Hyperbolic Buildings, Affine Buildings, and Automatic Groups

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1. Introduction

Two standard references for the theory of automatic groups are the book [12] and the paper [2]; each contains numerous examples of automatic groups. Perhaps the "canonical" class of automatic groups is the class of word hyperbolic groups of Gromov et al. [15]. As we shall see in Section 3, any building whose underlying Coxeter group is word hyperbolic is itself hyperbolic in the word metric. In particular, any finitely generated group acting co-compactly with finite stabilizers on such a building is word hyperbolic and thus automatic. Hence, from the viewpoint of automatic groups, it is natural to look next at actions on affine buildings.

The first result in this direction is provided by Gersten and Short [13], who prove that a finitely generated torsion-free group that acts co-compactly discretely by isometries on a Euclidean building of dimension 2 is automatic. It has often seemed likely that the restriction on dimension could be lifted. In this paper, we prove this for finitely generated groups that act simply transitively (and in a type-rotating way) on the vertices of a thick building of type $\tilde{A}_n$. We call such groups *$\tilde{A}_n$-groups*. In fact, we will show more. We shall see that these groups are bi-automatic. The structure in question is a symmetric automatic structure [12]. (The term "fully automatic" is used in [8] and in early versions of [12].) In particular, this implies that $\tilde{A}_n$-groups have solvable conjugacy problem [14]. Further, the structure consists of geodesics, and thus these groups have rational growth functions. It is shown in [7] that for $n \geq 2$ and for any prime power q there are examples of $\tilde{A}_n$-groups that are arithmetic lattices in $\mathrm{PGL}(n+1, \mathbf{F}_q((X)))$. This result generalizes earlier work for small n in [5] and [6]. For $n = 2, 3$ and for some small primes p, there are examples of $\tilde{A}_n$-groups that are arithmetic lattices in $\mathrm{PGL}(n+1, \mathbf{Q}_p)$ (see [6; 7; 21]).

The paper is organized as follows. Section 2 defines hyperbolic groups and reviews some basic background information, most of which can be found in [1]. In Section 3 we show that a building is hyperbolic (in an appropriate sense) if and only if its underlying Coxeter group is hyperbolic. Section 4 defines $\tilde{A}_n$-groups and reviews the necessary background from [5]. Section 5

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gives the proof that finitely generated $\tilde{A}_n$-groups are bi-automatic. Sections 4 and 5 require no knowledge of buildings; Section 3 uses elementary results that can be found in [3, IV.3].

2. Hyperbolic Groups

We say that a metric space (X, d) is a *geodesic metric space* if, for every $x, y \in X$, there is a path from x to y that realizes their distance. Such a path is called a *geodesic*. Following [15], we say that a geodesic metric space is *δ-hyperbolic* if, whenever P is a point on side α of a geodesic triangle with sides α, β, and γ, there is a point Q on $\beta \cup \gamma$ such that $d(P, Q) \leq \delta$. We say that (X, d) is *hyperbolic* if it is δ-hyperbolic for some δ.

Given any connected graph Γ, there is a natural metric in Γ. Take each edge of Γ to be isometric to the unit interval and take the path metric that this induces on Γ. Further, given any finitely generated group G, the choice of a finite generating set $\mathcal{G} = \{a_1, \dots, a_k\}$ turns G into a directed labeled connected graph $\Gamma = \Gamma_{\mathcal{G}}$. The vertices of Γ are the elements of G and the edges of Γ are $\{(g, ga) \mid g \in G, a \in \mathcal{G}\}$. We direct the edge (g, ga) from g to ga and label it with a. We assume that $\mathcal{G}$ is closed under inverses and identify (g, ga) with the inverse of (ga, g). Γ is called the *Cayley graph* of G with respect to $\mathcal{G}$.

We say that G is *hyperbolic* if $\Gamma = \Gamma_{\mathcal{G}}$ is hyperbolic. While this appears to depend on $\mathcal{G}$, in fact only the particular value of δ depends on $\mathcal{G}$.

We will want a standard fact about hyperbolic metric spaces. Given a path σ in X and given $0 < \lambda \leq 1$ and $0 \leq \epsilon$, we will say that σ is a *(λ, ϵ)-quasigeodesic* if, for every decomposition $\sigma = \alpha\beta\gamma$, the endpoints of β are separated by at least $\lambda\ell(\beta) - \epsilon$. (Here $\ell(\beta)$ denotes the length of β.) If X is a δ-hyperbolic metric space then there exists an $N = N(\delta, \lambda, \epsilon)$ such that, if σ is a (λ, ϵ)-quasigeodesic and τ is a geodesic with the same endpoints, then σ and τ lie in the N-neighborhood of each other [1, 3.3]. Geodesics are simply $(1, 0)$-quasigeodesics, and thus there is an $N = N(\delta, 1, 0)$ such that all geodesics joining common endpoints live in an N-neighborhood of each other. We call a pair of geodesics with common endpoints a *bigon*.

From this one can construct a proof that any geodesic metric space that is quasi-isometric to a hyperbolic space is itself hyperbolic, and that, in particular, hyperbolicity of a group is independent of the generating set. It is a standard result (see e.g. [4]) that if a finitely generated group G acts cocompactly by isometries and with finite stabilizers on a geodesic metric space (X, g), then every Cayley graph of G is quasi-isometric to X. In particular, when X is hyperbolic, so is G.

We have seen that the definition of a hyperbolic metric space requires checking that geodesic triangles are "thin", that is, that no side of a triangle is ever far from the union of the other two sides. Papasoglu [17; 18] has shown that in graphs it is only necessary to check bigons. Note that the endpoints of geodesics (and hence the endpoints of bigons) need not be vertices of Γ.

