

Cores of Ideals in 2-Dimensional Regular Local Rings

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1. Introduction

The main result of this paper is the explicit determination of the core of integrally closed ideals in 2-dimensional regular local rings. The core of an ideal I in a ring R was introduced by Judith Sally in the late 1980s and was alluded to in Rees and Sally's paper [RS]. Recall that a reduction of I is any ideal J for which there exists an integer n such that $JJ^n = I^{n+1}$ [NR]. In other words, J is a reduction of I if and only if I is integrally dependent on J . An ideal is integrally closed if it is not a reduction of any ideal properly containing it.

(1.1) DEFINITION. The core of an ideal I , denoted $\text{core}(I)$, is the intersection of all reductions of I .

In general, the core seems extremely difficult to determine and there are few computed examples. A priori, it is not clear whether it is zero. However, one can show that, in general, the core always contains a power of I . A proof of this for Buchsbaum rings can be found in [Tr, Prop. 5.1].

It is quite natural to study the core, partly due to the theorem of Briançon and Skoda (see [BS; LS; LT; L4; HH; RS; Sa; AH1; AH2; AHT]). A simple version of this theorem states that if R is a d -dimensional regular ring and I is any ideal of R , then the integral closure of I^d is contained in I . In particular, the integral closure of I^d is contained in $\text{core}(I)$. It is an important question to understand how the core of I relates to I . More generally, we are interested in approximating general m -primary ideals in local rings (R, m) by intersections of parameter ideals. We hope our results in dimension 2 will provide insight into the nature of the core in higher dimensions.

Some of the open questions regarding the core are as follows.

- (a) If I is integrally closed, is $\text{core}(I)$ also integrally closed?
- (b) If the completion \hat{R} of R is equidimensional, does $\text{core}(I)\hat{R}$ equal $\text{core}(\hat{I})$? More generally, how does the core behave under faithfully flat maps?

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- (c) How does $\text{core}(I^n)$ compare to $\text{core}(I)$?
 (d) If I and J are integrally closed ideals and $I \subseteq J$, is $\text{core}(I) \subseteq \text{core}(J)$?

We will assume throughout that R is a 2-dimensional regular local ring with maximal ideal m . An ideal I in such a ring equals xJ for some $x \in R$ and some ideal J which is either R or m -primary. Let S be the set of all reductions of J . Since every reduction of I is of the form xK for some reduction K of J ,

$$\text{core}(I) = \bigcap_{K \in S} xK = x \bigcap_{K \in S} K = x \cdot \text{core}(J).$$

Therefore, to compute the core of ideals in 2-dimensional regular local rings, we need only consider m -primary ideals.

We will also assume throughout that the residue field of R is infinite. This assumption serves two purposes: it allows us to choose good quadratic transformations (see Section 2), and it ensures that we have an abundant set of reductions of every ideal (see [NR, Thm. 1, p. 153]).

We determine in Section 3 the core of an integrally closed ideal I in terms of ideals of minors of any presentation of I . We prove that $\text{core}(I) = I \cdot \text{adj}(I)$, where $\text{adj}(I)$ is the adjoint of I as defined by Lipman in [L3]. We prove the equivalent statement that $\text{core}(I)$ is $\mathfrak{F}_1(I)\mathfrak{F}_2(I)$, where $\mathfrak{F}_i(I)$ is the i th Fitting ideal of I . We also answer several of the questions above. For example, in regular local rings of dimension 2 with an infinite residue field, we prove that if $J \subseteq I$ are integrally closed ideals then $\text{core}(J) \subseteq \text{core}(I)$. We also prove (again for 2-dimensional regular local rings) that the core of an integrally closed ideal is integrally closed.

