# On Virtually Projective Groups

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## 1. Introduction

Denote the maximal pro-2 Galois group of a field $K$ by $G_K(2)$. Fields $K$ for which $G_{K(\sqrt{-1})}(2)$ is a free pro-2 group were studied by various authors (see e.g. [ELP; Er; W1; W2; W3; E1]). The approach in these works is, to a large extent, arithmetical—leaning heavily on (among other things) properties of quadratic forms. The objective of the present note is twofold: In Sections 2–4 we extend the structure theory of such fields and, moreover, generalize some of their Galois-theoretic properties into a purely group-theoretic setting. Next we deal with the following (known) structure theorem: $G_{K(\sqrt{-1})}(2)$ is a free pro-2 group if and only if $G=G_K(2)$ is a free pro-2 product (in a natural sense) of groups of order 2 and of a free pro-2 group. This deep fact has been proven by Eršov [Er] and Ware [W3] using field-theoretic tools. It was generalized by Haran [H4] to an arbitrary pro-2 group $G$ (under a mild assumption arising from Artin–Schreier theory). The proof in [H4], however, uses heavy machinery: a cohomology theory for the category of the so-called Artin–Schreier structures, and the study of projective resolutions of profinite $G$-modules (these tools are also partly developed in [H2] and [H3]). Our second goal is thus to give a simplified proof of this fundamental fact, using only standard methods of *Galois* cohomology. This is done in Section 5, using the results of the previous sections.

Our approach is to explore the cohomological connections between a profinite group $G$ and its *real core* $N$; that is, $N$ is the closed subgroup generated by all the involutions in $G$. For $G$ as above (or, more generally, when $G$ is *virtually projective of real type;* cf. Sections 2–3) we obtain a short exact sequence relating $N$ to the Bockstein operator of $G$ (Corollary 3.4). Combined with an approximation property for $H^1(G,\mathbb{Z}/2\mathbb{Z})$, this is used to show that $G/N$ is projective—that is, has cohomological dimension $\le 1$ (Theorem 4.5; see also Remark 5.3(2)).

This latter fact is of particular interest in studying the structure of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$. Indeed, denote the field of totally real algebraic numbers by $\mathbb{Q}_{tr}$, let $\mathbb{Q}_{ab}$ be the maximal pro-abelian extension of $\mathbb{Q}$, and let $\bar{\mathbb{Q}}_{ab}=\mathbb{Q}_{ab}\cap\mathbb{R}=\mathbb{Q}_{ab}\cap\mathbb{Q}_{tr}$. Our results then imply that $\mathrm{Gal}(\mathbb{Q}_{tr}/\bar{\mathbb{Q}}_{ab})$

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is projective (Cor. 4.6). We remark in this connection that Fried, Haran, and Völklein [FHV] have proved that $G_{\mathbb{Q}_{tr}}$ is a free profinite product (in a natural sense) of groups of order 2 indexed by the Cantor space $\{0,1\}^{\omega}$. Another proof (and of a more general result) has later been given by Pop. On the other hand, the Kronecker–Weber theorem implies that $\mathrm{Gal}(\bar{\mathbb{Q}}_{ab}/\mathbb{Q}) \cong \mathbb{Z}_2 \times \prod_{p\,\mathrm{odd}} \mathbb{Z}_p^{\times}$.

## 2. Groups of Real Type

Denote the set of all closed subgroups of order 2 of a profinite group $G$ by $\mathfrak{D}(G)$. The topology of $G$ induces in a natural way a topology on $\mathfrak{D}(G)$. We write $H^i(G)$ for the Galois cohomology group $H^i(G, \mathbb{Z}/2\mathbb{Z})$, with $G$ acting trivially on $\mathbb{Z}/2\mathbb{Z}$. Thus $H^1(G)$ is the group of all continuous homomorphisms $G \to \mathbb{Z}/2\mathbb{Z}$. (For all unexplained notions in Galois cohomology we refer, e.g., to [Ri].)

Although our approach will be purely group-theoretic, it is motivated by Galois-theoretic arguments. In fact, our main results are not valid for arbitrary profinite groups, but only under the following four conditions, which reflect fundamental properties of canonical Galois groups of ordered fields (as explained below).

DEFINITION 2.1. We say that a profinite group $G$ has *real type* if there exists $\delta \in H^1(G)$ such that:

(i) $\mathrm{Ker}(\delta)$ is torsion-free;
(ii) for each $\Gamma \in \mathfrak{D}(G)$, $\Gamma = \{\sigma \in G \mid \Gamma^{\sigma} = \Gamma\}$;
(iii) if $G' \leq G$ and $\Gamma, \Gamma' \in \mathfrak{D}(G')$ are contained in the same open subgroups of $G'$ of index $\leq 2$, then they are conjugate in $G'$; and
(iv) for any $G' \leq G$ and $\psi \in H^1(G')$, $\psi \cup (\psi + \mathrm{Res}_{G'}\delta) = 0$ in $H^2(G')$.

REMARK 2.2. Suppose that $G$ is a profinite group of real type.

(1) Condition (i) implies that the torsion in $G$ consists only of involutions.
(2) By (i), 1 is not an accumulation point of involutions in $G$. Therefore, the set of involutions in $G$ is closed. It follows that this set—and hence also its homeomorphic copy $\mathfrak{D}(G)$—is a Boolean space (i.e., is Hausdorff, compact, and totally disconnected).
(3) $\mathfrak{D}(G)$ has a *compact* system of representatives for its conjugacy classes: this follows from the Booleanity of $\mathfrak{D}(G)$ and from (ii) by means of, for example, [Mel, (5.1)] or [H1, Lemma 5.2].
(4) A closed subgroup of a profinite group of real type has real type.

