The Subspace Problem for Weighted Inductive Limits of Spaces of Holomorphic Functions

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The aim of the present article is to solve in the negative a well-known open problem raised by Bierstedt, Meise, and Summers in [BMS1] (see also [BM1]). We construct a countable inductive limit of weighted Banach spaces of holomorphic functions, which is not a topological subspace of the corresponding weighted inductive limit of spaces of continuous functions. As a consequence, the topology of the weighted inductive limit of spaces of holomorphic functions cannot be described by the weighted sup-seminorms given by the maximal system of weights associated with the sequence of weights defining the inductive limit. The main step of our construction shows that a certain sequence space is isomorphic to a complemented subspace of a weighted space of holomorphic functions. To do this we make use of a special sequence of outer holomorphic functions and of the existence of radial limits of holomorphic bounded functions in the disc.

Weighted spaces and weighted inductive limits of spaces of holomorphic functions on open subsets of $\mathbb{C}^N$ ($N \in \mathbb{N}$) arise in many fields, such as linear partial differential operators, convolution equations, complex and Fourier analysis, and distribution theory. Since the structure of general locally convex inductive limits is rather complicated and many pathologies can occur, the applications of weighted inductive limits have been restricted. The reason was that it did not seem possible to describe the inductive limit, its topology, and in particular a fundamental system of seminorms in a way that permits direct estimates and computations. In the theory of Ehrenpreis [Eh] of "analytically uniform spaces", the topology of certain weighted inductive limits of spaces of entire functions, which are the Fourier–Laplace transforms of spaces of test functions or ultradistributions, were required to have a fundamental system of weighted sup-seminorms. Berenstein and Dostal [BD] reformulated the problem in a more general setting and used the term "complex representation". This corresponds exactly with the term "projective description" used by Bierstedt, Meise, and Summers [BMS1], which is the one we will utilize in this paper. In [BMS1] it was proved that countable weighted inductive limits of Banach spaces of holomorphic functions on arbitrary open subsets G of $\mathbb{C}^N$ admit such a canonical projective description by weighted

sup-seminorms whenever the linking maps between the generating Banach spaces are compact. This theorem extended previous work by Taylor [Ta] with a more functional analytic approach and was very satisfactory from the point of view of applications. It remained open whether the projective description theorem continued to hold for weighted inductive limits of spaces of holomorphic functions without any restriction on the linking maps. This problem is solved here.

1. Notation and Preliminaries

All vector spaces are defined over the complex scalar field $\mathbb{C}$. We denote by $\mathbb{R}^+$ (resp. $\mathbb{R}_0^+$) the space of strictly positive reals (resp. $\mathbb{R}^+ \cup \{0\}$).

Let $V = (v_k)_{k=1}^\infty$ be a decreasing sequence of continuous strictly positive weight functions defined on an open subset G of $\mathbb{C}^N$, $N \in \mathbb{N}$. We denote by $\mathcal{V}C(G)$ and $\mathcal{V}H(G)$ the inductive limits $\text{ind}_k Cv_k(G)$ and $\text{ind}_k Hv_k(G)$, where $Cv_k(G)$ (resp. $Hv_k(G)$) denotes the Banach space

$$\{f : G \to \mathbb{C} \text{ continuous (resp. holomorphic)} \,|$$
$$p_{v_k}(f) := \sup_{z \in G} v_k(z)|f(z)| < \infty\}.$$

The canonical embedding $\mathcal{V}H(G) \hookrightarrow \mathcal{V}C(G)$ is continuous; it is a well-known open problem whether the topologies of $\mathcal{V}H(G)$ and $\mathcal{V}C(G)$ coincide on $\mathcal{V}H(G)$. (See [BM1, Sec. 1] or [BB4, Sec. 4, Prob. 5].) This is a particular case of the so-called subspace problem for locally convex inductive limits.