In Section 4 we discuss the various relationships between the core of an ideal and the adjoint of an ideal. More specifically, we prove that $\text{adj}(I^n) = I^{n-1}\text{adj}(I)$, $\text{core}(I^n) = I^{2n-2}\text{core}(I)$, $\text{adj}(\text{core}(I)) = (\text{adj}(I))^2$, and $\text{core}(\text{core}(I)) = I(\text{adj}(I))^3$. Lipman also proved in [L4] that $\text{adj}(I^n) = I^{n-1}\text{adj}(I)$. Lipman's proof uses duality and a vanishing theorem; we give an elementary proof. For higher dimensions, Lipman conjectures that $\text{adj}(I^n) = I\text{adj}(I^{n-1})$ for all n larger than the analytic spread of I .

Section 2 contains the needed background about 2-dimensional regular local rings.

2. 2-Dimensional Regular Local Rings

Throughout, (R, m, k) will be a 2-dimensional regular local ring with maximal ideal m and an infinite residue field k . Two-dimensional regular local rings have many special properties which we will use heavily. We list these properties and the needed definitions. For an introductory treatment see [Hu], and for comprehensive treatments and related results see [Za; ZS, vol. 2, apx. 5; Ko; L1; L2; L3; Cu; Gö; HS; No; Re; Ve].

For any matrix A and any integer n , we will denote the ideal generated by the $n \times n$ minors of A by $I_n(A)$. For any nonzero ideal I , the *order* of I ,

denoted $\text{ord}(I)$, is the largest integer n such that $I \subseteq m^n$. Since the associated graded ring is a polynomial ring, ord is a valuation. The order of a reduction of I is the same as the order of I . The smallest number of elements generating an ideal I will be denoted as $\mu(I)$.

The multiplicity of an m -primary ideal I in an arbitrary Noetherian local ring (R, m) of dimension d is defined as follows: There exists a polynomial $p(n)$ of degree d such that, for large n , $p(n) = \text{length}(R/I^n)$; the multiplicity of I is then defined to be the coefficient of $n^d/d!$ in $p(n)$.

We now recall some facts about 2-dimensional regular local rings.

(2.1) (Hilbert-Burch) For every m -primary ideal I there is a free resolution

$$0 \rightarrow R^{n-1} \xrightarrow{A} R^n \xrightarrow{(x_1, \dots, x_n)} R \rightarrow R/I \rightarrow 0,$$

where A is an $n \times (n-1)$ matrix such that the maximal minors of A generate I . We will refer to A as a *presenting matrix* of I . In fact, if $I = (x_1, \dots, x_n)$, we may assume that the $(n-1) \times (n-1)$ minor of A obtained after deleting the i th row equals x_i . (Here, n needn't be the minimal number of generators of I .)

(2.2) For any $x \in m \setminus m^2$ and any height-2 prime ideal P in $R[\frac{m}{x}]$, the localization $R[\frac{m}{x}]_P$ is a regular local ring. We call any ring of this form a *quadratic transformation* of R .

If $S = R[\frac{m}{x}]$, $x \notin m^2$, and I is an ideal of R , then $IS = x^{\text{ord}(I)}I'$ for some ideal I' in S of height 2. The images of I' in the quadratic transformations of R are called the *quadratic transforms* of I .

If J is a reduction of I —that is, if there exists an integer n such that $JI^n = I^{n+1}$ —then the same equality still holds after extending ideals to $R[\frac{m}{x}]$. As $\text{ord}(I) = \text{ord}(J)$, the same equality still holds for the transforms of J and I . This implies that the transform of a reduction of an ideal is a reduction of the transform of the ideal.

For any matrix A with elements in R , $\frac{A}{x}$ denotes the matrix obtained by dividing every element of A by x . If I is an ideal of order $n-1$ and with n generators, and if A is an $n \times (n-1)$ presenting matrix of I , then $\frac{A}{x}$ is a not necessarily minimal presenting matrix of the transform I' of I , as we now explain. Let $I = (x_1, \dots, x_n)$ with A as in (2.1). Then consider the complex

$$0 \rightarrow S^{n-1} \xrightarrow{\frac{A}{x}} S^n \xrightarrow{\left(\frac{x_1}{x^{n-1}}, \dots, \frac{x_n}{x^{n-1}}\right)} S \rightarrow S/I' \rightarrow 0.$$

Notice that $I_{n-1}(\frac{A}{x}) = I'$ since the $(n-1)$ minor of $\frac{A}{x}$ deleting the i th row is exactly $\frac{x_i}{x^{n-1}}$. Then by the Buchsbaum–Eisenbud criterion for exactness [BE], $\frac{A}{x}$ is a (possibly nonminimal) presentation of I' .