We now explain the Galois-theoretic motivation for Definition 2.1. Let $\mathbf{P}$ be a collection of prime numbers with $2 \in \mathbf{P}$. Let $\Omega/K$ be a Galois extension with $\mathrm{char}\, K \neq 2$ and suppose that, for any $p \in \mathbf{P}$, $\Omega$ has no proper finite Galois extensions of $p$-power order. For example, when $\mathbf{P}$ is the collection of all prime numbers we can take $\Omega$ to be the separable closure $K_{\text{sep}}$ of $K$ or its solvable closure $K_{\text{sol}}$, and when $\mathbf{P}=\{2\}$ we can take $\Omega$ to be the maximal Galois 2-extension $K(2)$ of $K$. Set $G=\text{Gal}(\Omega/K)$. Becker proved in [Be, Chap. II, Sec. 2] that the complete analog of the Artin–Schreier theory holds also in this relative context; namely, for any ordering $P$ on $K$ there exists a relative real closure in $\Omega$ (i.e., a maximal ordered-field extension of $(K,P)$ inside $\Omega$) that is unique up to a unique $K$-isomorphism. (Becker makes the constant assumption that $K$ is formally real, but this is not needed for the results quoted here.) Moreover, the relative real closures of $K$ in $\Omega$ are precisely the subfields $\bar{K}$ of $\Omega/K$ such that $\text{Gal}(\Omega/\bar{K})\cong\mathbb{Z}/2\mathbb{Z}$, and this is the only possible torsion in $G$. Denote the set of all orderings on $K$ by $X_K$. Note that $\bar{K}$ induces $P\in X_K$ (i.e., $P=\bar{K}^2\cap K$) if and only if $\sqrt{a}\in\bar{K}$ for all $a\in P$.

Also, the cohomology exact sequence corresponding to the short exact sequence

$$1\longrightarrow\{\pm1\}\longrightarrow\Omega^\times\xrightarrow{2}\Omega^\times\longrightarrow1$$

of $G$-modules together with Hilbert's Theorem 90 give the Kummer isomorphism $K^\times/(K^\times)^2\cong H^1(G)$. Explicitly, $a(K^\times)^2$ corresponds under this isomorphism to the homomorphism $(a)_K\in H^1(G)$ given by $\sigma\mapsto\sigma(\sqrt{a})/\sqrt{a}$ for $\sigma\in G$. For $a_1,\dots,a_n\in K^\times$ let $\langle\langle a_1,\dots,a_n\rangle\rangle$ be the Pfister form $\langle1,a_1\rangle\otimes\cdots\otimes\langle1,a_n\rangle$, and let $I(K)$ be the fundamental ideal of the Witt ring $W(K)$ of $K$. A conjecture of Milnor [Mi] says that for any $m\geq0$ there is a well-defined isomorphism

$$\begin{gathered}I(K)^m/I(K)^{m+1}\cong H^m(G),\\ \langle\langle a_1,\dots,a_m\rangle\rangle+I(K)^{m+1}\mapsto(-a_1)_K\cup\cdots\cup(-a_m)_K.\end{gathered}\tag{$*$}$$

(In fact, Milnor conjectured this only for the extension $K_{\text{sep}}/K$; however, since $\Omega$ is quadratically closed, this statement already implies that $H^i(\Omega)=0$ for all $i$ and hence $\text{Inf}\colon H^m(G)\to H^m(G_K)$ is an isomorphism [Ri, Cor. 5.4, p. 177]). Milnor's conjecture has been proven in many cases, including $m\leq4$ (see [Ar; Mer; MS; JR; Ro]).

Proposition 2.3. *In this set-up and notation, $G$ has real type with $\delta=(-1)_K$.*

*Proof.* For (i) of Definition 2.1, note that $\text{Ker}(\delta)=\text{Gal}(\Omega/K(\sqrt{-1}))$ and that $X_{K(\sqrt{-1})}=\emptyset$. Conditions (ii) and (iii) reflect the fact that, for any subextension $K\subseteq L\subseteq\Omega$, any two real closures of $L$ in $\Omega$ with respect to $P\in X_L$ are $L$-isomorphic in a unique way (observe that $P$ is a union of cosets in $L^\times/(L^\times)^2$ and that these cosets correspond to open subgroups of $\text{Gal}(\Omega/L)$ of index $\leq2$). Finally, (iv) says that for every subextension $K\subseteq L\subseteq\Omega$ and every $a\in L^\times$ we have $(a)_L\cup(-a)_L=0$. To verify this (well-known) equality, use for example that $\langle\langle-a,a\rangle\rangle=0$ in $W(L)$ and the existence of an isomorphism $I(L)^2/I(L)^3\cong H^2(G')$ as in $(*)$. □

Remark 2.4. When $\text{char}\,K=2$ we have $X_K=\emptyset$, so $G$ is torsion-free. As remarked earlier, $H^2(G)\cong H^2(G_K)\cong H^2(G_K(2))$ via the inflation maps.

Theorem 3.3 of [Ri, p. 256] now implies that $H^2(G)=0$. Therefore conditions (i)–(iv) of Definition 2.1 hold with $\delta=0$.

REMARK 2.5. In this set-up, the real core $N$ of $G$ is $\mathrm{Gal}(\Omega/L)$, where $L$ is the intersection of all real closures of $K$ in $\Omega$ (and $L=\Omega$ if $X_K=\emptyset$). Therefore [Be, Thm. 7, p. 81] shows that $G/N$ is torsion-free, and is in fact the largest torsion-free quotient of $G$.

## 3. An Exact Sequence for Virtually Projective Groups

DEFINITION. A profinite group $G$ will be called *virtually projective* if it contains a projective open subgroup.