In order to describe the topology of the weighted inductive limits $\mathcal{V}C(G)$ and $\mathcal{V}H(G)$, Bierstedt, Meise, and Summers [BMS1] introduced the system of weights $\bar{V}$, associated with the sequence V,

$$\bar{V} = \{\bar{v} : G \to \mathbb{R}^+ \text{ continuous} \,|\, \forall k \in \mathbb{N} \,\exists C_k > 0 \text{ such that } \bar{v} \le C_k v_k\}.$$

The projective hull $C\bar{V}(G)$ (resp. $H\bar{V}(G)$) of $\mathcal{V}C(G)$ (resp. $\mathcal{V}H(G)$) is the locally convex space

$$\{f : G \to \mathbb{C} \text{ continuous (resp. holomorphic)} \,|$$
$$p_{\bar{v}}(f) := \sup_{z \in G} \bar{v}(z)|f(z)| < \infty \text{ for all } \bar{v} \in \bar{V}\},$$

endowed with the locally convex topology defined by the seminorms $p_{\bar{v}}$ as $\bar{v}$ varies in $\bar{V}$. Clearly the inclusions $\mathcal{V}C(G) \hookrightarrow C\bar{V}(G)$ and $\mathcal{V}H(G) \hookrightarrow H\bar{V}(G)$ are continuous. In [BMS1] it was proved that $\mathcal{V}C(G) = C\bar{V}(G)$ and $\mathcal{V}H(G) = H\bar{V}(G)$ hold algebraically, and that the two spaces in each equality have the same bounded sets. Moreover, one of the main results in [BMS1] shows that if V satisfies condition (S),

(S)   for all k there is l such that v_l/v_k vanishes at infinity on G,

then $\mathcal{V}H(G) = H\bar{V}(G)$ holds topologically and $\mathcal{V}H(G)$ is a topological subspace of $\mathcal{V}C(G)$. In [BM2], [Ba] and [BB3], the topological identity $\mathcal{V}C(G) = C\bar{V}(G)$ was characterized in terms of a condition (D) on the sequence V. We present here an example showing that if condition (S) does not hold, then the space $\mathcal{V}H(G)$ need not be a topological subspace of $\mathcal{V}C(G)$ and $\mathcal{V}H(G) = H\bar{V}(G)$ need not hold topologically.

In the construction of our example we need weighted inductive limits of spaces of sequences on $\mathbb{N}$. We recall the notation from [BMS2]. We will denote here by $\Lambda = (\lambda_k)_{k=1}^{\infty}$ a decreasing sequence of strictly positive weights on $\mathbb{N}$, $\lambda_{nk} := \lambda_k(n)$ for $k, n \in \mathbb{N}$. The corresponding weighted inductive limit is denoted by $k_{\infty} = \text{ind}_k \, l_{\infty}(\lambda_k)$. The system of weights associated with Λ is denoted by $\bar{\Lambda}$ and $\bar{\lambda} \in \bar{\Lambda}$ if and only if $\bar{\lambda}(n) > 0$ for every $n \in \mathbb{N}$ and for every $k \in \mathbb{N}$ there is $C_k > 0$ with $\bar{\lambda} \le C_k \lambda_k$ on $\mathbb{N}$. The projective hull of the inductive limit k_{∞} is denoted by K_{∞}, and it is the space

$$\{x = (x_n) \mid p_{\bar{\lambda}}(x) := \sup_{n \in \mathbb{N}} \bar{\lambda}(n)|x_n| < \infty \text{ for all } \bar{\lambda} \in \bar{\Lambda}\}.$$

The spaces K_{∞} and k_{∞} always coincide algebraically and have the same bounded sets; however, there are examples of sequences Λ such that K_{∞} and k_{∞} do not coincide topologically, K_{∞} has bounded sets that are not metrizable, and K_{∞} is not bornological (see [BMS2; BB1; Kö; Vo]). We refer to [BB4] for a survey article on spaces of type $\mathcal{V}C(X)$.

2. Main Construction

In this section we construct a sequence of weights $W = (w_k)_{k=1}^{\infty}$ on an open bounded set G_1 of $\mathbb{C}$ such that the projective hull $H\bar{W}(G_1)$ contains a complemented subspace isomorphic to a space of sequences K_{∞} that is not bornological. Consequently, the space $H\bar{W}(G_1)$ is not bornological and hence it does not coincide topologically with $\mathcal{W}H(G_1)$.

