phi(b)) \leq G_a^N(b)$$

for all $a, b \in N$. Therefore, if Φ is biholomorphic then the pluri-complex Green function is *invariant,* that is, the equality holds in the last inequality.

(2.2) For every $p \in M$, $G_p^M \in PS^M(p)$.

(2.3) When M is a hyperbolic Riemann surface, the function $-G_p^M(\cdot)$ is the usual Green function on M with pole at p.

Let $X \in T_pM$ be a holomorphic tangent vector at $p \in M$. Take a mapping $\varphi \in \text{Hol}(\epsilon E, M)$, $\epsilon > 0$, such that $\varphi(0) = p$ and $\varphi'(0) = X$. Define the *induced pseudometric P^M on M from the pluri-complex Green function* by

$$P^M(X) = \limsup_{\lambda \to 0} \frac{\exp G_p^M(\varphi(\lambda))}{|\lambda|}$$

(cf. [1; 2; 4; 6; 8]). The definition of $P^M(X)$ does not depend on the choice of the mapping φ, and P^M is a pseudometric on M, that is, a $[0, +\infty)$-valued function on the holomorphic tangent bundle TM satisfying $P^M(\lambda X) = |\lambda| P^M(X)$ for $X \in TM$, $\lambda \in \mathbf{C}$. From (2.1) we obtain the following.

(2.4) (Decreasing property). If $\Phi \in \text{Hol}(N, M)$ is a holomorphic mapping between complex manifolds N and M, then

$$P^M(\Phi_* X) \le P^N(X)$$

for all $X \in TN$. Therefore, if Φ is biholomorphic then the induced metric from the pluri-complex Green function is *invariant,* that is, the equality holds in the last inequality.

THEOREM A. *Let $\pi: N \to M$ be a covering mapping from a complex manifold N to another one M. For $p, q \in M$, let $a \in \pi^{-1}(p)$ and $\{b_0, b_1, \dots\} = \pi^{-1}(q)$. Then*

$$G_p^M(q) \ge \sum_{j \ge 0} G_a^N(b_j). \tag{2.5}$$

Proof. Fix a point $p \in M$, and denote the function of $q \in M$ defined in the right-hand side of (2.5) by $f(q)$. For $q \in M$, let $U \ni q$ and $V_j \ni b_j$ be neighborhoods such that $\pi^{-1}(U) = \bigcup_{j \ge 0} V_j$, V_j are disjoint, and $\pi|_{V_j}: V_j \to U$ are biholomorphic. Then

$$f(x) = \sum_{j \ge 0} G_a^N \circ (\pi|_{V_j})^{-1}(x) \quad \text{for } x \in U.$$

Since f is a limit of a decreasing sequence of plurisubharmonic functions on U, f is also plurisubharmonic on U. In particular, assume that $q = p$ and $b_0 = a$. Since

$$G_a^N(u) - \log\|z(u) - z(a)\| \le O(1)$$

as $u \to a$ for some coordinate z around a (see (2.2)), we have

$$f(x) - \log\|w(x) - w(p)\| \le G_a^N \circ (\pi|_{V_0})^{-1}(x) - \log\|w(x) - w(p)\| \le O(1)$$

as $x \to p$ for the coordinate $w = z \circ (\pi|_{V_0})^{-1}$ around p, so that $f \in PS^M(p)$. By definition, we have

$$G_p^M(q) \ge f(q) = \sum_{j \ge 0} G_a^N(b_j),$$

as desired. □

3. Regular Covering

For a manifold M and a point $m \in M$ we denote the fundamental group of M with reference point m by $\pi_1(M, m)$. A covering $\pi: N \to M$ is said to be *regular* if, for every $m \in M$ and every $n \in \pi^{-1}(m)$, the induced group $\pi_* \pi_1(N, n)$ is a normal subgroup of $\pi_1(M, m)$. In that case, it is known (see e.g. [7]) that, for every $m \in M$ and every pair of points n_1, n_2 in $\pi^{-1}(m)$, there exists a unique homeomorphism (called a *covering transformation*) Φ on N such that $\pi \circ \Phi = \pi$ and $\Phi(n_1) = n_2$. We note that every universal covering is regular.