THEOREM (Papasoglu). *Suppose that Γ is a graph and that there is a constant K such that if σ, σ' is a bigon then σ and σ' lie in a K-neighborhood of each other. Then Γ is hyperbolic.*

Notice that the hypothesis is equivalent to the a priori stronger hypothesis that there exists a K' such that if σ and σ' form a bigon then, for all t, $d(\sigma(t), \sigma'(t)) \leq K'$. For suppose that $\sigma(t)$ is within K of $\sigma'(t')$. Then the fact that these are geodesics emanating from a common point allows us to use the triangle inequality to see that $|t-t'| \leq K$. Hence, taking $K' = 2K$, we have $d(\sigma(t), \sigma'(t)) \leq K'$.

3. Hyperbolic Buildings

In [16] Moussong constructs actions of Coxeter groups on non-positively curved geodesic metric spaces. An account of this metric can be found in [10]. The metric spaces in question are locally Euclidean or locally hyperbolic complexes, and can be made negatively curved if and only if the Coxeter group in question is word hyperbolic. The actions are co-compact, by isometries and with finite stabilizers. As a scholium of his construction, one knows exactly which Coxeter groups are word hyperbolic.

THEOREM (Moussong). *Let (W, S) be a Coxeter system. Then the following are equivalent.*

(1) *W is word hyperbolic.*
(2) *W has no $\mathbf{Z} \times \mathbf{Z}$ subgroup.*
(3) *(W, S) does not contain an affine sub-Coxeter system of rank ≥ 3, and does not contain a pair of disjoint commuting sub-Coxeter systems whose groups are both infinite.*

Charney and Davis [9] have pointed out that by using Moussong's metric of nonpositive curvature one can give a building a metric of nonpositive curvature, and that the metric on the building is negatively curved if and only if the Coxeter group is word hyperbolic. (Construction of the metric on the building can be done along the lines of [3, VI.3].)

We give a similar characterization in terms of graphs. Given a building Δ, there is a metric on the set of chambers of Δ, and we will want a path metric space that reflects this metric. To do this we let Δ' be the graph dual to Δ. That is to say, the vertices of Δ' are the barycenters of the chambers of Δ. Two such vertices are connected by an edge when they lie in chambers with a common face. As usual, Δ' is metrized by considering each edge as isometric to the unit interval. Nonstuttering galleries of Δ correspond to edge-paths in Δ'. The decomposition of Δ into apartments induces a decomposition of Δ' into apartments that are isometric as labeled graphs to the Cayley graph of (W, S), the Coxeter system of Δ.

THEOREM 1. *Suppose Δ is a building whose apartments are the Coxeter complex of a word hyperbolic Coxeter group. Then Δ' is hyperbolic.*

REMARK. The converse is also true. That is, if Δ' is hyperbolic, the associated Coxeter group is word hyperbolic. This follows immediately from the fact that the embedding of the Cayley graph into Δ' is an isometry.

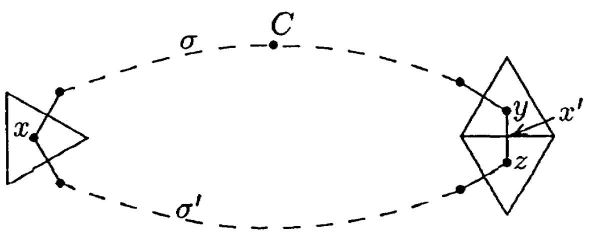
COROLLARY. *Suppose Δ is a building whose apartments are the Coxeter complex of a word hyperbolic Coxeter group, and suppose that G is a finitely generated group that acts simplicially, co-compactly with finite stabilizers on Δ. Then G is word hyperbolic.*

Proof. G acts on Δ, and the natural embedding of Δ' into Δ is equivariant with respect to this action. This induces an action of G on Δ'. Since there are finitely many G-orbits of chambers of Δ and finitely many G-orbits of codimension-1 faces, the induced action on Δ' is co-compact. Now G carries edges of Δ' to edges of Δ' and thus acts by isometries of the graph metric. Since the action of G has finite stabilizers, the (setwise) stabilizer of each chamber and codimension-1 face of Δ is finite. These are the stabilizers of the vertices and edges of Δ'. Hence, as we have outlined in Section 2, G is quasi-isometric to Δ' and thus word hyperbolic. □

Proof of Theorem 1. To prove the theorem, we consider a bigon σ, σ'. It suffices to show that there exists a K such that, if C is any point of σ, then C is within K of σ'. We distinguish three cases depending on whether both, one, or neither of the endpoints of this bigon are vertices.

Case 1: Both ends of the bigon are vertices. Given any two chambers of Δ, there is an apartment Σ containing them both. Any geodesic gallery in Δ connecting these two chambers lies in Σ (see [3, p. 88]). It now follows that σ and σ' lie in a common apartment of Δ', and by the hyperbolicity of the underlying Coxeter group, we are done.

Case 2: The geodesics σ and σ' begin at a vertex, but do not end at a vertex. We let x and x' be the beginning and endpoints of our bigon. We let y and z be the last vertices of σ and σ', respectively, and let τ and τ' be the initial segments of σ and σ' ending at y and z, respectively. It now follows that $\ell(\tau) = \ell(\tau')$ and that x' is the midpoint of an edge e. If $y = z$ then τ and τ' lie in a common apartment, and we are done by Case 1. Thus we may assume $y \neq z$. Now τ and τ' cannot lie in a common apartment. For if this were so, $\tau e \tau'^{-1}$ would label a relator of odd length in the underlying Coxeter group, and this is impossible.



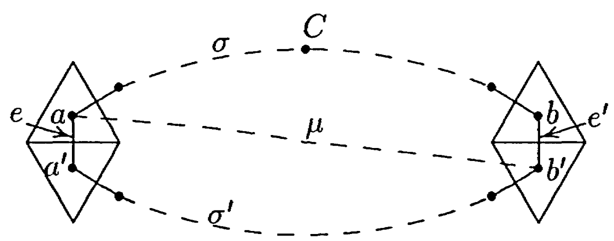
Case 2

Let Σ be an apartment containing σ. We consider $C = \sigma(t)$ and may suppose that this is a vertex lying on τ. We let $\rho = \rho_{\Sigma,C}$ be the retraction onto Σ centered at C (see e.g. [3, IV.3]). Now $d(y, \rho(z)) \leq 1$, and since ρ does not increase distance we have

$$n-1 \leq d(x, \rho(z)) \leq n,$$

where $n = \ell(\tau)$. Now consider the path $\rho(\tau')$. This may not be an edgepath, as some edge of τ' may be folded by ρ. However its length as a path is still n. We thus have a path of length n whose endpoints lie at distance at least $n-1$. It follows that $\rho(\tau')$ is a $(1,1)$-quasigeodesic. Since $\rho(\tau')$ lies in Σ, it now follows from the hyperbolicity of Σ that $\rho(\tau')$ lies close to τ. We can thus find $C' = \tau'(t')$ such that $\rho(C')$ lies close to C. Since $\rho = \rho_{\Sigma,C}$ preserves distance from C, C' lies close to C and we are done.