We first select a decreasing sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$ of strictly positive functions $\lambda_k(n) = \lambda_{nk}$, $n, k \in \mathbb{N}$, on $\mathbb{N}$ such that $1/n^2 < \lambda_{nk} \le 1$ for all n and k, the corresponding space K_{∞} is not bornological and contains bounded sets that are not metrizable. For example, combine [Kö, Sect. 31.7] with [BM1, Thm. 9] and [BB1]. In this case the system of weights $\bar{\Lambda}$ associated with Λ satisfies: For each $\bar{\mu} \in \bar{\Lambda}$ there are $\bar{\lambda} \in \bar{\Lambda}$ and $C > 0$ with $\bar{\mu} \le C\bar{\lambda}$ and $1/n^2 < \bar{\lambda}(n) \le 1$ for all $n \in \mathbb{N}$. Indeed, given $\bar{\mu} \in \bar{\Lambda}$, we select $c_k > 0$ such that $\bar{\mu} \le \inf_k c_k \lambda_k$. We put $d_k = \max(c_k, 1)$ for all k and we set $\bar{\lambda} = \min(\inf_k d_k \lambda_k, \lambda_1) \in \bar{\Lambda}$. Accordingly, K_{∞} has a fundamental system of seminorms P given by multiples of elements $\bar{\lambda} \in \bar{\Lambda}$ satisfying $1/n^2 < \bar{\lambda}(n) \le 1$ for all $n \in \mathbb{N}$.

Let $G_1 = \{z \in \mathbb{C} \mid 1/2 < |z| < 1, 0 < \arg z < \pi\}$ and define the system $W = (w_k)_{k=1}^{\infty}$ of weight functions on G_1 by $w_k(re^{i\theta}) = \hat{w}_k(\theta)$, where $\hat{w}_k : [0, \pi[\to \mathbb{R}^+$ satisfies $\hat{w}_k(\theta) = \lambda_{nk}$ for $\theta \in I_n := [\theta_n - 1/(2^5 n^2), \theta_n + 1/(2^5 n^2)]$ with $\theta_n := 1/(2n)$ for all $n \in \mathbb{N}$, and $\hat{w}_k$ is extended affinely for other θ.

Now we define a sequence of elements of $H(G_1)$ which will be essential in our construction. For all $n \in \mathbb{N}$, let $\epsilon_n > 0$ be a number satisfying $\epsilon_n < 2^{-n-16}n^{-6}$; because of the choice of (λ_{nk}) we have, in particular,

$$\epsilon_n < 2^{-n-16}n^{-4}\lambda_{nk}.$$

For all n, let e_n be an analytic function on the disc defined by

$$e_n(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log \varphi_n(\theta)\, d\theta\right),$$

where $\varphi_n : [0, 2\pi] \to \mathbb{R}^+$ is the measurable function

$$\varphi_n(\theta) := \begin{cases} 1 & \text{for } \theta \in J_n := [\theta_n - \epsilon_n, \theta_n + \epsilon_n] \\ \epsilon_n 2^{-m-4} & \text{for } \theta \in J_m, \ m \neq n \\ \epsilon_n & \text{for other } \theta. \end{cases}$$

In fact, by [Ru, Thm. 17.16], $e_n \in H^\infty$ and $|e_n^*(e^{i\theta})| = \varphi_n(\theta)$, where $e_n^*(e^{i\theta}) :=$ $\lim_{r \to 1} e_n(re^{i\theta})$ holds for a.e. $\theta \in [0, 2\pi]$. We also denote by e_n the restrictions of e_n to G_1.

For $n \in \mathbb{N}$, let

$$D_n := \{z \in G_1 \mid |z - e^{i\theta_n}| < 1/(50n^2)\}, \qquad C_n := G_1 \backslash D_n.$$

We have $D_n \subset \{z \in G_1 \mid z = re^{i\theta}, \ \theta \in I_n\}$ and, moreover, $|e^{i\theta} - z| > 1/(2^6 n^2)$ for $\theta \in J_n$ and $z \in C_n$. Since

$$\int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \, d\theta = 2\pi,$$

we can apply the Jensen inequality ([Ru, Thm. 3.3]; take exp for the convex function and $(1 - |z|^2)/(2\pi |e^{i\theta} - z|^2) \, d\theta$ for the probability measure) to derive, for $z \in C_n$,

$$\begin{aligned} |e_n(z)| &\leq \exp\left(\frac{1}{2\pi} \operatorname{Re} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \varphi_n(\theta) \, d\theta \right) \\ &= \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \log \varphi_n(\theta) \, d\theta \right) \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \varphi_n(\theta) \, d\theta \\ &\leq \frac{1}{2\pi} \int_{J_n} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \, d\theta + \frac{\epsilon_n}{2\pi} \int_{[0, 2\pi] \backslash J_n} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \, d\theta \\ &\leq \pi^{-1} \epsilon_n 2^{12} n^4 + \epsilon_n \leq 2^{-4-n}. \end{aligned}$$

Analogously, one can show that $|e_n(z)| \leq 1$ for all $z \in G_1$.