THEOREM B. *Let $\pi: N \to M$ be a regular covering mapping from a complex manifold N to another one M. For $p, q \in M$, let $\{a = a_0, a_1, \ldots\} = \pi^{-1}(p)$ and $\{b = b_0, b_1, \ldots\} = \pi^{-1}(q)$. Then*

$$G_p^M(q) \geq \sum_{j \geq 0} G_a^N(b_j) = \sum_{j \geq 0} G_{a_j}^N(b).$$

Proof. Let $\mathrm{Aut}_\pi(N)$ be the covering transformation group of the regular covering $\pi: N \to M$. For every $j \geq 0$, take $\Phi_j \in \mathrm{Aut}_\pi(N)$ such that $\Phi_j(a_j) = a_0$. Then $\mathrm{Aut}_\pi(N) = \{\Phi_j; j \geq 0\}$ and

$$\{\Phi_j(b_0); j \geq 0\} = \pi^{-1}(q) = \{b_0, b_1, \ldots\}.$$

Since $\mathrm{Aut}_\pi(N) \subset \mathrm{Aut}(N)$, it follows from the invariant property (2.1) of the pluri-complex Green function that

$$\sum_{j \geq 0} G_{a_j}^N(b) = \sum_{j \geq 0} G_{a_0}^N(\Phi_j(b_0)) = \sum_{j \geq 0} G_{a_0}^N(b_j).$$

The proof is completed. □

THEOREM C. *Let $\pi: N \to M$ be a regular covering mapping from a complex manifold N to another one M. For $p \in M$, let $\{a, a_1, a_2, \ldots\} = \pi^{-1}(p)$. Then*

$$P^N(X) \geq P^M(\pi_* X) \geq C(p) P^N(X)$$

for all holomorphic tangent vectors $X \in T_a N$ at $a \in N$, where

$$C(p) = \prod_{j \geq 1} \exp G_a^N(a_j) = \prod_{j \geq 1} \exp G_{a_j}^N(a).$$

The function $C(p)$ is $[0, 1]$-valued and does not depend on the choice of $a \in \pi^{-1}(p)$.

Here we mean that $C(p) = 1$ when $\pi^{-1}(p) = \{a\}$, a singleton.

Proof. Take a function $\varphi: \epsilon E \to N$, holomorphic such that $\varphi(0) = a$ and $\varphi'(0) = X$. It follows from Theorem B that

$$\exp G_p^M(\pi \circ \varphi(\lambda)) \geq \exp G_a^N(\varphi(\lambda)) \prod_{j \geq 1} \exp G_{a_j}^N(\varphi(\lambda)).$$

Divide both sides by $|\lambda|$ and take the superior limit as $\lambda \to 0$. Since the functions $\exp G_{a_j}^N$ are plurisubharmonic, the functions $\exp G_{a_j}^N \circ \varphi$ are subharmonic around $\lambda = 0$, so that

$$\limsup_{\lambda \to 0} G_{a_j}^N(\varphi(\lambda)) = G_{a_j}^N(\varphi(0))$$

(see e.g. [11; 13]). Therefore, noting that $\pi\circ\varphi(0)=p$ and $(\pi\circ\varphi)'(0)=\pi_* X$, by definition we obtain

$$P^M(\pi_* X)\geq P^N(X)\prod_{j\geq 1}\exp G^N_{a_j}(a)=P^N(X)C(p).$$

For every j, take $\Phi_j\in \mathrm{Aut}_\pi(N)$ so that $\Phi_j(a_j)=a$. By a similar argument as in the proof of Theorem B, we have

$$\{\Phi_j(a); j\geq 1\}=\pi^{-1}(p)\backslash\{a\}=\{a_1,a_2,\ldots\},$$

so that

$$\prod_{j\geq 1}\exp G^N_{a_j}(a)=\prod_{j\geq 1}\exp G^N_{\Phi_j(a_j)}(\Phi_j(a))=\prod_{j\geq 1}\exp G^N_a(\Phi_j(a))=\prod_{j\geq 1}\exp G^N_a(a_j).$$