Case 3: Neither end of the bigon is a vertex. In this case we let a and b (resp. a' and b') be the first and last vertices of σ (resp. σ'), and let τ (resp. τ') be the segment of σ (resp. σ') connecting these first and last vertices. If $a = a'$ or $b = b'$ or both then we are reduced to previous cases, so we can assume $a \neq a'$ and $b \neq b'$. We let e be the edge from a to a' and let e' be the edge from b to b'. It is easy to check that $\ell(\tau) = \ell(\tau')$. We take $n = \ell(\tau)$. Thus $d(a, b')$ is either $n-1$, n, or $n+1$. We let μ be a geodesic from a to b'.



Case 3

If $d(a, b') = n-1$ then $\mu e'^{-1}$ and τ form a bigon whose ends are the vertices a and b. Likewise, $e^{-1}\mu$ and τ' form a bigon whose ends are the vertices a' and b'. Applying Case 1 twice takes care of this situation.

If $d(a, b') = n$, we let y and y' be the midpoints of e and e'. We then have a bigon whose ends are a and y' and whose sides consist of τ and μ, each with a half of e' appended. Similarly, we have a bigon with ends b' and y and sides μ^{-1} and τ'^{-1} with the halves of e appended. Now we can apply Case 2 twice.

Finally, if $d(a, b') = n+1$ then μ and $\tau e'$ form a bigon, as do μ and $e\tau'$; once again, we can apply Case 1 twice.

4. Review of $\tilde{A}_n$-Groups

$\tilde{A}_n$-groups were introduced for general $n \geq 2$ in [5], after earlier work on the case $n = 2$ in [6]. (One-dimensional buildings are trees, and are thus hyperbolic.) Recall that a building is a *labelable* complex—that is, each vertex v of a building Δ of type $\tilde{A}_n$ has a *type* $\tau(v) \in \{0, 1, \ldots, n\}$, with each chamber having one vertex of each type. If g is an automorphism of Δ, and if there is an integer c such that $\tau(gv) = \tau(v)+c \pmod{n+1}$ for each vertex v, then g is called *type-rotating*. Such automorphisms form a subgroup of index at most 2 in the group of all automorphisms of Δ.

If K is a field with discrete valuation, then there is a thick building Δ_K of type $\tilde{A}_n$ associated with K (see [19, Sec. 9.2] or [3, Sec. V.8]), and the group $\mathrm{PGL}(n+1, K)$ acts transitively and in a type-rotating way on Δ_K. A group is said to be an *$\tilde{A}_n$-group* if it acts simply transitively on the vertices of a thick building of type $\tilde{A}_n$ in a type-rotating way.

We now describe $\tilde{A}_n$-groups. Let Π be a projective geometry of dimension $n \geq 2$ (see e.g. [11, p. 24] or [20, p. 105]). For $i = 1, \ldots, n$, let $\Pi_i = \{x \in \Pi\colon \dim(x) = i\}$. To avoid unnecessary abstraction, the reader may assume that Π is the set $\Pi(\mathbf{V})$ (partially ordered by inclusion) of nontrivial proper subspaces of an $(n+1)$-dimensional vector space $\mathbf{V}$ over a field k, and that $\dim(x)$ refers to the dimension of the subspace x of $\mathbf{V}$. For when $n \geq 3$, or when $n = 2$ and Π is desarguesian, Π must be isomorphic to $\Pi(\mathbf{V})$ for some $\mathbf{V}$ ([11, pp. 27–28] or [20, p. 203]). Let $\lambda\colon \Pi \to \Pi$ be an involution such that $\lambda(\Pi_i) = \Pi_{n+1-i}$ for $i = 1, \ldots, n$, and let $\mathfrak{T}$ be an *$\tilde{A}_n$-triangle presentation* compatible with λ. This means that $\mathfrak{T}$ is a set of triples (u, v, w), where $u, v, w \in \Pi$, such that:

(A) given $u, v \in \Pi$, we have $(u, v, w) \in \mathfrak{T}$ for some $w \in \Pi$ if and only if $\lambda(u)$ and v are distinct and incident;
(B) if $(u, v, w) \in \mathfrak{T}$, then $(v, w, u) \in \mathfrak{T}$;
(C) if $(u, v, w_1) \in \mathfrak{T}$ and $(u, v, w_2) \in \mathfrak{T}$, then $w_1 = w_2$;
(D) if $(u, v, w) \in \mathfrak{T}$, then $(\lambda(w), \lambda(v), \lambda(u)) \in \mathfrak{T}$;
(E) if $(u, v, w) \in \mathfrak{T}$, then $\dim(u)+\dim(v)+\dim(w) = n+1$ or $2(n+1)$; and
(F) if $(x, y, u) \in \mathfrak{T}'$ and $(x', y', \lambda(u)) \in \mathfrak{T}'$, then for some $w \in \Pi$ we have that $(y', x, w) \in \mathfrak{T}'$ and $(y, x', \lambda(w)) \in \mathfrak{T}'$.

Here $\mathfrak{T}'$ denotes the "half" of $\mathfrak{T}$ consisting of the triples $(u, v, w) \in \mathfrak{T}$ for which $\dim(u)+\dim(v)+\dim(w) = n+1$. If $u, v \in \Pi$, then $(u, v, w) \in \mathfrak{T}'$ for some $w \in \Pi$ if and only if $\lambda(u) \supsetneq v$. We also write $\mathfrak{T}''$ for $\mathfrak{T} \setminus \mathfrak{T}'$.