In the proof of our main result in this section we need the following technical lemma. It shows that given an arbitrary weight function $\bar{w}' \in \bar{W}$, we can choose a dominating weight function on G_1 that has some specific properties.

LEMMA 1. *Given a weight function $\bar{w}' \in \bar{W}$, we can find a weight function $\bar{w} \in \bar{W}$ with the following properties.*

(1°) *There exists $C > 0$ such that $C\bar{w}' \leq \bar{w} \leq 1$.*

(2°) *If $\bar{w}'' \in \bar{W}$ is defined as w_k except that λ_{nk} is replaced by $1/n^2$, we have $\bar{w}'' \leq \bar{w}$.*

(3°) *The weight $\bar{w}$ is constant on every D_n, so that*

$$\bar{\lambda}(n) : \mathbb{N} \to \mathbb{R}^+, \quad \bar{\lambda}(n) := \bar{w}(z), \ z \in D_n,$$

satisfies $1/n^2 \leq \bar{\lambda}(n)$ for all k and n and $\bar{\lambda} \in \bar{\Lambda}$.

Proof. Let $\rho(n) := \max\{1/n^2, \sup\{\bar{w}'(z) \mid \arg(z) \in I_n\}\}$ and define $\bar{w}^{(1)}$ as w_k but replace λ_{nk} by $\rho(n)$. Now it is easy to see that the weight $\bar{w}$, defined by

$$\bar{w}(z) = \min\{w_1(z), \max\{\bar{w}^{(1)}, \bar{w}'(z)\}\}$$

for $z \in G_1$, has all the desired properties; the property $(3°)$ follows from the facts that $\bar{w}^{(1)}$ and w_1 are constants on D_n and $\bar{w}^{(1)} \geq \bar{w}'$ on D_n. $\square$

Our next lemma is essentially known.

LEMMA 2.   *Let E and F be complete locally convex spaces. Let $\psi: E \to F$ and $\phi: F \to E$ be continuous linear maps such that $\phi\psi: E \to E$ satisfies the following condition: there exist a fundamental system of seminorms P on E and $0 < \delta < 1$ such that*

$$p((\phi\psi - \mathrm{id}_E)x) \leq \delta p(x) \quad \forall x \in E \; \forall p \in P.$$

Then E is isomorphic to a complemented subspace of F.

Proof. We put $B := \phi\psi - \mathrm{id}_E$ and define $A: E \to E$ by

$$Ax := \sum_{n=0}^{\infty} (-1)^n B^n x, \quad x \in E.$$

Then A is a well-defined continuous linear operator on E. Indeed, for $x \in E$, the series $\sum_{n=0}^{\infty}(-1)^n B^n x$ is absolutely summable in E and, for $p \in P$ and $x \in E$, we have

$$p\left(\sum_{n=0}^{\infty} (-1)^n B^n x\right) \leq \sum_{n=0}^{\infty} p(B^n x) \leq \sum_{n=0}^{\infty} \delta^n p(x) \leq C p(x).$$

Moreover $\phi(\psi A) = \mathrm{id}_E$. Indeed,

$$\phi(\psi A) = (\phi\psi)A = (\mathrm{id}_E + B) \sum_{n=0}^{\infty} (-1)^n B^n = \mathrm{id}_E.$$

This implies that $(\psi A)\phi$ is a projection on F whose image is isomorphic to E (see e.g. [Hor, pp. 122-3]). $\square$

THEOREM 3.   *The space $H\bar{W}(G_1)$ contains a complemented subspace isomorphic to the nonbornological space K_∞. In particular, $H\bar{W}(G_1)$ does not coincide topologically with the weighted inductive limit $\mathcal{W}H(G_1)$.*

Proof. We construct continuous linear maps $\psi: K_\infty \to H\bar{W}(G_1)$ and $\phi: H\bar{W}(G_1) \to K_\infty$ satisfying the assumptions of Lemma 2.