(2.3) An ideal I is contracted from $R[\frac{m}{x}]$ for some $x \in m \setminus m^2$ if and only if $\text{ord}(I) = \mu(I) - 1$ (see [Hu, Prop. 2.3]). Any x chosen such that its leading form in the associated graded ring $\text{gr}_m(R)$ does not divide the leading term of the image of some element of I of degree $\text{ord}(I)$ will suffice (see [Hu, proof

of Prop. 2.3]). Because the residue field is infinite, we can always choose such an x for a finite set of ideals at a time. Thus the appropriate quadratic transformations always exist; in the rest of the paper we ignore the issue of existence.

For any m -primary ideal I of R , the multiplicity of I is strictly larger than the multiplicity of the transform I' of I in any localization of $R[\frac{m}{x}]$ if x is chosen such that I is contracted from $R[\frac{m}{x}]$ (see [Hu, Prop. 3.6; Ko, p. 33]). It will be implicit throughout that our transformations are chosen carefully.

After a finite number of quadratic transformations, an integrally closed m -primary ideal transforms to a power of the maximal ideal.

(2.4) We will need to use several more properties of integrally closed ideals.

- (a) Let I be integrally closed and m -primary. Then $\text{ord}(I) = \mu(I) - 1$ (see [Hu, Props. 2.3 & 3.1]). Thus every integrally closed m -primary ideal in R contracts from some $R[\frac{m}{x}]$. (This is false for an arbitrary m -primary ideal.)
- (b) The product of two integrally closed ideals in R is integrally closed (see [Za, Part II, 12]).
- (c) For any reduction (x_1, x_2) of an integrally closed m -primary ideal I , $(x_1, x_2)I = I^2$ (see [LT, Cor. 5.4; Hu, Thm. 5.1; HS, Thm. 3.1]).
- (d) The quadratic transform of an integrally closed m -primary ideal is integrally closed, and is either the whole ring or has height 2 [Hu].

(2.5) More generally, let R be a 2-dimensional regular local ring and let $R \subseteq S$, where S is a 2-dimensional regular local ring between R and its field of fractions. If I is an ideal of R , we define $I^S := (\text{gcd}(IS))^{-1}IS$, the S -transform of I . This will be a height-2 ideal of S (or possibly S itself). The *point basis* of I is the family of integers $\{\text{ord}_S(I^S)\}_{R \subseteq S}$, where S runs over all the 2-dimensional regular local rings containing R and contained in its fraction field. There are only finitely many S for which $\text{ord}_S(I^S)$ is nonzero; these S are called the *base points* of I (see [L3, p. 225]). Two m -primary ideals of R have the same point basis if and only if their integral closures are equal [L2, p. 209, (1.10)]. It will follow from our theorems that the core of an integrally closed ideal I is exactly the integrally closed ideal whose point basis is $\{\max\{0, 2 \text{ord}_S(I^S) - 1\}\}_{R \subseteq S}$.

3. The Core

We analyze a free resolution of an m -primary ideal $I = (x_1, \dots, x_n)$ more carefully:

$$\mathbb{F}: 0 \longrightarrow R^{n-1} \xrightarrow{A} R^n \xrightarrow{(x_1, \dots, x_n)} R \longrightarrow R/I \longrightarrow 0, \quad (1)$$

where A is an $n \times (n-1)$ matrix. The columns of A generate all the relations on x_1, \dots, x_n . Assume that (x_1, x_2) is a reduction of I and let B be the submatrix of A consisting of the last $n-2$ rows.