REMARK 3.1. Suppose that $G$ is virtually projective of real type, and let $\delta$ satisfy condition (i) of Definition 2.1. Then $\mathrm{Ker}(\delta)$ is projective. Indeed, take a projective open subgroup $U$ of $G$. Then $U\cap \mathrm{Ker}(\delta)$ is projective [Ri, Prop. 2.1, p. 204]. Since $\mathrm{Ker}(\delta)$ is torsion-free, a theorem of Serre ([Se]; see also [H2]) implies that $\mathrm{cd}(\mathrm{Ker}(\delta))=\mathrm{cd}(U\cap\mathrm{Ker}(\delta))\le 1$, as claimed. Of course, if $\mathrm{Ker}(\delta)$ is projective then it is torsion-free [Ri, Cor. 2.3 & 2.4, pp. 208–209]. Consequently, $G$ is virtually projective of real type if and only if there exists $\delta\in H^1(G)$ such that $\mathrm{cd}(\mathrm{Ker}(\delta))\le 1$ and conditions (ii)–(iv) of Definition 2.1 hold.

By a well-known theorem of Artin, an element $x$ of a field $K$ is positive with respect to all orderings if and only if it is a sum of squares in $K$. The latter condition can be restated as follows: For some $n\ge 0$, the $(n+1)$-fold Pfister form $\langle\langle -x,1,\dots,1\rangle\rangle$ is 0 in $W(K)$. Thus, in the spirit of Section 2, Proposition 3.3 (to follow) with $m=1$ is a group-theoretic generalization of Artin's classical result (see (*)). For the proof we will need a technical lemma.

LEMMA 3.2. *Let $G$ be a profinite group, $n\ge 0$ and $0\ne\psi_i\in H^n(G)$, $i\in I$. Then there exists a closed subgroup $F$ of $G$ that is minimal with respect to the conditions* $\mathrm{Res}_F\,\psi_i\ne 0$, *$i\in I$. Moreover, $F$ is pro-2.*

*Proof.* For the first claim it suffices, by Zorn's lemma, to show that the collection $\mathcal{C}$ of all closed subgroups $F$ of $G$ such that $\mathrm{Res}_F\,\psi_i\ne 0$ for every $i\in I$ is closed under intersections of chains (with respect to inclusion). Indeed, let $F_j$, $j\in J$, be a chain in $\mathcal{C}$ and let $F=\bigcap_{j\in J}F_j$. We have

$$H^n(F)=\varinjlim H^n(F/F\cap M)\cong\varinjlim H^n(FM/M),$$

with $M$ ranging over all open normal subgroups of $G$, and likewise for each $F_j$ [Ri, Cor. 4.2, p. 114]. But for each such $M$ there exists a $j\in J$ such that $FM=F_jM$. We conclude that $F\in\mathcal{C}$.

The second assertion follows using a standard restriction–corestriction argument (cf. [Ri, Cor. 6.9, p. 141]). □

For any group $\Gamma$ of order 2, we clearly have $H^1(\Gamma)\cong\mathbb{Z}/2\mathbb{Z}$. Let $\delta_\Gamma$ be the nonzero element of $H^1(\Gamma)$. Then the cohomology ring $H^*(\Gamma)=\bigoplus_{n=0}^{\infty}H^n(\Gamma)$

is isomorphic as a graded ring to the polynomial ring $\mathbb{Z}/2\mathbb{Z}[T]$ graded by the usual degree map, with $\delta_\Gamma$ being mapped to $T$ [Sn, Thm. 3.9, p. 18]. In other words, $H^n(\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$ (as additive groups) for any $n \geq 0$, and under these isomorphisms the cup products $H^n(\Gamma) \times H^m(\Gamma) \to H^{n+m}(\Gamma)$ correspond to multiplication in the ring $\mathbb{Z}/2\mathbb{Z}$. For a profinite group $G$, for $\delta \in H^1(G)$ and for $n \geq 0$, we write $\delta^n = \delta \cup \cdots \cup \delta \in H^n(G)$.

PROPOSITION 3.3. *Let $G$ be a profinite group, let $N$ be its real core, and let $\delta \in H^1(G)$ satisfy conditions (i) and (iv) of Definition 2.1. Then, for every $\varphi \in H^m(G)$, we have $\mathrm{Res}_N \varphi = 0$ if and only if $\varphi \cup \delta^n = 0$ for some $n \geq 0$.*

*Proof* (cf. [AEJ, Lemma 2.2]). Suppose that $\varphi \cup \delta^n = 0$ for some $n \geq 0$. Then, for every $\Gamma \in \mathfrak{D}(G)$, we have $(\mathrm{Res}_\Gamma \varphi) \cup (\mathrm{Res}_\Gamma \delta)^n = \mathrm{Res}_\Gamma(\varphi \cup \delta^n) = 0$ in $H^{m+n}(\Gamma)$. Also, condition (iv) with $G' = \Gamma$ and with $\psi$ being the nonzero element of $H^1(\Gamma)$ implies that $\mathrm{Res}_\Gamma \delta \neq 0$. Therefore $\mathrm{Res}_\Gamma \varphi = 0$. Since $\Gamma \in \mathfrak{D}(G)$ was arbitrary, $\mathrm{Res}_N \varphi = 0$.