First define $\psi: K_\infty \to H\bar{W}(G_1)$ by $\psi(a) := \sum_{n=1}^{\infty} a_n e_n$ for $a = (a_n)_{n=1}^{\infty} \in K_\infty$. To see that ψ is well-defined and continuous, we fix $\bar{w}' \in \bar{W}$ and select $\bar{w} \in \bar{W}$ and $\bar{\lambda} \in \bar{\Lambda}$ as in Lemma 1. If $(a_n)_{n=1}^{\infty}$ is a sequence of scalars such that

$$\sup_n \bar{\lambda}(n)|a_n| = 1,$$

we have $|a_n| \leq n^2$ for all n. Every $z_0 \in G_1$ has a neighborhood U that intersects at most one of the sets D_n. It follows from the estimates of $|e_n(z)|$ established after the definition of e_n that $\sum a_n e_n(z)$ converges uniformly for

$z \in U$ and thus defines a holomorphic function on G_1. Moreover, denoting $D := \bigcap_{n=1}^{\infty} C_n$, by the choice of (a_n) we have

$$p_{\bar{w}}\left(\sum_{n=1}^{\infty} a_n e_n\right) \leq \sup_{z \in G_1} \bar{w}(z) \sum_{n=1}^{\infty} \bar{\lambda}(n)^{-1}|e_n(z)|$$

$$= \sup_{m \in \mathbb{N}} \sup_{z \in D_m} \left(\bar{w}(z) \sum_{\substack{n \in \mathbb{N} \\ n \neq m}} \bar{\lambda}(n)^{-1}|e_n(z)| + \bar{w}(z)\bar{\lambda}(m)^{-1}|e_m(z)|\right)$$

$$+ \sup_{z \in D} \bar{w}\left(z \sum_{n=1}^{\infty} \bar{\lambda}(n)^{-1}|e_n(z)|\right)$$

$$\leq \sup_{m \in \mathbb{N}}\left(\sum_{n=1}^{\infty} \bar{\lambda}(n)^{-1}2^{-4-n} + \bar{\lambda}(m)\bar{\lambda}(m)^{-1}\right)$$

$$+ \sum_{n=1}^{\infty} \bar{\lambda}(n)^{-1}2^{-4-n} \leq 3 = 3\sup_n \bar{\lambda}(n)|a_n|.$$

This shows that the map ψ is continuous.

To define $\phi\colon H\bar{W}(G_1) \to K_\infty$ we need radial values of elements of $H\bar{W}(G_1)$. We fix $f \in H\bar{W}(G_1)$. There is a $k \in \mathbb{N}$ such that $f \in Hw_k(G_1)$. Given $n \in \mathbb{N}$, since the weight w_k is constant in $\{re^{i\theta}\colon \theta \in I_n, 1/2 < r < 1\}$ it follows that $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ exists a.e. for $\theta \in J_n$ (see [Hof, pp. 34ff] or [Du, p. 170]). Accordingly, we define $f^*(e^{i\theta})$ a.e. $\theta \in J_n$, which is an element of $L^\infty(J_n)$. Observe that the radial limits of e_n in each J_n are the restriction of the ones of e_n in the disc, and that, if $(a_n) \in K_\infty$, it follows from the inequalities established in the first part of this proof that $(\sum a_n e_n)^*(e^{i\theta}) = \sum a_n e_n^*(e^{i\theta})$. We set $\chi_n(\theta) := \exp[-i \arg e_n^*(e^{i\theta})]$ and define $\phi\colon H\bar{W}(G_1) \to K_\infty$ by

$$\phi(f) := \left((2\epsilon_n)^{-1} \int_{J_n} f^*(e^{i\theta})\chi_n(\theta)\,d\theta\right)_{n \in \mathbb{N}}.$$

We first check that ϕ is well-defined and continuous. Given $\bar{\lambda} \in \bar{\Lambda}$ with $1/n^2 \leq \bar{\lambda}(n) \leq 1$ for all n, we define the weight $\bar{w} \in \bar{W}$ as w_k, but replacing λ_{nk} by $\bar{\lambda}(n)$. If $f \in H\bar{W}(G_1)$ and $n \in \mathbb{N}$ we have

$$\bar{\lambda}(n)(2\epsilon_n)^{-1}\left|\int_{J_n} f^*(e^{i\theta})\chi_n(\theta)\,d\theta\right| \leq \bar{\lambda}(n)\sup_{J_n}|f^*(e^{i\theta})| \leq \sup_{z \in G_1} \bar{w}(z)|f(z)|.$$

This shows $\phi(f) \in K_\infty$ and the continuity of ϕ.