We form the associated group $\Gamma_{\mathfrak{T}}$ with a generating set indexed by Π:

$$\Gamma_{\mathfrak{T}} = \langle \{a_v\}_{v \in \Pi} \mid (1)\ a_{\lambda(v)} = a_v^{-1} \text{ for all } v \in \Pi,$$
$$(2)\ a_u a_v a_w = 1 \text{ for all } (u, v, w) \in \mathfrak{T} \rangle.$$

It was shown in [5] that the Cayley graph of $\Gamma_{\mathfrak{T}}$ with respect to the generators a_v, $v \in \Pi$, is the 1-skeleton of a thick building $\Delta_{\mathfrak{T}}$ of type $\tilde{A}_n$. Clearly, $\Gamma_{\mathfrak{T}}$ acts, by left multiplication, simply transitively on the set of vertices of $\Delta_{\mathfrak{T}}$. Conversely, if Γ is a group of type-rotating automorphisms of a thick building Δ of type $\tilde{A}_n$, and acts simply transitively on the vertices of Δ, then $\Gamma \cong \Gamma_{\mathfrak{T}}$ and $\Delta \cong \Delta_{\mathfrak{T}}$ for some $\tilde{A}_n$-triangle presentation $\mathfrak{T}$. These results generalized earlier work [6] on the case $n = 2$.

In this paper, Π is assumed to be finite. The number of $x \in \Pi_1$ incident with any given $y \in \Pi_2$ is denoted $q+1$, and is independent of y. Here q is called the *order* of Π, and when $\Pi = \Pi(\mathbf{V})$, q is the number of elements in the field k. It is shown in [7] that, for every $n \geq 2$ and for any prime power q, $\tilde{A}_n$-triangle presentations exist when $\Pi = \Pi(\mathbf{V})$.

Let L denote the set of all strings $u_1 u_2 \cdots u_\ell$ over Π such that

$$\lambda(u_i) + u_{i+1} = \mathbf{V} \quad \text{for } i = 1, \ldots, \ell - 1.$$

(The notation assumes that $\Pi = \Pi(\mathbf{V})$, but in general, "$\lambda(u_i) + u_{i+1} = \mathbf{V}$" is interpreted as "there is no $x \in \Pi$ such that $\lambda(u_i) \subset x$ and $u_{i+1} \subset x$", where we write $y \subset x$ if x and y are incident, and $\dim(y) \leq \dim(x)$.) Theorem 2.2 in [5] states that $u = u_1 \cdots u_\ell \mapsto \bar{u} = a_{u_1} \cdots a_{u_\ell}$ is a bijection $L \to \Gamma_3$. Moreover, in the notation of [12], each string $u_1 \cdots u_\ell \in L$ is *geodesic,* and so the number ℓ is the *word length* $|g|$ of $g = \bar{u}$. If $g \in \Gamma_3$, $u \in L$, and $g = \bar{u}$, then u is called the *normal form* of g. We shall also refer to strings in L as being *in normal form.* When g is the identity element 1, its normal form is the empty word, and its word length is 0, by definition. We write $d(g, g')$ for $|g^{-1}g'|$, the distance from g to g' in the *word metric.*

5. Finitely Generated $\tilde{A}_n$-Groups Are Automatic

Throughout this section, let $\Gamma = \Gamma_3$ be a finitely generated $\tilde{A}_n$-group.

We start by showing that L is a regular language. We can define a finite state automaton M accepting L as follows: The set S of states of M is Π together with an initial state s_0 (not in Π) and a single failure state s_1 (not in Π, and distinct from s_0); the alphabet of M is Π; the transition function μ of M is given by $\mu(s_0, x) = x$ and $\mu(s_1, x) = s_1$ for $x \in \Pi$, while for $x, y \in \Pi$ we set $\mu(x, y) = y$ if $\lambda(x) + y = \mathbf{V}$ and $\mu(x, y) = s_1$ otherwise; the set Y of accept states of M is $\Pi \cup \{s_0\}$. Clearly $L = L(M)$.

Note that Π is a set of semigroup generators for Γ_3 that is closed under inversion (as $a_{\lambda(x)} = a_x^{-1}$). Moreover, L has the *uniqueness property* (i.e., $u \mapsto \bar{u}$ is a bijection $L \to \Gamma_3$), is obviously prefix-closed, and is symmetric (i.e., $u_1 \cdots u_\ell \in L$ implies that $\lambda(u_\ell) \cdots \lambda(u_1) \in L$). This last property of L and our Theorem 2 below imply that Γ_3 is *bi-automatic,* and in fact *symmetric automatic* or *fully automatic.*

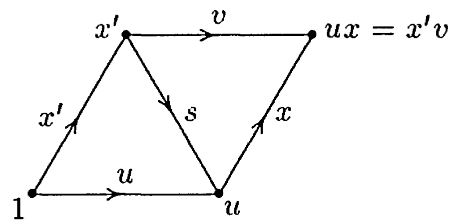
If $u = u_1 \cdots u_\ell \in L$ let $u(0)$ be the identity element 1 in Γ, and for $1 \leq t \leq \ell$ let $u(t)$ denote the element $a_{u_1} \cdots a_{u_t}$ of Γ. For $t > \ell$, let $u(t) = a_{u_1} \cdots a_{u_\ell} = \bar{u}$. Let $x \in \Pi$, and let $v_1 \cdots v_k$ be the normal form of $\overline{ux} = a_{u_1} \cdots a_{u_\ell} a_x$. By [12, Thm. 2.3.5], to show that Γ_3 is automatic it is enough to show that, for some k (independent of u and x), $d(u(t), v(t)) \leq k$ holds for each integer $t \geq 0$. This property is called the *k-fellow traveler property* [2]. In fact, we show this property holds for $k = 1$.