Recall that for ideals I and J in a ring R , $I:J$ denotes the ideal $\{r \in R \mid rJ \subseteq I\}$. With this we state a result from linkage theory.

(3.1) LEMMA. *With notation as above, $(x_1, x_2):I = I_{n-2}(B)$.*

Proof. Let \mathbb{K} be the Koszul complex

$$\mathbb{K}: 0 \longrightarrow R \xrightarrow{\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}} R^2 \xrightarrow{(x_1, x_2)} R \longrightarrow R/(x_1, x_2) \longrightarrow 0.$$

There is an obvious complex homomorphism $\lambda: \mathbb{K} \rightarrow \mathbb{F}$ with λ_0 being the identity on R and λ_1 taking (r_1, r_2) to $(r_1, r_2, 0, \dots, 0)$. Let \mathbb{M} be the mapping cone of the dual of $\lambda: \mathbb{K} \rightarrow \mathbb{F}$. It is easy to see that \mathbb{M} is acyclic (see [PS, Prop. 2.6]), that $H_0(\mathbb{M}) = R/((x_1, x_2):I)$, and that the presenting matrix of $H_0(\mathbb{M})$ is B^{tr} (the transpose of B). In particular, B^{tr} is injective and by the Hilbert–Burch theorem, $(x_1, x_2):I = I_{n-2}(B)$. \square

(3.2) LEMMA. *Let I be a contracted ideal. Then*

$$\mu(I_{n-2}(B)) - 1 = \text{ord}(I_{n-2}(B)) = n - 2;$$

that is, $I_{n-2}(B)$ is contracted from a quadratic transformation.

Proof. The elementary column operations on A do not change $I_i(A)$ or $I_i(B)$ for any integer i . The same is true for the elementary row operations on rows of A which do not modify the rows of B by subtracting multiples of the first two rows of A .

Suppose that some entry a_{ij} in A is a unit. This means that x_i is contained in the ideal $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Thus, as x_1, x_2 are part of a minimal generating set of I , any a_{1j} or a_{2j} being a unit would imply that, for some $i \geq 3$, a_{ij} is also a unit. Thus, if some a_{ij} is a unit, we may assume that $i \geq 3$. Then by elementary row and column operations as prescribed above, and by possibly changing the element x_i , we may without loss of generality assume that all the other entries of A in the i th row or in the j th column are zero. Then the matrix A' obtained from A by deleting the i th row and the j th column is still a presenting matrix of I , but we are leaving out the redundant i th generator of I . Note that for all integers i , $I_i(A) = I_{i-1}(A')$ and $I_i(B) = I_{i-1}(B')$, where B' is the submatrix of A' consisting of all but the first two rows.

Thus we may assume that $n = \mu(I)$. Since all the entries of A lie in m , $\text{ord}(I_{n-2}(B)) \geq n - 2$. If $\text{ord}(I_{n-2}(B)) \geq n - 1$, then $I_{n-2}(B) \subseteq m^{n-1}$. By expanding x_1 (resp. x_2) (as determinants) along the second (resp. first) row, we see that $(x_1, x_2) \subseteq mI_{n-2}(B) \subseteq m^n$. Thus (now we use that I is contracted)

$$n - 1 = \mu(I) - 1 = \text{ord}(I) = \text{ord}((x_1, x_2)) \geq n,$$

which is a contradiction. So $\text{ord}(I_{n-2}(B)) = n - 2$.

Furthermore, B^{tr} presents $I_{n-2}(B)$ and every entry of B is in the maximal ideal, so by the Hilbert–Burch theorem $\mu(I_{n-2}(B)) = n - 1$. \square

Thus, if x is such that $I_{n-2}(B)$ is contracted from $R\left[\frac{m}{x}\right]$, then this lemma says that

$$I_{n-2}(B) = I_{n-2}(B)R\left[\frac{m}{x}\right] \cap R = x^{n-2} \cdot I_{n-2}\left(\frac{B}{x}\right)R\left[\frac{m}{x}\right] \cap R.$$

Furthermore, we have the following proposition.