It remains to demonstrate that $\phi\psi - \mathrm{id}_{K_\infty}$ satisfies the condition in Lemma 2. First observe that, for all $n \in \mathbb{N}$,

$$(2\epsilon_n)^{-1}\int_{J_n} e_n^*(e^{i\theta})\chi_n(\theta)\,d\theta = 1.$$

On the other hand, for $n \in \mathbb{N}$ fixed: $|e_j^*(e^{i\theta})| = \epsilon_j 2^{-n-4}$ for all $j \in \mathbb{N}$; $j \neq n$; a.e. $\theta \in J_n$; and the series $\sum a_j \epsilon_j$ converges absolutely for $(a_n) \in K_\infty$. Indeed, select $\bar{\lambda} \in \bar{\Lambda}$ with $1/n^2 \leq \bar{\lambda}(n) \leq C$ for all $n \in \mathbb{N}$. We have $S := \sup_j \bar{\lambda}(j)|a_j| < \infty$ and $\epsilon_j < 2^{-j-16}j^{-6} < 2^{-j-16}j^{-4}\bar{\lambda}(j)$, so $\sum |a_j|\epsilon_j < S\sum 2^{-j-16}$. In particular, for every $\bar{\lambda}$ as before,

$$\Sigma |a_j| \epsilon_j \le (1/8) \sup_n \bar{\lambda}(n) |a_n|.$$

Moreover,

$$(\phi\psi - \mathrm{id}_{K_\infty})(a_n) = \left((2\epsilon_n)^{-1} \sum_{j \ne n} a_j \int_{J_n} e_j^*(e^{i\theta}) \chi_n(\theta) \, d\theta \right)_{n \in \mathbb{N}}.$$

If $\bar{\lambda} \in \bar{\Lambda}$ satisfies $1/n^2 \le \bar{\lambda}(n) \le 1$ for all $n \in \mathbb{N}$, and if $a \in K_\infty$ is such that $p_{\bar{\lambda}}(a) = \sup_n \bar{\lambda}(n) |a_n| = 1$, then for $n \in \mathbb{N}$ we have

$$\left| \bar{\lambda}(n)(2\epsilon_n)^{-1} \sum_{j \ne n} a_j \int_{J_n} e_j^*(e^{i\theta}) \chi_n(\theta) \, d\theta \right|$$

$$\le \bar{\lambda}(n)(2\epsilon_n)^{-1} \sum_{j \ne n} |a_j| \int_{J_n} |e_j^*(e^{i\theta}) \chi_n(\theta)| \, d\theta$$

$$\le \bar{\lambda}(n) \sum_{j \ne n} |a_j| \sup_{\theta \in J_n} |e_j^*(e^{i\theta})| \le 2^{-n-4} \bar{\lambda}(n) \sum_{j \ne n} |a_j| \epsilon_j \le \frac{1}{128}.$$

Therefore

$$p_{\bar{\lambda}}((\phi\psi - \mathrm{id}_{K_\infty})(a)) \le (1/128) p_{\bar{\lambda}}(a).$$

Since the multiples of the weights $\bar{\lambda} \in \bar{\Lambda}$ with $1/n^2 \le \bar{\lambda}(n) \le 1$ for all $n \in \mathbb{N}$ form a fundamental system of seminorms of the space K_∞, the conclusion follows from Lemma 2. $\qquad\square$

3. The Subspace Problem

In this section we set $G = G_1 \times \mathbb{C} \subset \mathbb{C}^2$. We construct a decreasing sequence $V = (v_k)_{k=1}^\infty$ of weight functions on G such that $\mathcal{V}C(G) = C\bar{V}(G)$ holds topologically, but $\mathcal{V}H(G)$ is not a topological subspace of $\mathcal{V}C(G)$. Moreover, the projective hull $H\bar{V}(G)$ is not even a (DF) space.

If $z_1 \in G_1$, we write $d(z_1)$ to denote the distance of z_1 to the complement of G_1. We define the system $V = (v_k)_{k=1}^\infty$ of weight functions on G by

$$v_k(z_1, z_2) = w_k(z_1) u_k(z_1, |z_2|),$$

where $u_k: G_1 \times \mathbb{R}_0^+ \to \mathbb{R}^+$ is defined by

$$u_k(z_1, t) = \begin{cases} (1+t)^{-(k-1)/2k}, & t \ge k+1, \\ (1+1/d(z_1)+t)^{-(k-1)/2k}, & t \le k, \end{cases}$$

and, for each fixed z_1, $u_k(z_1, t)$ is extended affinely for $k < t < k+1$. It is easy to see that the functions v_k are continuous on G.