Conversely, assume that $\varphi \cup \delta^n \neq 0$ for all $n \geq 0$. Lemma 3.2 yields a closed subgroup $F$ of $G$ that is minimal with respect to the conditions $\mathrm{Res}_F(\varphi \cup \delta^n) \neq 0$, $n \geq 0$. Furthermore, $F$ is pro-2. Suppose that $\mathrm{rank}(F) \geq 2$. Then $H^1(F)$ has at least four elements [Ri, Thm. 6.8, p. 237]. Therefore we can find $0, \mathrm{Res}_F \delta \neq \psi \in H^1(F)$. Thus $F' = \mathrm{Ker}(\psi)$ and $F'' = \mathrm{Ker}(\psi + \mathrm{Res}_F \delta)$ are open subgroups of $F$ of index 2. The minimality of $F$ yields $n \geq 0$ such that $\mathrm{Res}_{F'}(\varphi \cup \delta^n) = 0$ and $\mathrm{Res}_{F''}(\varphi \cup \delta^n) = 0$. By [Ar, Satz 4.5], the following two sequences are exact:

$$H^{m+n-1}(F) \xrightarrow{\cup \psi} H^{m+n}(F) \xrightarrow{\mathrm{Res}} H^{m+n}(F');$$

$$H^{m+n-1}(F) \xrightarrow{\cup(\psi + \mathrm{Res}_F \delta)} H^{m+n}(F) \xrightarrow{\mathrm{Res}} H^{m+n}(F'').$$

Therefore,

$$\mathrm{Res}_F(\varphi \cup \delta^n) \in H^{m+n-1}(F) \cup \psi;$$

$$\mathrm{Res}_F(\varphi \cup \delta^n) \in H^{m+n-1}(F) \cup (\psi + \mathrm{Res}_F \delta).$$

Taking cup products with $\psi + \mathrm{Res}_F \delta$ and $\psi$, respectively, and using condition (iv) of Definition 2.1, we have

$$\mathrm{Res}_F(\varphi \cup \delta^n) \cup (\psi + \mathrm{Res}_F \delta) = \mathrm{Res}_F(\varphi \cup \delta^n) \cup \psi = 0.$$

By subtracting we obtain $\mathrm{Res}_F(\varphi \cup \delta^{n+1}) = 0$, a contradiction.

Consequently, $\mathrm{rank}(F) \leq 1$. But $F$ cannot be 1 or $\mathbb{Z}_2$, since in these cases $H^2(F) = 0$ [Ri, Thm. 6.5, p. 235], implying that $\mathrm{Res}_F(\varphi \cup \delta) = 0$. By Remark 2.2(1), $F \cong \mathbb{Z}/2\mathbb{Z}$, whence $F \leq N$. As $\mathrm{Res}_F \varphi \neq 0$, also $\mathrm{Res}_N \varphi \neq 0$. □

Define $\beta_G \colon H^1(G) \to H^2(G)$ (the *Bockstein operator* of $G$) by $\beta_G(\varphi) = \varphi \cup \varphi$.

COROLLARY 3.4. *Let $G$ be a virtually projective profinite group of real type, and let $N$ be its real core. Then the following sequence is exact:*

$$0 \longrightarrow H^1(G/N) \xrightarrow{\mathrm{Inf}} H^1(G) \xrightarrow{\beta_G} H^2(G) \longrightarrow 0.$$

*Proof.* Take $\delta \in H^1(G)$ as in Definition 2.1. If $\delta = 0$ then $G$ is projective (Remark 3.1); hence $N=1$ and $H^2(G)=0$, so the assertion is in this case trivial. Assume that $\delta \neq 0$ and consider again Arason's exact sequence [Ar, Satz 4.5]:

$$\cdots \longrightarrow H^n(\mathrm{Ker}(\delta)) \xrightarrow{\mathrm{Cor}} H^n(G) \xrightarrow{\cup\delta} H^{n+1}(G) \xrightarrow{\mathrm{Res}} H^{n+1}(\mathrm{Ker}(\delta)) \longrightarrow \cdots.$$

Since $\mathrm{cd}(\mathrm{Ker}(\delta)) \leq 1$, the map $\cup\delta\colon H^n(G) \to H^{n+1}(G)$ is surjective for $n=1$ and bijective for $n \geq 2$. For $\varphi \in H^1(G)$ we therefore obtain from Proposition 3.3 that $\mathrm{Res}_N \varphi = 0$ if and only if $\varphi \cup \delta = 0$. By condition (iv) of Definition 2.1, this is equivalent to $\beta_G(\varphi) = 0$. The assertion now follows from the exactness of the sequence [Ri, Cor. 5.4, p. 177]:

$$0 \longrightarrow H^1(G/N) \xrightarrow{\mathrm{Inf}} H^1(G) \xrightarrow{\mathrm{Res}} H^1(N). \qquad \square$$

## 4. Virtually Projective Groups Modulo the Real Core

Let $G$ be a profinite group, and suppose that $\mathfrak{X}$ is a compact system of representatives for the conjugacy classes of $\mathfrak{D}(G)$. For each $\varphi \in H^1(G)$ set $A(\mathfrak{X}, \varphi) = \{\Gamma \in \mathfrak{X} \mid \varphi(\Gamma) = 0\}$.

EXAMPLE. Consider the field-theoretic setting of Section 2. Then there is a bijection between $\mathfrak{X}$ and the ordering space $X_K$, where $\Gamma = \mathrm{Gal}(\Omega/\bar{K}) \in \mathfrak{X}$ corresponds to $P \in X_K$ if and only if $\bar{K}$ induces $P$. For $a \in K^\times$ we thus have $a \in P$ if and only if $\sqrt{a} \in \bar{K}$, or (equivalently) the homomorphism $(a)_K$ is trivial on $\Gamma$. Therefore $A(\mathfrak{X}, (a)_K)$ may be identified with $\{P \in X_K \mid a \in P\}$. Thus, part (b) of the following lemma is a group-theoretic version of the strong approximation property (in the sense of [Pr]) for virtually projective groups of real type.

LEMMA 4.1. *Let $G$ and $\mathfrak{X}$ be as above and let $C$ be a clopen subset of $\mathfrak{X}$.*

(a) *If $G$ satisfies condition (iii) of Definition 2.1, then $C$ is a finite Boolean combination of sets of the form $A(\mathfrak{X}, \varphi)$, $\varphi \in H^1(G)$.*

(b) *If $G$ is virtually projective of real type, then $C = A(\mathfrak{X}, \varphi)$ for some $\varphi \in H^1(G)$.*

*Proof.* (a) Let $\Gamma \in C$. By assumption, $\{\Gamma\}$ is an intersection of sets of the form $A(\mathfrak{X}, \varphi)$. Since these sets are closed in $\mathfrak{X}$ and since $\mathfrak{X} \setminus C$ is compact, some finite intersection $A(\mathfrak{X}, \varphi_1) \cap \cdots \cap A(\mathfrak{X}, \varphi_n)$, $\varphi_1, \ldots, \varphi_n \in H^1(G)$, to which $\Gamma$ belongs is fully contained in $C$. Thus, the compact space $C$ is covered by such intersections. Since they are open, $C$ is the union of only finitely many of them.