(3.3) PROPOSITION. *Let I be an integrally closed m -primary ideal and let A and B be as in (1). Then the m -primary ideal $I_{n-2}(B)$ equals $I_{n-2}(A)$ and is integrally closed.*

Proof. By the elementary row and column operations we may assume without loss of generality that $n = \mu(I)$ (see the discussion in the proof of Lemma 3.2). We know that $I_{n-2}(B)$ is contracted from some $R\left[\frac{m}{x}\right]$. As $\frac{A}{x}$ presents the integrally closed ideal

$$\frac{I}{x^{\text{ord}(I)}} \text{ in } R\left[\frac{m}{x}\right] \text{ and since } \left(\frac{x_1}{x^{\text{ord}(I)}}, \frac{x_2}{x^{\text{ord}(I)}}\right)$$

is a reduction of I' , by induction on the multiplicity of I (cf. (2.3)),

$$I_{n-2}\left(\frac{A}{x}\right)R\left[\frac{m}{x}\right]_P = I_{n-2}\left(\frac{B}{x}\right)R\left[\frac{m}{x}\right]_P$$

is integrally closed for any height-2 prime ideal P in $R\left[\frac{m}{x}\right]$. Now note that the order of $I_{n-2}(A)$ equals $\text{ord}(I_{n-2}(B)) = n - 2$. Hence $I_{n-2}\left(\frac{A}{x}\right)$ (resp. $I_{n-2}\left(\frac{B}{x}\right)$) is the transform of the m -primary ideal $I_{n-2}(A)$ (resp. $I_{n-2}(B)$). It follows that $I_{n-2}\left(\frac{A}{x}\right)$ and $I_{n-2}\left(\frac{B}{x}\right)$ have height at least 2 and so the above means that $I_{n-2}\left(\frac{A}{x}\right)$ equals $I_{n-2}\left(\frac{B}{x}\right)$ and that this ideal is integrally closed in $R\left[\frac{m}{x}\right]$. Thus

$$\begin{aligned} I_{n-2}(A) &\subseteq x^{n-2}I_{n-2}\left(\frac{A}{x}\right)R\left[\frac{m}{x}\right] \cap R \\ &= x^{n-2}I_{n-2}\left(\frac{B}{x}\right)R\left[\frac{m}{x}\right] \cap R \\ &= I_{n-2}(B) \\ &\subseteq I_{n-2}(A), \end{aligned}$$

and equality holds throughout. Moreover, these containments show that $I_{n-2}(A)$ is contracted from the integrally closed ideal $x^{n-2}I_{n-2}\left(\frac{B}{x}\right)R\left[\frac{m}{x}\right]$, hence $I_{n-2}(A)$ is integrally closed. \square

We record here the following corollary.

(3.4) COROLLARY. *Let I be an integrally closed m -primary ideal and let A and B be as in (1). Then $I_j(A)$ equals $I_j(B)$ and is integrally closed for all $j = 0, \dots, n - 2$.*

Proof. If $n = 2$ then there is nothing to show, so assume that $n > 2$. By Proposition 3.3, $I_{n-2}(B) = I_{n-2}(A)$ is integrally closed. As B^{tr} presents $I_{n-2}(B)$,

induction on n shows that $I_j(B) = I_j(B^{\text{tr}})$ is integrally closed for all $j = 0, \dots, n - 3$. By induction on multiplicity of ideals (cf. (2.3)),

$$\begin{aligned} I_j(A) &\subseteq x^j \cdot I_j\left(\frac{A}{x}\right)R\left[\frac{m}{x}\right] \cap R \\ &= x^j \cdot I_j\left(\frac{B}{x}\right)R\left[\frac{m}{x}\right] \cap R \\ &= I_j(B) \\ &\subseteq I_j(A). \end{aligned} \quad \square$$

Proposition 3.3, together with Lemma 3.1, gives the next proposition.