Now let $u = u_1 \cdots u_\ell \in L$, and let $x \in \Pi$. The main work below is to describe the normal form $v = v_1 \cdots v_k$ of $a_{u_1} \cdots a_{u_\ell} a_x$. The description of this normal form is complicated by the fact that, given $u, x \in \Pi$, there are five mutually exclusive cases to consider: $\lambda(u) = x$, $\lambda(u)+x = \mathbf{V}$, $\lambda(u) \supsetneq x$, $\lambda(u) \subsetneq x$, and (finally) $\lambda(u)$ and x are distinct and nonincident, with $\lambda(u)+x \neq \mathbf{V}$.

We start with a lemma that concerns the last of these cases. In this lemma and in what follows, we consistently identify a string u over Π with its image $\bar{u}$ in $\Gamma_{\mathfrak{T}}$, sometimes writing "u in $\Gamma_{\mathfrak{T}}$" for emphasis, when thinking of u as $\bar{u}$.

LEMMA 1. *Suppose that $u, x \in \Pi$, that $\lambda(u)$ and x are distinct and nonincident, and that $\lambda(u)+x = \lambda(s)$ $(\neq \mathbf{V})$. Thus we can write $(s, \lambda(u), x') \in \mathfrak{T}'$ and $(s, x, \lambda(v)) \in \mathfrak{T}'$ for some $x', v \in \Pi$, and $ux = x'v$ in Γ. Then $x'v$ is in normal form. That is, $\lambda(x')+v = \mathbf{V}$. Conversely, if two triples $(s, \lambda(u), x')$, $(s, x, \lambda(v)) \in \mathfrak{T}'$ are given, with $\lambda(x')+v = \mathbf{V}$, then $\lambda(u)$ and x must be distinct and nonincident, with $\lambda(u)+x = \lambda(s)$.*

We illustrate this lemma with a diagram in the Cayley graph of $\Gamma_{\mathfrak{T}}$.



Proof. First, $\lambda(x') \neq v$. Otherwise, $x' = \lambda(v)$, and so Axioms (B) and (C) imply that $\lambda(u) = x$, contrary to hypothesis. We shall henceforth use Axioms (A)–(E) in the definition of an $\tilde{A}_n$-triangle presentation without comment, but refer to Axiom (F) when it is used. Next, suppose that $\lambda(x') \neq v$, but that $\lambda(x')$ and v are incident. Thus $(x', v, w) \in \mathfrak{T}$ for some $w \in \Pi$. If $(x', v, w) \in \mathfrak{T}'$, then $(w, x', v) \in \mathfrak{T}'$, $(s, x, \lambda(v)) \in \mathfrak{T}'$, and Axiom (F) together imply that

$$(x, w, y) \in \mathfrak{T}' \quad \text{and} \quad (x', s, \lambda(y)) \in \mathfrak{T}' \quad \text{for some } y \in \Pi.$$

Thus $y = u$. But then $(u, x, w) \in \mathfrak{T}$, so that $\lambda(u)$ and x are incident, contrary to hypothesis. If $(x', v, w) \in \mathfrak{T}''$, then $(s, \lambda(u), x') \in \mathfrak{T}'$, $(\lambda(w), \lambda(v), \lambda(x')) \in \mathfrak{T}'$, and Axiom (F) imply that $(\lambda(v), s, y) \in \mathfrak{T}'$ and $(\lambda(u), \lambda(w), \lambda(y)) \in \mathfrak{T}'$ for some $y \in \Pi$. Then $y = x$, and so $(x, w, u) \in \mathfrak{T}''$, so that again $\lambda(u)$ and x are incident, contrary to hypothesis.

Suppose that $\lambda(x') \neq v$, that $\lambda(x')$ and v are not incident, and that

$$\lambda(x')+v = \lambda(s') \neq \mathbf{V}.$$

Thus we have that $(s', \lambda(x'), x'') \in \mathfrak{T}'$ and $(s', v, \lambda(v')) \in \mathfrak{T}'$ for some $x'', v' \in \Pi$. Now $(s, \lambda(u), x') \in \mathfrak{T}'$, $(x'', s', \lambda(x')) \in \mathfrak{T}'$, and Axiom (F) show that $(s', s, z) \in \mathfrak{T}'$ and $(\lambda(u), x'', \lambda(z)) \in \mathfrak{T}'$ for some $z \in \Pi$. Similarly, $(\lambda(v'), s', v) \in \mathfrak{T}'$ and $(s, x, \lambda(v)) \in \mathfrak{T}'$ imply that $(x, \lambda(v'), z') \in \mathfrak{T}'$ and $(s', s, \lambda(z')) \in \mathfrak{T}'$ for some $z' \in \Pi$. Thus $\lambda(z') = z$. Hence $(\lambda(u), x'', \lambda(z)) \in \mathfrak{T}'$ and $(x, \lambda(v'), \lambda(z)) \in \mathfrak{T}'$, so that $z \supset \lambda(u), x$. Hence $z \supset \lambda(u)+x = \lambda(s)$. But $(s', s, z) \in \mathfrak{T}'$ implies that $z \subsetneq \lambda(s)$. This contradiction completes the proof of the first part of the lemma.

Consider the converse part. First, $\lambda(u)$ and x must be distinct, for otherwise $x' = \lambda(v)$ must hold, which is impossible because $\lambda(x')+v = \mathbf{V}$. Next, $\lambda(u) \supsetneqq x$ cannot hold. For otherwise $(u,x,y) \in \mathfrak{I}'$ for some $y \in \Pi$. This, Axiom (F), and $(\lambda(u), x', s) \in \mathfrak{I}'$ then imply that $(s,x,z) \in \mathfrak{I}'$ and $(y, x', \lambda(z)) \in \mathfrak{I}'$ hold for some $z \in \Pi$. Thus $z = \lambda(v)$, so that $(y, x', v) \in \mathfrak{I}'$, which implies that $\lambda(x') \supset v$, again a contradiction. Similarly, $\lambda(u) \subsetneqq x$ leads to a contradiction. Finally, the hypotheses imply that $\lambda(u)+x \subset \lambda(s)$. If $\lambda(u)+x \neq \lambda(s)$ then write $\lambda(u)+x = \lambda(s')$; we can then find $x'', v' \in \Pi$ such that $(s', \lambda(u), x'')$, $(s', x, \lambda(v')) \in \mathfrak{I}'$. But then $x'v = ux = x''v'$, and both $x'v$ and $x''v'$ are in normal form (by the hypotheses, and by the first part of the lemma, respectively). Uniqueness of normal forms now shows that $x'' = x'$, so that $s' = s$, a contradiction. This completes the proof. □