(b) By (a), it is enough to show that the family of sets $A(\mathfrak{X}, \varphi)$, $\varphi \in H^1(G)$, is closed under finite unions and complements. So suppose that $\varphi_1, \varphi_2 \in H^1(G)$ and take $\varphi_3 \in H^1(G)$ such that $\varphi_1 \cup \varphi_2 = \beta_G(\varphi_3)$ (Corollary 3.4). For each

$\Gamma \in \mathfrak{D}(G)$, $(\mathrm{Res}_\Gamma \varphi_1) \cup (\mathrm{Res}_\Gamma \varphi_2) = \beta_\Gamma(\mathrm{Res}_\Gamma \varphi_3)$. As $\beta_\Gamma$ is an isomorphism, $A(\mathfrak{X}, \varphi_1) \cup A(\mathfrak{X}, \varphi_2) = A(\mathfrak{X}, \varphi_3)$. Also, for $\delta \in H^1(G)$ as in Definition 2.1, we have $\mathfrak{X} \setminus A(\mathfrak{X}, \varphi_1) = A(\mathfrak{X}, \varphi_1 + \delta)$. □

For the next lemma, again let $G$ be a profinite group, let $\delta \in H^1(G)$, and suppose that $\mathrm{Ker}(\delta)$ is torsion-free. Also let $\mathfrak{X}$ be a compact subset of $\mathfrak{D}(G)$. Let $\hat{\prod}_{\Gamma \in \mathfrak{X}} H^1(\Gamma)$ be the subgroup of the direct product $\prod_{\Gamma \in \mathfrak{X}} H^1(\Gamma)$ consisting of all elements $(\varphi_\Gamma)_{\Gamma \in \mathfrak{X}}$ such that $\{\Gamma \in \mathfrak{X} \mid \varphi_\Gamma = 0\}$ is clopen in $\mathfrak{X}$. (Since $H^1(\Gamma) = \{0, \mathrm{Res}_\Gamma \delta\}$ for $\Gamma \in \mathfrak{X}$, this also coincides with the terminology of [El, Sec. 2].)

LEMMA 4.2. *Let $G$ and $\mathfrak{X}$ be as above, let $N$ be the real core of $G$, and let $\varphi \in H^1(N)$. Then* $(\mathrm{Res}_\Gamma \varphi)_{\Gamma \in \mathfrak{X}} \in \hat{\prod}_{\Gamma \in \mathfrak{X}} H^1(\Gamma)$.

*Proof.* Since $\mathrm{Ker}(\varphi)$ is the intersection of all open subgroups $U$ of $G$ containing it, there exists such $U$ with $\mathrm{Ker}(\varphi) = U \cap N$. Therefore

$$\{\Gamma \in \mathfrak{X} \mid \varphi(\Gamma) = 0\} = \{\Gamma \in \mathfrak{X} \mid \Gamma \le U\}$$

is clopen in $\mathfrak{X}$, as required. □

LEMMA 4.3. *Let $G$ be a virtually projective group of real type, and let $\mathfrak{X}$ be a compact system of representatives for the conjugacy classes of $\mathfrak{D}(G)$. Then* $\mathrm{Res}\colon H^1(G) \to \hat{\prod}_{\Gamma \in \mathfrak{X}} H^1(\Gamma)$ *is surjective.*

*Proof.* Take $(\varphi_\Gamma)_{\Gamma \in \mathfrak{X}} \in \hat{\prod}_{\Gamma \in \mathfrak{X}} H^1(\Gamma)$. Thus, $C = \{\Gamma \in \mathfrak{X} \mid \varphi_\Gamma = 0\}$ is clopen in $\mathfrak{X}$. Lemma 4.1(b) yields $\varphi \in H^1(G)$ such that $C = A(\mathfrak{X}, \varphi)$. Then $\varphi_\Gamma = \mathrm{Res}_\Gamma \varphi$ for all $\Gamma \in \mathfrak{X}$. □

Combining the previous observations, we obtain the following.

PROPOSITION 4.4. *Let $G$ be a virtually projective group of real type, and let $N$ be its real core. Then:*

(a) $\mathrm{Res}\colon H^1(G) \to H^1(N)^G$ *is surjective;*
(b) $\mathrm{Res}\colon H^2(G) \to H^2(N)$ *is injective; and*
(c) $H^2(G/N) = 0$.

*Proof.* Let $\mathfrak{X}$ be a compact system of representatives for the conjugacy classes of $\mathfrak{D}(G)$ (Remark 2.2(3)). To prove (a), take $\varphi \in H^1(N)^G$. Lemma 4.2 and Lemma 4.3 yield $\psi \in H^1(G)$ such that $\mathrm{Res}_\Gamma \varphi = \mathrm{Res}_\Gamma \psi$ for all $\Gamma \in \mathfrak{X}$. For each $\sigma \in G$ we trivially have $\psi^\sigma = \psi$, and (by the assumption on $\varphi$) also $\varphi^\sigma = \varphi$. Hence $\varphi(\Gamma^\sigma) = \varphi(\Gamma) = \psi(\Gamma) = \psi(\Gamma^\sigma)$. Since $N = \langle \Gamma^\sigma \mid \Gamma \in \mathfrak{X}, \sigma \in G \rangle$, we get $\varphi = \mathrm{Res}_N \psi$, as required.