Let $\{b,b_1,b_2,\ldots\}=\{a,a_1,a_2,\ldots\}$. We want to show that

$$\prod_{j\geq 1}\exp G^N_{b_j}(b)=\prod_{j\geq 1}\exp G^N_{a_j}(a).$$

Take a covering transformation Φ such that $\Phi(a)=b$. Then $\{\Phi(a_j); j\geq 1\}=\{b_1,b_2,\ldots\}$, so that

$$\prod_{j\geq 1}\exp G^N_{a_j}(a)=\prod_{j\geq 1}\exp G^N_{\Phi(a_j)}(\Phi(a))=\prod_{j\geq 1}\exp G^N_{\Phi(a_j)}(b)=\prod_{j\geq 1}\exp G^N_{b_j}(b).$$

The proof is completed. □

4. Consequences of Poletzkiĭ's Interpretation

We need Poletzkiĭ's interpretation of the pluri-complex Green function. Let $f\in \mathrm{Hol}(E,M)$ and $t\in E$. By $\omega(f,t)$ we denote *the multiplicity of f at t*; that is, $\omega(f,t)$ is the order of zero at t of the function $z\circ f-z(f(t))$, where z is a holomorphic coordinate around $f(t)\in M$. That is, if $g(\lambda)=z\circ f(\lambda)-z(f(t))$ then

$$\omega(f,t)=\min\{n; g^{(n)}(t)\neq 0\}.$$

We note that the definition of $\omega(f,t)$ does not depend on the choice of the coordinate z. Indeed, let w be another coordinate around $f(t)$, and define $\Phi=(w-w(f(t)))\circ(z-z(f(t)))^{-1}$ in a neighborhood of 0 in $\mathbf{C}^m$, $m=\dim M$. Then $w\circ f-w(f(t))=\Phi\circ g$. We want to prove

$$\min\{n; g^{(n)}(t)\neq 0\}=\min\{n; (\Phi\circ g)^{(n)}(t)\neq 0\}. \tag{4.1}$$

Let $\omega=\min\{n; g^{(n)}(t)\neq 0\}$ and let $n<\omega$. We have the chain rule

$$(\Phi\circ g)^{(n)}(t)=\sum_{\{P_1,\ldots,P_u\}\in\Pi(n)}\ \sum_{j_1,\ldots,j_u}(\partial^{j_1}\cdots\partial^{j_u}\Phi)\circ g(t)g_{j_1}^{(\#P_1)}(t)\cdots g_{j_u}^{(\#P_u)}(t),$$

where $\Pi(n)$ is the family of all partitions of the set $\{1,\ldots,n\}$, $\partial^j=\partial/\partial z_j$, $z=(z_1,\ldots,z_m)$, $g=(g_1,\ldots,g_m)$, each j_i runs through $1,\ldots,m$, and $\#P_i$ is the number of the set P_i (cf. [5]). Since for all $\{P_1,\ldots,P_u\}\in\Pi(n)$ and for all $i\in\{1,\ldots,u\}$, $\#P_i\leq n$, it follows that $(\Phi\circ g)^{(n)}(t)=0$. Therefore

$$\min\{n; (\Phi\circ g)^{(n)}(t)\neq 0\}\geq\omega=\min\{n; g^{(n)}(t)\neq 0\}.$$

Since $g = \Phi^{-1} \circ (\Phi \circ g)$, we have

$$\min\{n; (\Phi \circ g)^{(n)}(t) \neq 0\} \leq \min\{n; g^{(n)}(t) \neq 0\},$$

and have proved (4.1). For $f \in \mathrm{Hol}(E, M)$ and $q \in M$, define

$$u_f(q) = \begin{cases} \sum_{t \in f^{-1}(q)} \omega(f, t) \log|t| & \text{if } f^{-1}(q) \neq \emptyset, \\ 0 & \text{if } f^{-1}(q) = \emptyset. \end{cases}$$

Poletzkiĭ [14] proved the following.

THEOREM. *For any domain M in $\mathbf{C}^n$ and any $p, q \in M$,*

$$G_p^M(q) = \inf\{u_f(p); f \in \mathrm{Hol}(E, M), f(0) = q\}.$$

We note that, by virtue of Poletzkiĭ's theorem, we can prove Myrberg's theorem mentioned in Section 1 for planar cases.