(3.5) PROPOSITION. *For an m -primary integrally closed ideal I and any reduction (x_1, x_2) of I , $(x_1, x_2) : I = I_{n-2}(A)$ for any $n \times (n - 1)$ presenting matrix A of I .* □

There are other ideals for which the results above also hold. For example, let I be (y^2, x^4, yx^3) in the power series ring in two variables x and y over a field. I has order 2 and is generated by three elements. It is easy to see that (y^2, x^4) is a reduction of I . A presenting matrix of I is

$$A = \begin{pmatrix} x^3 & 0 \\ 0 & -y \\ -y & x \end{pmatrix}.$$

The matrix B is just the last row of A . Note that $I_1(A) = (x, y) = I_1(B)$ but that I is not integrally closed, as x^2y is in the integral closure but not in I .

We now prove several corollaries for integrally closed ideals. For some statements we need only the weaker hypothesis that for a finite number of reductions (x_1, x_2) of I , the corresponding matrices B satisfy $I_{n-2}(B) = I_{n-2}(A)$. However, in order to avoid messy hypotheses, we state the results only for integrally closed ideals.

(3.6) COROLLARY. *For an integrally closed ideal I ,*

$$I \cdot I_{n-2}(B) = (x_1, x_2)I_{n-2}(B) = (x_1, x_2)I_{n-2}(A)$$

for any reduction (x_1, x_2) of I , where A and B are defined as in (1). In particular, $I_{n-2}(A)I$ is contained in $\text{core}(I)$.

Proof. We have to show the first equality and in particular that the left-hand side is contained in the right-hand side. Note that $I \cdot I_{n-2}(B) = I \cdot ((x_1, x_2) : I)$ is contained in (x_1, x_2) . In fact, when the matrices A and B are replaced by generic matrices A^* and B^* , $I^* \cdot I_{n-2}(B^*) \subseteq (x_1^*, x_2^*)I_{n-2}(A^*)$ where I^* is the ideal of $(n - 1)$ minors and x_1^*, x_2^* are $(n - 1)$ -size minors, as in (1). (This can be seen easily by using $(x_1^*, \dots, x_n^*)A^* = 0$ and summing the rows of this equation times suitable $(n - 3)$ -size minors of B^* , or alternatively by using the straightening relations on the minors.) Thus $I \cdot I_{n-2}(B) \subseteq (x_1, x_2) \cdot I_{n-2}(A)$.

But in our case this equals the right-hand side. The last claim follows as any reduction contains a reduction generated by two elements (see [NR, Thm. 1, p. 153]). \square

(3.7) COROLLARY. *Let I be integrally closed and let (x_1, x_2) be a reduction of I . Then $(x_1^2, x_2^2) : I^2 = I_{n-2}(B)(x_1, x_2)$.*

Proof. We use Corollary 3.6. Because

$$I_{n-2}(B)(x_1, x_2)I^2 = I_{n-2}(B)(x_1, x_2)^3 \subseteq (x_1^2, x_2^2),$$

the right-hand side is contained in the left-hand side. Now let $a \in (x_1^2, x_2^2) : I^2$. In particular, $ax_1x_2 \in (x_1^2, x_2^2)$, so $a \in (x_1, x_2)$. Write $a = a_1x_1 + a_2x_2$. Then $ax_2I \subseteq (x_1^2, x_2^2)$ implies that $a_1x_1x_2I \subseteq (x_1^2, x_2^2)$, so $a_1 \in (x_1^2, x_2^2) : x_1x_2I \subseteq (x_1, x_2) : I = I_{n-2}(B)$. Similarly, $a_2 \in I_{n-2}(B)$. Hence a is contained in the ideal on the right. \square

In what follows, we prove that $I_{n-2}(A)I$ is the core of I when I is integrally closed.