Next consider the following commutative diagram, whose row and column are exact (by the Hochschild–Serre spectral sequence [Ri, Cor. 5.4, p. 177] and Corollary 3.4, respectively):

$$\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & H^1(G/N) & & & & & & \\
 & & & /\!\!/ & \downarrow{\scriptstyle \mathrm{Inf}} & & & & & & \\
0 \longrightarrow & H^1(G/N) & \xrightarrow{\mathrm{Inf}} & & H^1(G) & \xrightarrow{\mathrm{Res}} & H^1(N)^G & \xrightarrow{\mathrm{tr}} & H^2(G/N) & \xrightarrow{\mathrm{Inf}} & H^2(G). \\
 & & & & \downarrow{\scriptstyle \beta_G} & & \downarrow{\scriptstyle \beta_N} & & & & \\
 & & & & H^2(G) & \xrightarrow{\mathrm{Res}} & H^2(N) & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array} \quad (1)$$

By Corollary 3.4 again (applied to $N$ instead of $G$; see Remark 2.2(4)), $\beta_N$: $H^1(N) \to H^2(N)$ is an isomorphism; hence $\beta_N: H^1(N)^G \to H^2(N)$ is injective. We conclude from (1) that Res: $H^2(G) \to H^2(N)$ is injective, as asserted in (b).

Finally, the exactness of the row in (1) together with (a) implies that Inf: $H^2(G/N) \to H^2(G)$ is injective. The composition of this monomorphism with the monomorphism of (b) is, however, trivial (by definition). Therefore $H^2(G/N) = 0$, as claimed in (c). □

THEOREM 4.5. *Let G be a virtually projective group of real type, and let N be its real core. Then G/N is projective.*

*Proof.* Fix a prime number $p$. We need to show that $\mathrm{cd}_p(G/N) \leq 1$. Replacing $G$ by a subgroup $G_p$ containing $N$ such that $G_p/N$ is a $p$-Sylow subgroup of $G/N$, we may assume that $G/N$ is a pro-$p$ group (see Remark 2.2(4)). Thus, it is enough to prove that $H^2(G/N, \mathbb{Z}/p\mathbb{Z}) = 0$ [Ri, Cor. 4.2, p. 220]. Proposition 4.4(c) shows this for $p = 2$. For $p$ odd, let $\delta$ be as in Definition 2.1. Then $\mathrm{cd}_p(G) = \mathrm{cd}_p(\mathrm{Ker}(\delta)) \leq 1$, by [Ri, Prop. 2.1, p. 204] and Remark 3.1. Therefore $H^2(G, \mathbb{Z}/p\mathbb{Z}) = 0$. Furthermore, since $N$ is generated by involutions while $\mathbb{Z}/p\mathbb{Z}$ has no involutions, $H^1(N, \mathbb{Z}/p\mathbb{Z}) = 0$. By the Hochschild–Serre spectral sequence again, the sequence

$$H^1(N, \mathbb{Z}/p\mathbb{Z})^G \xrightarrow{\mathrm{tr}} H^2(G/N, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\mathrm{Inf}} H^2(G, \mathbb{Z}/p\mathbb{Z})$$

is exact. Therefore $H^2(G/N, \mathbb{Z}/p\mathbb{Z}) = 0$, as required. □

COROLLARY 4.6. $\mathrm{Gal}(\mathbb{Q}_{\mathrm{tr}}/\bar{\mathbb{Q}}_{\mathrm{ab}})$ *is projective.*

*Proof.* By [Ri, Thm. 8.8, pp. 302–303], $G_{\mathbb{Q}_{\mathrm{ab}}}$ is projective. Since $[\mathbb{Q}_{\mathrm{ab}} : \bar{\mathbb{Q}}_{\mathrm{ab}}] = 2$, Proposition 2.3 (with $\Omega$ being the algebraic closure $\tilde{\mathbb{Q}}$ of $\mathbb{Q}$) thus implies that $G_{\bar{\mathbb{Q}}_{\mathrm{ab}}}$ is virtually projective of real type. By Remark 2.5, the real core $N$ of $G_{\bar{\mathbb{Q}}_{\mathrm{ab}}}$ is generated by the subgroups $G_{\bar{K}}$, with $\bar{K}$ ranging over all real closures of $\bar{\mathbb{Q}}_{\mathrm{ab}}$. Furthermore, since all real closures of $\mathbb{Q}$ are isomorphic and $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}}/\mathbb{Q})$ is abelian, the unique ordering on $\mathbb{Q}$ has a unique extension to $\bar{\mathbb{Q}}_{\mathrm{ab}}$. It follows that the real closures of $\bar{\mathbb{Q}}_{\mathrm{ab}}$ are just the real closures of $\mathbb{Q}$, so $N = G_{\mathbb{Q}_{\mathrm{tr}}}$. We conclude from Theorem 4.5 that $\mathrm{Gal}(\mathbb{Q}_{\mathrm{tr}}/\bar{\mathbb{Q}}_{\mathrm{ab}}) \cong G_{\bar{\mathbb{Q}}_{\mathrm{ab}}}/G_{\mathbb{Q}_{\mathrm{tr}}}$ is projective. □

Let $\hat{F}_\omega$ be the free profinite group of rank $\aleph_0$.

QUESTION 4.7. Is $\mathrm{Gal}(\mathbb{Q}_{tr}/\bar{\mathbb{Q}}_{ab}) \cong \hat{F}_\omega$?

It is suspected that $G_{\bar{\mathbb{Q}}_{ab}}$ is a free profinite product (in a natural sense) of groups of order 2 and of $\hat{F}_\omega$ (this is a strengthened version of the celebrated conjecture of Shafarevich, stating that $G_{\mathbb{Q}_{ab}} \cong \hat{F}_\omega$). If so, then the following observation would answer Question 4.7 affirmatively.