THEOREM D. *If $\pi: E \to M$ is a covering of a domain M in $\mathbf{C}$, then for $p, q \in M$ we have*

$$G_p^M(q) = \sum_{j \geq 0} G_a^E(b_j) = \sum_{j \geq 0} G_{a_j}^E(b),$$

where $\{a = a_0, a_1, \ldots\} = \pi^{-1}(p)$ and $\{b = b_0, b_1, \ldots\} = \pi^{-1}(q)$.

Proof. Take $\varphi_b \in \mathrm{Aut}(E)$ defined by

$$\varphi_b(\lambda) = \frac{\lambda - b}{\bar{b}\lambda - 1}, \quad \lambda \in E.$$

Since $\pi \circ \varphi_b \in \mathrm{Hol}(E, M)$ and $\pi \circ \varphi_b(0) = p$, we may consider the quantity $u_{\pi \circ \varphi_b}(q)$. Since $\pi \circ \varphi_b$ is locally biholomorphic, $\omega(\pi \circ \varphi_b, t) = 1$ for every $t \in (\pi \circ \varphi_b)^{-1}(p)$. It follows that

$$\begin{aligned} u_{\pi \circ \varphi_b}(p) &= \sum_{t \in (\pi \circ \varphi_b)^{-1}(p)} \log|t| \\ &= \sum_{\lambda \in \pi^{-1}(p)} \log|\varphi_b(\lambda)| \\ &= \sum_{j \geq 0} \log|\varphi_b(a_j)| = \sum_{j \geq 0} \log|\varphi_{a_j}(b)| \\ &= \sum_{j \geq 0} G_{a_j}^E(b). \end{aligned}$$

By Poletzkiĭ's theorem we have

$$\begin{aligned} G_p^M(q) &= \inf\{u_f(p); f \in \mathrm{Hol}(E, M), f(0) = q\} \\ &\leq u_{\pi \circ \varphi_b}(p) = \sum_{j \geq 0} G_{a_j}^E(b). \end{aligned}$$

Since the opposite inequality $G_p^M(q) \geq \sum_{j \geq 0} G_{a_j}^E(b)$ always holds (Theorem B), we obtain one of the equalities in the theorem. The other follows from the regularity of π and Theorem B, and the proof is completed. □

By using Poletskiĭ's theorem, we can prove the following counterpart of Theorem A.

THEOREM A'. *Let $\pi: N \to M$ be a covering mapping from a domain N in $\mathbf{C}^n$ to another one M. For $p, q \in M$, let $\{a_0, a_1, \dots\} = \pi^{-1}(p)$ and $b \in \pi^{-1}(q)$. Then*

$$G_p^M(q) \geq \sum_{j\geq 0} G_{a_j}^N(b).$$

Proof. Let $f \in \mathrm{Hol}(E, M)$ with $f(0) = q$. Take a lifting $g: E \to N$ of f with $g(0) = b$. Since $f = \pi \circ g$,

$$f^{-1}(p) = g^{-1}(\pi^{-1}(p)) = \bigcup_{j\geq 0} g^{-1}(a_j) \quad \text{(disjoint union)}.$$

For $t \in g^{-1}(a_j)$, the multiplicity $\omega(g, t)$ is given by the order of zero at t of the function $g - a_j$. Take a neighborhood V of a_j such that $\pi|_V$ is a homeomorphism, and set $z = (\pi|_V)^{-1}$. Then z is a coordinate around p. Since $z \circ f - z(p) = g - a_j$ in a neighborhood of t, we see that $\omega(f, t) = \omega(g, t)$. Thus

$$\begin{aligned} u_f(p) &= \sum_{t \in f^{-1}(p)} \omega(f, t) \log|t| \\ &= \sum_{j\geq 0} \sum_{t \in g^{-1}(a_j)} \omega(g, t) \log|t| \\ &= \sum_{j \geq 0} u_g(a_j) \\ &\geq \sum_{j\geq 0} G_{a_j}^N(b). \end{aligned}$$