(3.8) LEMMA. *Let I be an integrally closed m -primary ideal in a 2-dimensional regular local ring (R, m) with infinite residue field. Let x_1 be part of a minimal reduction of I ; that is, there exists x_2 (actually, infinitely many by [NR, Thm. 1, p. 153]) such that (x_1, x_2) is a reduction of I . Then*

$$\bigcap_{x_2} (x_1, x_2) = (x_1) + I_{n-2}(A)I$$

for any $n \times (n-1)$ presenting matrix A of I , where x_2 varies over all elements such that (x_1, x_2) is a reduction of I .

Proof. The right-hand side is contained in the left-hand side, by Corollary 3.6. It is easy to see that, in general, if $J_1 \subseteq J_2$ are m -primary ideals then to prove the equality $J_1 = J_2$ it is necessary and sufficient to prove the equality $J_1 = J_2 \cap (J_1 : m)$. Thus, to prove the lemma let α be arbitrary in the intersection of $\bigcap_{x_2} (x_1, x_2)$ and $((x_1) + I_{n-2}(A)I) : m$. We will show that α is contained in $(x_1) + I_{n-2}(A)I$.

Fix one x_2 and write $\alpha = rx_1 + sx_2$. Note that, for any $x \in m$, $xsx_2 = x\alpha - xr x_1 \in (x_1) + I_{n-2}(A)I = (x_1) + I_{n-2}(A)x_2$. The last equality follows from Corollary 3.6. Because x_1, x_2 form a regular sequence, $xs \in I_{n-2}(A) + (x_1) = I_{n-2}(A)$. Since this holds for all $x \in m$, $s \in I_{n-2}(A) : m$. We will show that s is contained in $I_{n-2}(A)$, which will finish the proof of Lemma 3.8.

Suppose that s is not in $I_{n-2}(A) = (x_1, x_2) : I$. Then there exists an element x_3 of I such that (x_1, x_3) is a minimal reduction of I and such that $s \notin (x_1, x_2) : x_3$. This also follows by [NR, Thm. 1, p. 153]: The set of all elements x_3 in I such that (x_1, x_3) is a reduction of I generates all of I and is Zariski-open in I/mI ; the condition $s \notin (x_1, x_2) : x_3$ means we must also avoid the proper subideal $((x_1, x_2) : s) \cap I$ in I . It also follows by [NR] that there

exist infinitely many units u such that $(x_1, x_2 + ux_3)$ is a reduction of I . For each such u write $\alpha = r_u x_1 + s_u(x_2 + ux_3)$.

For any $x \in m$,

$$xs_u(x_2 + ux_3) = x\alpha - xr_u x_1 \in I_{n-2}(A)I + (x_1) = I_{n-2}(A)(x_2 + ux_3) + (x_1).$$

As $x_1, x_2 + ux_3$ is a regular sequence, xs_u is contained in $I_{n-2}(A)$ for all u . Since this holds for all $x \in m$ and all allowable u , the s_u generate a R/m vector space in $R/I_{n-2}(A)$. Let u_1, \dots, u_l be such that the images of s_{u_1}, \dots, s_{u_l} form a basis of this vector space. Now choose a u different from all the u_i ; we can do this because we have infinitely many units at our disposal. Then there exist $l_i \in R$ such that $us_u - \sum l_i u_i s_{u_i} \in I_{n-2}(A)$.

First suppose that $s_u - \sum l_i s_{u_i}$ is contained in $I_{n-2}(A)$. Then in the vector space $(I_{n-2}(A) : m) / I_{n-2}(A)$ (the images there marked with an overbar $\bar{}$) we get $\overline{us_u} = \sum \overline{l_i u_i s_{u_i}}$ and $\overline{s_u} = \sum \overline{l_i s_{u_i}}$ which is only possible if all the $\overline{l_i}$ are zero and hence only if $\overline{s_u} = 0$. But then $\alpha = r_u x_1 + s_u(x_2 + ux_3)$ is contained in $(x_1) + I_{n-2}(A)I$, which proves the lemma in the case $s_u - \sum l_i s_{u_i} \in I_{n-2}(A)$.