PROPOSITION 4.8. *Let $G$ be a profinite group with real core $N$, let $D$ be a closed subgroup of $G$ that is generated by involutions, and let $F$ be a closed subgroup of $G$ with $F \cong \hat{F}_\omega$. Suppose that $G$ is the free profinite product $G = D * F$. Then $G/N \cong \hat{F}_\omega$.*

*Proof.* Let $F_{ab}$ be the maximal pro-abelian quotient of $F$, and let $\pi: G = D*F \to F_{ab}$ and $\theta: G \to G/N$ be the natural projections. Since $F$ is torsion-free, [HR, Thm. A] or [Mel, Prop. 4.9] implies that any involution in $G$ is conjugate to an involution in $D$. Hence, since $\pi(D) = 1$, also $\pi(N) = 1$, so $\pi$ breaks through an epimorphism $\bar{\pi}: G/N \to F_{ab}$. We conclude that $\mathrm{rank}(G/N) \geq \mathrm{rank}(F_{ab}) = \aleph_0$. On the other hand, $\theta(D) = 1$, so $G/N = \theta(G) = \theta(F)$, implying that $\mathrm{rank}(G/N) = \aleph_0$.

Now let $\alpha: B \to A$ be an epimorphism of finite groups, and let $\varphi: G/N \to A$ be a continuous epimorphism. By a result of Iwasawa [FJ, Cor. 24.2] it suffices to find a continuous epimorphism $\bar{\beta}: G/N \to B$ such that $\varphi = \alpha \circ \bar{\beta}$ on $G/N$ (note that this latter property with $A = 1$ implies that every finite group is a quotient of $G/N$). Since $F$ is free, we can find a continuous epimorphism $\beta_0: F \to B$ such that $\varphi \circ \theta = \alpha \circ \beta_0$ on $F$. The universal property of $G = D * F$ yields an epimorphism $\beta: G \to B$ extending $\beta_0$ such that $\beta(D) = 1$. Then $\varphi \circ \theta = \alpha \circ \beta$ on $G$. Since $\beta(N) = 1$, $\beta$ factors through an epimorphism $\bar{\beta}: G/N \to B$ with $\beta = \bar{\beta} \circ \theta$, implying $\varphi = \alpha \circ \bar{\beta}$. □

## 5. Connections with Real Projectivity

Let $\mathfrak{X}$ be a collection of closed subgroups of a pro-2 group $G$. We say that $G$ is a *free pro-2 product* of $\mathfrak{X}$ if every continuous map $\bigcup_{\Gamma \in \mathfrak{X}} \Gamma \to H$, where $H$ is a pro-2 group, that induces a homomorphism on each $\Gamma \in \mathfrak{X}$ uniquely extends to a continuous homomorphism $G \to H$. Using the preceding results, we now obtain the following version of (the difficult part of) [H4, Thm. A].

THEOREM 5.1. *Let $G$ be a virtually projective pro-2 group of real type, and let $N$ be the real core of $G$. Then $G$ is a free pro-2 product of $\mathfrak{X} \cup \{F\}$ for a compact system $\mathfrak{X}$ of representatives for the conjugacy classes of $\mathfrak{D}(G)$ and for a free pro-2 closed subgroup $F$ of $G$ satisfying $F \cong G/N$.*

*Proof.* By Theorem 4.5, $G/N$ is projective and hence a free pro-2 group. Consequently, there exists a closed subgroup $F$ of $G$ that is mapped isomorphically onto $G/N$ by the natural projection $G \to G/N$ [Ri, Prop. 3.1,

p. 211]. Choose $\mathfrak{X}$ as in the statement of the theorem (Remark 2.2(3)). In light of [E1, Prop. 4.3], we need to prove that $\mathrm{Res}\colon H^1(G)\to H^1(F)\times\hat{\prod}_{\Gamma\in\mathfrak{X}}H^1(\Gamma)$ is bijective and that $\mathrm{Res}\colon H^2(G)\to\prod_{\Gamma\in\mathfrak{X}}H^2(\Gamma)$ is injective (observe that $H^2(F)=0$).

The injectivity at $H^1$ follows from the fact that $G=FN=\langle F,\Gamma^\sigma\mid\Gamma\in\mathfrak{X},\ \sigma\in G\rangle$. To prove surjectivity at $H^1$, consider the commutative diagram

$$\begin{array}{ccccc} H^1(G/N) & \xrightarrow{\mathrm{Inf}} & H^1(G) & \xrightarrow{\mathrm{Res}_N} & H^1(N) \\ & {\scriptstyle\cong}\searrow^{\iota} & \downarrow{\scriptstyle \mathrm{Res}_F} & & \\ & & H^1(F) & & \end{array} \tag{2}$$

with exact upper row [Ri, Cor. 5.4, p. 177]. Let

$$\psi\in H^1(F)\quad\text{and}\quad(\chi_\Gamma)_{\Gamma\in\mathfrak{X}}\in\hat{\prod}_{\Gamma\in\mathfrak{X}}H^1(\Gamma).$$

Take $\chi\in H^1(G)$ such that $\mathrm{Res}_\Gamma\chi=\chi_\Gamma$ for all $\Gamma\in\mathfrak{X}$ (Lemma 4.3). Also take $\psi_1\in H^1(G/N)$ such that $\iota(\psi_1)=\psi-\mathrm{Res}_F\chi$. For $\varphi=\mathrm{Inf}\,\psi_1+\chi$ we have

$$\mathrm{Res}_F\,\varphi=\iota(\psi_1)+\mathrm{Res}_F\,\chi=\psi.$$

Furthermore, for each $\Gamma\in\mathfrak{X}$ we have

$$\mathrm{Res}_\Gamma(\mathrm{Inf}\,\psi_1)=\mathrm{Res}_\Gamma((\mathrm{Res}_N\circ\mathrm{Inf})(\psi_1))=0,$$

hence $\mathrm{Res}_\Gamma\,\varphi=\chi_\Gamma$. We conclude that $\mathrm{Res}\colon H^1(G)\to H^1(F)\times\hat{\prod}_{\Gamma\in\mathfrak{X}}H^1(\Gamma)$ is indeed surjective.