Bierstedt and Meise [BM2] introduced the following condition (M) on the sequence $V = (v_k)_{k=1}^\infty$: For each $k \in \mathbb{N}$ and each subset Y of G which is not relatively compact, there exists $k' = k'(k, Y) > k$ with $\inf_{y \in Y} v_{k'}(y)/v_k(y) = 0$. They proved that this condition is equivalent to the fact that $C\bar{V}(G)$ induces the compact open topology on each bounded subset and that condition (M) implies the topological identity $\mathcal{V}C(G) = C\bar{V}(G)$. Moreover, if V satisfies (M) then $H\bar{V}(G)$ is a Montel space. It was an open problem (see also [Bi]) whether $\mathcal{V}H(G)$ is a Montel space when V satisfies (M). In fact,

if $\mathcal{V}H(G)$ is a Montel space, then $\mathcal{V}H(G) = H\bar{V}(G)$ holds topologically by a direct application of the Baernstein open mapping lemma as in [BMS1].

PROPOSITION 4.   *The sequence $V = (v_k)_{k=1}^{\infty}$ satisfies condition* (M). *Consequently, $\mathcal{V}C(G) = C\bar{V}(G)$ holds algebraically and topologically, and $H\bar{V}(G)$ is a Montel space with metrizable bounded sets.*

Proof. Let $k \in \mathbb{N}$ be given, and let Y be a subset of G which is not relatively compact. We have two possibilities: either

(i) $\exists (z^{(m)}) = ((z_1^{(m)}, z_2^{(m)})) \subset Y$: $\sup_m |z_2^{(m)}| = \infty$; or

(ii) $\exists (z^{(m)}) = ((z_1^{(m)}, z_2^{(m)})) \subset Y$: $|z_2^{(m)}| \le M$ and $\inf_m d(z_1^{(m)}) = 0$.

In the first case it follows easily from the definition of u_k that, taking $k' = k+1$,

$$\sup_{m \in \mathbb{N}} \frac{v_k(z^{(m)})}{v_{k+1}(z^{(m)})} \ge \sup_{m \in \mathbb{N}} \frac{u_k(z_1^{(m)}, |z_2^{(m)}|)}{u_{k+1}(z_1^{(m)}, |z_2^{(m)}|)}$$

$$\ge \sup_{\substack{m \in \mathbb{N} \\ |z_2^{(m)}| > k+2}} |z_2^{(m)}|^{-(k-1)/2k + k/2(k+1)} = \infty.$$

In case (ii), we choose $k' > M$ and $k' > k+1$ to obtain

$$\sup_{m \in \mathbb{N}} \frac{v_k(z^{(m)})}{v_{k'}(z^{(m)})} \ge \sup_{m \in \mathbb{N}} \left(\frac{1}{d(z_1^{(m)})} + 1 + M \right)^{-(k-1)/2k} \cdot \left(\frac{1}{d(z_1^{(m)})} \right)^{k/2(k+1)} = \infty.   \quad \square$$

It is very easy to see that every $f \in H\bar{V}(G)$ is constant with respect to the second variable. Indeed, if $f \in H\bar{V}(G)$ then there are $k \in \mathbb{N}$ and $C > 0$ such that $p_{v_k}(f) \le C$, so that for every fixed $z_1 \in G_1$ we have

$$C \ge \sup_{z_2 \in \mathbb{C}} w_k(z_1) u_k(z_1, |z_2|) |f(z_1, z_2)|$$

$$\ge w_k(z_1) \sup_{z_2 \in \mathbb{C}} \left(1 + \frac{1}{d(z_1)} + |z_2| \right)^{-1/2} |f(z_1, z_2)|.$$

Now it is an elementary fact of complex analysis that a holomorphic g: $\mathbb{C} \to \mathbb{C}$, satisfying $\sup\{(|z| + c_0)^{-1/2} |g(z)| \,|\, z \in \mathbb{C}\} < \infty$ for some constant c_0, must be constant. Accordingly, f must be constant with respect to z_2. Now we define A: $H\bar{V}(G) \to H\bar{W}(G_1)$ by $Af(z_1) = f(z_1, 0)$. To show that A is well-defined, we observe that

$$p_{w_k}(Af) := \sup_{z \in G_1} w_k(z) |f(z, 0)| \le \sup_{z \in G_1} w_k(z) C_k u_k(z, k+1) |f(z, 0)|$$

$$\le C_k p_{v_k}(f)$$

for all $f \in Hv_k(G)$ and $C_k := (k+2)^{(k-1)/(2k)}$.