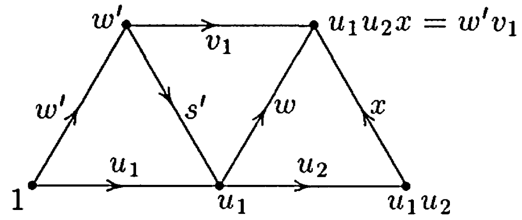
LEMMA 2. *Let $u_1, u_2, x \in \Pi$, with u_1u_2 in normal form. In deriving the normal form of u_1u_2x, we have the following five possibilities.*

(1) *If $\lambda(u_2)+x = \mathbf{V}$, then u_1u_2x is in normal form.*
(2) *If $\lambda(u_2) = x$, then the normal form of u_1u_2x is u_1.*
(3) *If $\lambda(u_2) \supsetneqq x$, then $(u_2, x, \lambda(w)) \in \mathfrak{I}'$ for some $w \in \Pi$, and the normal form of u_1u_2x is u_1w.*
(4) *If $\lambda(u_2) \subsetneqq x$, then $(u_2, x, \lambda(w)) \in \mathfrak{I}''$ for some $w \in \Pi$. Thus $u_1u_2x = u_1w$ in Γ. There are now the following possibilities: either*
   (a) *$\lambda(u_1)+w = \mathbf{V}$, in which case the normal form of u_1u_2x is u_1w; or*
   (b) *$\lambda(u_1) \supsetneqq w$, in which case $(u_1, w, \lambda(w')) \in \mathfrak{I}'$ for some $w' \in \Pi$, and the normal form of u_1u_2x is w'; or*
   (c) *$\lambda(u_1)$ and w are distinct and nonincident, with $\lambda(u_1)+w \neq \mathbf{V}$. Then, writing $\lambda(u_1)+w = \lambda(s')$, there exist unique $w', v_1 \in \Pi$ such that $(s', \lambda(u_1), w') \in \mathfrak{I}'$ and $(s', w, \lambda(v_1)) \in \mathfrak{I}'$, and the normal form of u_1u_2x is $w'v_1$.*
(5) *If $\lambda(u_2)$ and x are distinct and nonincident, with $\lambda(u_2)+x \neq \mathbf{V}$, then, writing $\lambda(u_2)+x = \lambda(s)$, there are unique $x', v_2 \in \Pi$ such that*

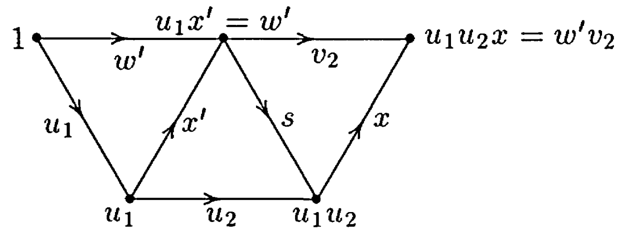
$$(s, \lambda(u_2), x') \in \mathfrak{I}' \quad \text{and} \quad (s, x, \lambda(v_2)) \in \mathfrak{I}'.$$

   Thus $u_1u_2x = u_1x'v_2$ in Γ. There are now the following possibilities: either
   (a) *$\lambda(u_1)+x' = \mathbf{V}$, in which case the normal form of u_1u_2x is $u_1x'v_2$; or*
   (b) *$\lambda(u_1) \supsetneqq x'$, in which case $(u_1, x', \lambda(w')) \in \mathfrak{I}'$ for some $w' \in \Pi$, and the normal form of u_1u_2x is $w'v_2$; or*
   (c) *$\lambda(u_1)$ and x' are distinct and nonincident, with $\lambda(u_1)+x' \neq \mathbf{V}$. Then, writing $\lambda(u_1)+x' = \lambda(s')$, there are unique $x'', v_1 \in \Pi$ such that $(s', \lambda(u_1), x'') \in \mathfrak{I}'$ and $(s', x', \lambda(v_1)) \in \mathfrak{I}'$, and the normal form of u_1u_2x is $x''v_1v_2$.*

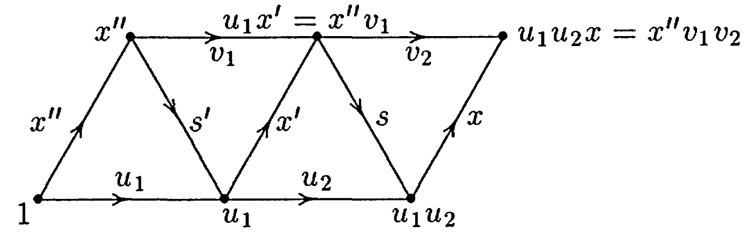
We illustrate the more complicated cases as follows.



Case 4(c)



Case 5(b)



Case 5(c)

Proof. The assertions in (1) and (2) are obvious.

Let us consider the situation in (3). Here $(u_2, x, \lambda(w)) \in \mathfrak{I}'$ implies that $w \supset u_2$. Thus $\lambda(u_1)+w \supset \lambda(u_1)+u_2 = \mathbf{V}$, and so u_1w is in normal form.