In light of Lemma 4.2, we have the commutative diagram

$$\begin{array}{ccccccc} 0\to H^1(G/N) & \xrightarrow{\mathrm{Inf}} & H^1(G) & & \xrightarrow{\beta_G} & & H^2(G) \to 0 \\ & & {\scriptstyle\mathrm{Res}_N}\downarrow & & & & \downarrow{\scriptstyle\mathrm{Res}} \\ & & H^1(N)^G & \xrightarrow{\mathrm{Res}} & \hat{\prod}_{\Gamma\in\mathfrak{X}}H^1(\Gamma) & \xrightarrow{(\beta_\Gamma)_{\Gamma\in\mathfrak{X}}} & \prod_{\Gamma\in\mathfrak{X}}H^2(\Gamma). \end{array}$$

By Corollary 3.4, its upper row is exact. The right lower horizontal arrow is an isomorphism at each factor, and hence is injective. Since $N=\langle\Gamma^\sigma\mid\Gamma\in\mathfrak{X},\ \sigma\in G\rangle$, the left lower horizontal arrow is also injective. We conclude from this and from the exactness of the row in (2) that the right vertical arrow is injective. □

We say that a pro-2 group $G$ is a *real-free* pro-2 group if it is a free pro-2 product of a collection $\mathfrak{X}\cup\{F\}$, where $\mathfrak{X}$ is a compact subset of $\mathfrak{D}(G)$ and $F$ is a free pro-2 closed subgroup of $G$. Then $\mathfrak{D}(G)=\{\Gamma^\sigma\mid\Gamma\in\mathfrak{X},\ \sigma\in G\}$ [E2, Lemma 5.3], so $\mathfrak{D}(G)$ is compact. Recall that a profinite group $G$ is *real projective* (cf. [HJ]) if:

(i) for any epimorphism $\alpha\colon B\to A$ of finite groups and for any continuous homomorphism $\varphi\colon G\to A$ there exists a continuous homomorphism $\beta\colon G\to B$ such that $\varphi=\alpha\circ\beta$, provided that for each $\Gamma\in\mathfrak{D}(G)$ there exists a homomorphism $\beta_\Gamma\colon\Gamma\to B$ such that $\varphi=\alpha\circ\beta_\Gamma$ on $\Gamma$; and

(ii) $\mathfrak{D}(G)$ is compact.

Thus, a profinite group $G$ is projective if and only if it is real projective and torsion-free. Also, a pro-2 group is real projective if and only if it is a real-free pro-2 group [H1, Prop. 4.2].

COROLLARY 5.2. *A profinite group is virtually projective of real type if and only if it is real projective.*

*Proof.* Suppose that $G$ is a virtually projective group of real type, and let $G_p$ be any $p$-Sylow subgroup of $G$. By Remark 2.2(4), $G_p$ is virtually projective of real type. If $p=2$, then $G_p$ is a real-free pro-2 group, by Theorem 5.1. If $p$ is an odd prime then, by Remark 2.2(1) and [Ri, Prop. 2.1, p. 204], $G_p$ is projective. Therefore, in both cases $G_p$ is real projective. Proposition 5.5 of [H1] now implies that $G$ is real projective.

Conversely, if $G$ is real projective then $G$ is the absolute Galois group of some field $K$ of characteristic 0 [HJ, Thm. 10.2]. Then $G_{K(\sqrt{-1})}$ is projective, by [HJ, Cor. 10.5] and by Artin–Schreier theory. We conclude from this and from Proposition 2.3 that $G$ is virtually projective of real type. (It is also not difficult to give a purely group-theoretic proof of this fact, using the methods of [HJ].) □

REMARK 5.3. (1) Suppose that $G$ is virtually projective and that there exists a $\delta \in H^1(G)$ for which only conditions (i) and (ii) of Definition 2.1 are satisfied. By [H4, Prop. 2.2] and [HJ, Thm. 10.2], $G \cong G_K$ for some field $K$ of characteristic 0. Proposition 2.3 implies that $G$ has real type. Thus, one can find $\delta \in H^1(G)$ for which conditions (iii) and (iv) of Definition 2.1 also hold. However, I do not know a direct proof of this.

(2) As observed by Pop, it is easy to derive Theorem 4.5 when the assumption that $G$ is virtually projective of real type is replaced by the assumption that it is real projective. Indeed, let $\pi\colon G \to G/N$ be the natural epimorphism ($N$ being the real core). Let $\alpha\colon B \to A$ be an epimorphism of finite groups and let $\varphi\colon G/N \to A$ be a continuous homomorphism. By [Ri, Prop. 3.1, p. 21], we need to find a continuous homomorphism $\gamma\colon G/N \to B$ such that $\varphi = \alpha\circ\gamma$. Corollary 6.2 of [HJ] yields a finite group $B'$ such that $\mathfrak{D}(B') = \emptyset$ and for which there exists an epimorphism $\theta\colon B' \to B$. As $(\varphi\circ\pi)(\mathfrak{D}(G)) = \{1\}$, the real projectivity assumption yields a continuous homomorphism $\beta\colon G \to B'$ such that $\varphi\circ\pi = \alpha\circ\theta\circ\beta$. By the choice of $B'$, $\beta(\mathfrak{D}(G)) = \{1\}$, whence $\beta(N) = 1$. Therefore $\beta = \bar{\beta}\circ\pi$ for some continuous homomorphism $\bar{\beta}\colon G/N \to B'$. We conclude that $\varphi = \alpha\circ\theta\circ\bar{\beta}$, as desired.

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