Given $g \in H\bar{W}(G_1)$, we define $\bar{g}$: $G \to \mathbb{C}$ by $\bar{g}(z_1, z_2) = g(z_1)$ for all $(z_1, z_2) \in G$. To show that $\bar{g} \in H\bar{V}(G)$, we fix $k \in \mathbb{N}$ with $g \in Hw_k(G_1)$. We then have the estimate

$$\sup_{(z_1, z_2) \in G} v_k(z_1, z_2) |\bar{g}(z_1, z_2)| \le \sup_{z_1 \in G_1} w_k(z_1) |g(z_1)|,$$

since $0 \leq u_k \leq 1$. This shows that A is bijective and $A^{-1}: Hw_k(G_1) \to Hv_k(G)$ is continuous for every $k \in \mathbb{N}$. By the closed graph theorem for (LB)-spaces, this also yields that $A: \mathcal{V}H(G) \to \mathcal{W}H(G_1)$ is a topological isomorphism. Moreover, $A^{-1}: H\overline{W}(G_1) \to H\overline{V}(G)$ is continuous. Indeed, if $\overline{v} \in \overline{V}$ is given then we define $\overline{w}(z_1) = \sup_{z_2 \in \mathbb{C}} \overline{v}(z_1, z_2)$ for $z_1 \in G_1$. For all k, we have

$$\overline{w}(z_1) = \sup_{z_2 \in \mathbb{C}} \overline{v}(z_1, z_2) \leq \sup_{z_2 \in \mathbb{C}} C_k v_k(z_1, z_2) \leq C_k w_k(z_1),$$

hence $\overline{w} \in \overline{W}$. Moreover,

$$p_{\overline{v}}(f) = \sup_{(z_1, z_2) \in G} \overline{v}(z_1, z_2) |f(z_1, 0)|$$

$$\leq \sup_{z_1 \in G_1} \{ |f(z_1, 0)| \sup_{z_2 \in \mathbb{C}} \{ \overline{v}(z_1, z_2) \} \} = p_{\overline{w}}(Af).$$

On the other hand, $A: H\overline{V}(G) \to H\overline{W}(G_1)$ is not continuous. In fact, $H\overline{W}(G_1)$ and $H\overline{V}(G)$ cannot be isomorphic, since the former contains a complemented subspace isomorphic to K_∞ (by Theorem 3) and hence contains bounded sets that are not metrizable; whereas every bounded subset of $H\overline{V}(G)$ is metrizable, by Proposition 4.

THEOREM 5. *The space* $H\overline{V}(G)$ *is not bornological;* $\mathcal{V}H(G) = H\overline{V}(G)$ *does not hold topologically; and* $\mathcal{V}H(G)$ *is not a topological subspace of* $\mathcal{V}C(G)$ *and is not a Montel space. Moreover,* $H\overline{V}(G)$ *is not a (DF)-space.*

Proof. If $H\overline{V}(G)$ is bornological (or, equivalently, if $\mathcal{V}H(G) = H\overline{V}(G)$ holds topologically), then the linear map $A^{-1}: H\overline{W}(G_1) \to \mathcal{V}H(G)$ is continuous. Consequently the identity $\text{id} = AA^{-1}: H\overline{W}(G_1) \to \mathcal{W}H(G_1)$ is continuous. This implies $H\overline{W}(G_1)$ is bornological, which contradicts Theorem 3.

By Proposition 4, $\mathcal{V}C(G) = C\overline{V}(G)$ holds topologically. Since $H\overline{V}(G)$ is clearly a topological subspace of $C\overline{V}(G)$, we conclude that $\mathcal{V}H(G)$ is not a topological subspace of $\mathcal{V}C(G)$. If $\mathcal{V}H(G)$ were Montel, we could apply directly the Baernstein open mapping lemma (see e.g. [PB, 8.6.8(5)]) to conclude that $\mathcal{V}H(G)$ would be a topological subspace of $\mathcal{V}C(G)$; a contradiction.

Finally, assume $H\overline{V}(G)$ is a (DF)-space. Since it is a complete (DF) space which is Montel, $H\overline{V}(G)$ is bornological (cf. [PB, 8.3.48]). This is a contradiction. □

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