Let us consider the situation in (4). In this case, again $w \neq \lambda(u_1)$. For otherwise, $(u_2, x, u_1) = (u_2, x, \lambda(w)) \in \mathfrak{I}''$, contradicting the hypothesis that u_1u_2 is in normal form. Also, $\lambda(u_1) \subsetneq w$ cannot hold; otherwise, $(\lambda(w), \lambda(u_1), w') \in \mathfrak{I}'$ for some $w' \in \Pi$. Now $(\lambda(x), \lambda(u_2), w) \in \mathfrak{I}'$, $(\lambda(u_1), w', \lambda(w)) \in \mathfrak{I}'$, and Axiom (F) imply that $(w', \lambda(x), y)$ and $(\lambda(u_2), \lambda(u_1), \lambda(y))$ are in $\mathfrak{I}'$ for some $y \in \Pi$. This again contradicts the hypothesis that u_1u_2 is in normal form. We are left with the three possibilities 4(a), 4(b), and 4(c). The assertions in 4(a) and 4(b) are obvious; in 4(c), that the word $w'v_1$ is in normal form is immediate from Lemma 1.

Let us consider the situation in (5). In this case, $\lambda(u_1)$ cannot equal x'. For otherwise $(s, \lambda(u_2), \lambda(u_1)) = (s, \lambda(u_2), x') \in \mathfrak{I}'$, contradicting the hypothesis that u_1u_2 is in normal form. Also $\lambda(u_1) \subsetneq x'$ cannot occur. For otherwise, $(\lambda(x'), \lambda(u_1), w') \in \mathfrak{I}'$ for some $w' \in \Pi$. Now Axiom (F) together with the fact that $(s, \lambda(u_2), x') \in \mathfrak{I}'$ and $(\lambda(u_1), w', \lambda(x')) \in \mathfrak{I}'$ imply that $(w', s, y) \in \mathfrak{I}'$ and

$(\lambda(u_2), \lambda(u_1), \lambda(y)) \in \mathfrak{I}'$ for some $y \in \Pi$. This last result again contradicts the hypothesis that u_1u_2 is in normal form. We are left with the three possibilities 5(a), 5(b), and 5(c). The assertion in 5(a) is obvious, by Lemma 1.

Consider the situation in 5(b). Note that $(x', \lambda(w'), u_1), (x', s, \lambda(u_2)) \in \mathfrak{I}'$ and the converse part of Lemma 1 imply that $\lambda(w') + s = \lambda(x')$. Because $(s, x, \lambda(v_2)) \in \mathfrak{I}'$, we have $v_2 \supset s$, and so

$$\lambda(w') + v_2 = \lambda(w') + s + v_2 = \lambda(x') + v_2 = \mathbf{V},$$

the last equality holding by Lemma 1. So $w'v_2$ is in normal form.

Finally, consider the situation in 5(c). First observe that $\lambda(v_1) + s = \lambda(x')$. This follows from the converse part of Lemma 1, because $(x', \lambda(v_1), s')$ and $(x', s, \lambda(u_2))$ are in $\mathfrak{I}'$ and because, using $(s', \lambda(u_1), x'') \in \mathfrak{I}'$, we have that $\lambda(s') + u_2 \supset \lambda(u_1) + u_2 = \mathbf{V}$. Now $(s, x, \lambda(v_2)) \in \mathfrak{I}'$, so that $v_2 \supset s$. Hence

$$\lambda(v_1) + v_2 = \lambda(v_1) + s + v_2 = \lambda(x') + v_2 = \mathbf{V},$$

the last equality holding by Lemma 1. Lemma 1 also shows that $\lambda(x'') + v_1 = \mathbf{V}$, and so $x''v_1v_2$ is in normal form. □

REMARK. In later work, we shall need a converse to Lemma 2. Let us write $x +' y = z$ for $x, y, z \in \Pi$ if x, y are distinct and nonincident, with $x + y = z$. Then, by the converse part of Lemma 1, we have $\lambda(w') +' \lambda(x) = \lambda(w)$ in part 4(b), $\lambda(v_1) +' \lambda(x) = \lambda(w)$ in part 4(c), $\lambda(w') +' s = \lambda(x')$ in part 5(b), and $\lambda(v_1) +' s = \lambda(x')$ in part 5(c). Provided these conditions are added, the converses of parts (4) and (5) hold. In 5(b), for example, if $u_1, u_2, x, s, x', v_2, w' \in \Pi$ and triples $(s, \lambda(u_2), x'), (s, x, \lambda(v_2)) \in \mathfrak{I}'$ are given with $\lambda(w') +' s = \lambda(x')$ and $\lambda(w') + v_2 = \mathbf{V}$, then u_1u_2 is in normal form and $\lambda(u_2) +' x = \lambda(s)$. The first of these is immediate from Lemma 1, as $(x', \lambda(w'), u_1), (x', s, \lambda(u_2)) \in \mathfrak{I}'$. To see that $\lambda(u_2) +' x = \lambda(s)$, observe that $\lambda(x') + v_2 \supset \lambda(w') + v_2 = \mathbf{V}$ and so the converse part of Lemma 1 is applicable.

The Normal Form $v_1 \cdots v_k$ of $\overline{ux}$

We can now describe how to obtain the normal form $v_1 \cdots v_k$ of $\overline{ux} = a_{u_1} \cdots a_{u_\ell} a_x$ given $u_1 \cdots u_\ell \in L$ and $x \in \Pi$. When $\lambda(u_\ell) + x = \mathbf{V}$, this is obviously $u_1 \cdots u_\ell x$. If $\lambda(u_\ell) = x$, then the normal form of $\overline{ux}$ is clearly $u_1 \cdots u_{\ell-1}$. If $\lambda(u_\ell) \supsetneqq x$ then $(u_\ell, x, \lambda(w)) \in \mathfrak{I}'$ for some $w \in \Pi$, and the normal form of $\overline{ux}$ is then $u_1 \cdots u_{\ell-1} w$, by part (3) of Lemma 2.