Since f is arbitrary,

$$G_p^M(q) \geq \sum_{j\geq 0} G_{a_j}^N(b),$$

as desired. □

5. Examples

Let $\mathbf{D}_m = \{|w_1|^2 + |w_2|^{2/m} < 1\} \subset \mathbf{C}^2$ with m a positive integer. If $m > 1$, then $\mathbf{D}_m$ is called a *Thullen domain* and $\mathbf{D}_1$ is the unit ball $\mathbf{B}$. We recall some properties on automorphisms of $\mathbf{B}$. For any $a, z \in \mathbf{B}$, set

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - \|a\|^2}\,(z - P_a z)}{1 - \langle z, a\rangle},$$

where $P_a z = (\langle z, a\rangle / \|a\|^2) a$ and $\langle \cdot, \cdot \rangle$ is the natural hermitian inner product on $\mathbf{C}^2$. Then $\varphi_a \in \mathrm{Aut}(\mathbf{B})$, $\varphi_a \circ \varphi_a = \mathrm{id}$, and $\varphi_a(a) = 0$. We have

$$\langle \varphi_a(z), \varphi_a(w)\rangle = 1 - \frac{(1 - \|a\|^2)(1 - \langle z, w\rangle)}{(1 - \langle z, a\rangle)(1 - \langle a, w\rangle)}, \tag{5.1}$$

$$\varphi_{a*}(a; X) = \left(0; \frac{-P_a X - \sqrt{1 - \|a\|^2}\,(X - P_a X)}{1 - \|a\|^2}\right) \tag{5.2}$$

(cf. e.g. [8]). If $N = \mathbf{B} \backslash \{z_2 = 0\}$ and $M = \mathbf{D}_m \backslash \{w_2 = 0\}$, then $\pi: N \ni (z_1, z_2) \mapsto (z_1, z_2^m) \in M$ is a regular covering mapping, so that Theorem C implies

$$C(p)P^N(X) \le P^M(\pi_* X) \le P^N(X) \tag{5.3}$$

for $p \in M$, $a \in \pi^{-1}(p)$, and $X \in T_a N$. Since plurisubharmonic functions that are bounded above are uniquely extendable beyond a pluri-polar set, it follows that

$$G_a^N = G_a^{\mathbf{B}} \quad \text{for } a \in N, \tag{5.4}$$

$P^M = P^{\mathbf{D}_m}$, and $P^N = P^{\mathbf{B}}$. Furthermore, $P^{\mathbf{B}}$ coincides with the Kobayashi metric $K^{\mathbf{B}}$ on $\mathbf{B}$ (cf. e.g. [1; 5]). If $a = (0, b) \in N$, $0 < b < 1$, $p = (0, b^m) \in M$, and $X = (a; (X_1, X_2)) \in T_a N = \{a\} \times \mathbf{C}^2$, then (5.3) becomes

$$C(p)K^{\mathbf{B}}(a; (X_1, X_2)) \le P^{\mathbf{D}_m}(p; (X_1, mb^{m-1}X_2)) \le K^{\mathbf{B}}(a; (X_1, X_2)). \tag{5.5}$$

If $\epsilon = e^{2\pi i/m}$, then by definition we have

$$\begin{aligned} C(p) &= \prod_{j=1}^{m-1} \exp G^N_{(0,b)}(0, \epsilon^j b) \\ &= \prod_{j=1}^{m-1} \exp G^{\mathbf{B}}_{(0,b)}(0, \epsilon^j b) \\ &= \prod_{j=1}^{m-1} \exp G^{\mathbf{B}}_{\varphi_{(0,b)}(0,b)}(\varphi_{(0,b)}(0, \epsilon^j b)) \\ &= \prod_{j=1}^{m-1} \exp G^{\mathbf{B}}_0(\varphi_{(0,b)}(0, \epsilon^j b)) \\ &= \prod_{j=1}^{m-1} \|\varphi_{(0,b)}(0, \epsilon^j b)\| \\ &= \prod_{j=1}^{m-1} \left(1 - \frac{(1-b^2)^2}{|1-\epsilon^j b^2|^2}\right)^{1/2} \\ &= \frac{(2b)^{m-1}(1-b^2)}{1-b^{2m}} \prod_{j=1}^{m-1} \sin\frac{j\pi}{m}. \end{aligned}$$