If $\lambda(u_\ell)$ and x are distinct and nonincident, with $\lambda(u_\ell) + x \neq \mathbf{V}$, then, writing $\lambda(u_\ell) + x = \lambda(s_\ell)$, we have $(s_\ell, \lambda(u_\ell), x_1) \in \mathfrak{I}'$ and $(s_\ell, x, \lambda(v_\ell)) \in \mathfrak{I}'$ for some $x_1, v_\ell \in \Pi$. Then $u_1 \cdots u_\ell x = u_1 \cdots u_{\ell-1} x_1 v_\ell$ in Γ. This situation may be repeated several times, with $\lambda(u_{\ell-1})$ and x_1 distinct and nonincident, with $\lambda(u_{\ell-1}) + x_1 \neq \mathbf{V}$, and so forth. Suppose this situation is repeated exactly i times. Then we find $s_{\ell-\nu}, x_{\nu+1}, v_{\ell-\nu} \in \Pi$ for $\nu = 0, \ldots, i-1$ such that $\lambda(u_{\ell-\nu}) + x_\nu = \lambda(s_{\ell-\nu})$, $(s_{\ell-\nu}, \lambda(u_{\ell-\nu}), x_{\nu+1}) \in \mathfrak{I}'$, and $(s_{\ell-\nu}, x_\nu, \lambda(v_{\ell-\nu})) \in \mathfrak{I}'$ for $\nu = 0, \ldots, i-1$ (writing $x_0 = x$). Then in Γ we have

$$u_1 \cdots u_\ell x = u_1 \cdots u_{\ell-i} x_i v_{\ell-i+1} \cdots v_\ell. \tag{5.1}$$

Then, by part (5) of Lemma 2, either the word on the right in (5.1) is in normal form or $\lambda(u_{\ell-i}) \supsetneq x_i$, in which case $(u_{\ell-i}, x_i, \lambda(w)) \in \mathfrak{Z}'$ for some $w \in \Pi$ and the normal form of $\overline{ux}$ is

$$u_1 \cdots u_{\ell-i-1} w v_{\ell-i+1} \cdots v_\ell. \tag{5.2}$$

Finally, if $\lambda(u_\ell) \subsetneq x$, then $(u_\ell, x, \lambda(w)) \in \mathfrak{Z}''$ for some $w \in \Pi$. Thus $u_1 \cdots u_\ell x = u_1 \cdots u_{\ell-1} w$ in Γ. A sequence of exactly $i \geq 0$ steps such as led to (5.1) may now occur, so that we find elements $s_{\ell-\nu}, w_\nu, v_{\ell-\nu}$ for $\nu = 1, \ldots, i$ such that $\lambda(u_{\ell-\nu}) + w_{\nu-1} = \lambda(s_{\ell-\nu})$, $(s_{\ell-\nu}, \lambda(u_{\ell-\nu}), w_\nu) \in \mathfrak{Z}'$, and $(s_{\ell-\nu}, w_{\nu-1}, \lambda(v_{\ell-\nu})) \in \mathfrak{Z}'$ for $\nu = 1, \ldots, i$ (writing $w_0 = w$). Then in Γ we have

$$u_1 \cdots u_\ell x = u_1 \cdots u_{\ell-i-1} w_i v_{\ell-i} \cdots v_{\ell-1}. \tag{5.3}$$

By parts (4) and (5) of Lemma 2, either the word on the right in (5.3) is in normal form or $\lambda(u_{\ell-i-1}) \supsetneq w_i$, in which case $(u_{\ell-i-1}, w_i, \lambda(w')) \in \mathfrak{Z}'$ for some $w' \in \Pi$ and the normal form of $\overline{ux}$ is

$$u_1 \cdots u_{\ell-i-2} w' v_{\ell-i} \cdots v_{\ell-1}. \tag{5.4}$$

We are now ready to prove the fellow traveler property.

THEOREM 2. *Let $\Gamma_\mathfrak{Z}$ be a finitely generated $\tilde{A}_n$-group. Let $u = u_1 \cdots u_\ell \in L$, let $x \in \Pi$, and let $v_1 \cdots v_k \in L$ be the normal form of $\overline{ux} = a_{u_1} \cdots a_{u_\ell} a_x$. Then for each integer $t \geq 0$, we have $d(u(t), v(t)) \leq 1$. That is, either $a_{u_1} \cdots a_{u_t} = a_{v_1} \cdots a_{v_t}$ or $a_{u_1} \cdots a_{u_t} = a_{v_1} \cdots a_{v_t} a_{x_t}$ for some $x_t \in \Pi$. Thus $\Gamma_\mathfrak{Z}$ is an automatic group.*

Proof. Let us discuss the most complicated case in detail, leaving the other cases to the reader. Suppose that $\lambda(u_\ell) \subsetneq x$, and that the normal form v of $\overline{ux}$ is (5.4). For $0 \leq t \leq \ell - i - 2$, we have $a_{u_1} \cdots a_{u_t} = a_{v_1} \cdots a_{v_t}$. Now

$$\begin{aligned} v(\ell - i - 1) = a_{v_1} \cdots a_{v_{\ell-i-1}} &= a_{u_1} \cdots a_{u_{\ell-i-2}} a_{w'} \\ &= a_{u_1} \cdots a_{u_{\ell-i-2}} a_{u_{\ell-i-1}} a_{w_i} \\ &= u(\ell - i - 1) a_{w_i} \end{aligned}$$

and $w_i \in \Pi$. For $1 \leq \nu \leq i$, we have

$$\begin{aligned} v(\ell - \nu) &= a_{u_1} \cdots a_{u_{\ell-i-2}} a_{w'} a_{v_{\ell-i}} \cdots a_{v_{\ell-\nu}} \\ &= a_{u_1} \cdots a_{u_{\ell-i-2}} a_{u_{\ell-i-1}} a_{w_i} a_{v_{\ell-i}} \cdots a_{v_{\ell-\nu}} \\ &= a_{u_1} \cdots a_{u_{\ell-i-2}} a_{u_{\ell-i-1}} a_{u_{\ell-i}} a_{u_{\ell-i+1}} \cdots a_{u_{\ell-\nu}} a_{w_{\nu-1}} \\ &= u(\ell - \nu) a_{w_{\nu-1}} \end{aligned}$$

and $w_{\nu-1} \in \Pi$. For $t \geq \ell$, $v(t) = \overline{ux} = u(t) a_x$. This completes the proof. □

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