In the second equality we used (5.4); the third equality, the invariance of the pluri-complex Green function for automorphisms; and the sixth equality, equation (5.1). (For the fifth equality, see e.g. [3, p. 2] or [8, p. 119].) Since the Kobayashi metric is invariant for automorphisms, we have

$$\begin{aligned} K^{\mathbf{B}}(a; (X_1, X_2)) &= K^{\mathbf{B}}(\varphi_{a*}(a; (X_1, X_2))) \\ &= \|\varphi_{a*}(a; (X_1, X_2))\| \\ &= \frac{\sqrt{(1-b^2)|X_1|^2 + |X_2|^2}}{1-b^2}. \end{aligned}$$

Here we have used (5.2). (For the second equality, see e.g. [1, p. 644] or [8, p. 119].) Observing $\mathbf{D}_m \subset \mathbf{B}$, we have $P^{\mathbf{B}} \le P^{\mathbf{D}_m}$. Summarizing these, we have the following estimates for the invariant metric $P^{\mathbf{D}_m}$ on the Thullen domain $\mathbf{D}_m$.

PROPOSITION E. *For $0<b<1$ and $(X_1,X_2)\in\mathbf{C}^2$, it holds that*

$$\max\{C_b K^{\mathbf{B}}((0,b^{1/m});(X_1,m^{-1}b^{1/m-1}X_2)),K^{\mathbf{B}}((0,b);(X_1,X_2))\}$$
$$\le P^{\mathbf{D}_m}((0,b);(X_1,X_2))\le K^{\mathbf{B}}((0,b^{1/m});(X_1,m^{-1}b^{1/m-1}X_2)),$$

where

$$K^{\mathbf{B}}((0,b);(X_1,X_2))=\frac{\sqrt{(1-b^2)|X_1|^2+|X_2|^2}}{1-b^2};$$

$$C_b=C((0,b))=\frac{2^{m-1}b^{1-1/m}(1-b^{2/m})}{1-b^2}\prod_{j=1}^{m-1}\sin\frac{j\pi}{m}.$$

We note that $P^{\mathbf{D}_m}((0,0);(X_1,X_2))$ coincides with

$$\mu^{\mathbf{D}_m}(X_1,X_2)=\inf\{\lambda>0;(X_1,X_2)\in\lambda\mathbf{D}_m\},$$

the Minkowski functional for $\mathbf{D}_m$, and that for every $(b_1,b_2)\in\mathbf{D}_m$ there exists a $\Phi\in\mathrm{Aut}(\mathbf{D}_m)$ such that $\Phi(b_1,b_2)=(0,|b_2|(1-|b_1|^2)^{-m/2})$ (cf. e.g. [8, p. 273]). We also note that if $m\ge 3$ then $\mathbf{D}_m$ is not convex.

Now, assume $m=2$. Then $\mathbf{D}_2$ is convex, so that $P^{\mathbf{D}_2}=K^{\mathbf{D}_2}$. The explicit formula for $K^{\mathbf{D}_2}$ is well known [8, p. 274]:

$$K^{\mathbf{D}_2}((0,b^2);(X_1,2bX_2))=\begin{cases}\dfrac{\sqrt{(1-b^2)|X_1|^2+|X_2|^2}}{1-b^2}, & |X_2|\le b|X_1|,\\[2ex] \dfrac{2b|X_2|}{1-b^4}, & X_1=0.\end{cases}$$

Since $C(p)=2b/(1+b^2)$ for $p=(0,b^2)$, $m=2$, this example shows that, in view of (5.5), in the inequalities (5.3) there exist $X^1,X^2\in T_aN\setminus\{0\}$ such that $P^M(\pi_*X^1)=P^N(X^1)$ and $P^M(\pi_*X^2)=C(p)P^N(X^2)$.

The last example also shows that in higher-dimensional cases the equality

$$G_p^M(q)=\sum_{j\ge 0}G_{a_j}^N(b)$$

in Theorem B does not hold in general, contrary to the 1-dimensional case. For if the equality holds, then the argument in the proof of Theorem C implies the equality

$$P^M(\pi_*X)=C(p)P^N(X),\quad X\in T_aN.$$

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