Next suppose that $s_u - \sum l_i s_{u_i} \notin I_{n-2}(A)$. Let $l = \sum l_i - 1$. Then

$$\begin{aligned} lrx_1 + lsx_2 &= l\alpha \\ &= \sum l_i \alpha - \alpha \\ &= \sum l_i (r_{u_i} x_1 + s_{u_i} (x_2 + u_i x_3)) - (r_u x_1 + s_u (x_2 + ux_3)) \\ &= \sum l_i r_{u_i} x_1 - r_u x_1 + \sum l_i s_{u_i} x_2 - s_u x_2 + (\sum l_i u_i s_{u_i} - s_u u) x_3. \end{aligned}$$

Since $\sum l_i u_i s_{u_i} - s_u u$ is in $I_{n-2}(A)$, we may write

$$(\sum l_i u_i s_{u_i} - s_u u) x_3 = t_1 x_1 + t_2 x_2 \quad \text{for some } t_i \in I_{n-2}(A).$$

Thus

$$lrx_1 + lsx_2 = \sum l_i r_{u_i} x_1 - r_u x_1 + \sum l_i s_{u_i} x_2 - s_u x_2 + t_1 x_1 + t_2 x_2.$$

It follows that

$$x_1(lr - \sum l_i r_{u_i} + r_u - t_1) = x_2(\sum l_i s_{u_i} - s_u - ls + t_2)$$

and hence that $\sum l_i s_{u_i} - s_u - ls + t_2 \in (x_1)$. This forces l to be a unit, for otherwise ls is contained in $I_{n-2}(A)$ and hence $\sum l_i s_{u_i} - s_u \in I_{n-2}(A)$, contradicting the hypotheses.

Thus l is a unit and the above says that

$$s \in (x_1, t_2) + (\{s_u, s_{u_i}\}) \subseteq I_{n-2}(A) + (\{s_u, s_{u_i}\}).$$

As $s_u, s_{u_i} \in (x_1, x_2) : x_3$, this proves that $s \in (x_1, x_2) : x_3$, which contradicts the choice of x_3 and finishes the proof of the lemma. \square

A corollary is the following main result of this paper.

(3.9) THEOREM. *If I is integrally closed in a regular local ring of dimension 2 with infinite residue field, and if A is an $n \times (n-1)$ presenting matrix of I , then $\text{core}(I) = I_{n-2}(A)I$.*

Proof. By Corollary 3.6, we need only prove that the left-hand side is contained in the right-hand side, so let α be an element of $\text{core}(I)$. Then, by Lemma 3.8, for any reduction (x_1, x_2) of I ,

$$\alpha \in ((x_1) + I_{n-2}(A)I) \cap ((x_2) + I_{n-2}(A)I).$$

So $\alpha = rx_1 + \alpha_1 = sx_2 + \alpha_2$ for some α_i in $I_{n-2}(A)I$. Thus, by Corollary 3.6, $sx_2 - rx_1 = \alpha_1 - \alpha_2 = t_1x_1 + t_2x_2$ for some $t_i \in I_{n-2}(A)$. Hence $(s - t_2)x_2 = -(r + t_1)x_1$. It follows that $s \in (t_2) + (x_1) \subseteq I_{n-2}(A)$. Similarly, $r \in I_{n-2}(A)$. \square

Thus, once we find a presenting matrix of I , it is easy to calculate the core of I .

Now we mention a construction from [RS] that is related to the core. Let $I = (a_1, \dots, a_n)$ be an m -primary ideal. If d is the dimension of R , let Y_{ij} ($i = 1, \dots, d, j = 1, \dots, n$) be indeterminates over R . Then define the ring S to be the localization at the extension of m of the polynomial ring in all the Y_{ij} over R . With this, Rees and Sally define $X(I) = (\sum_{j=1}^n a_j Y_{ij} \mid i = 1, \dots, d)S \cap R$ and show that $X(I)$ is contained in the core of I . By using the theorem above and some technical lemmas from [RS] one can obtain the reverse inclusion also. Thus another characterization of the core is:

$$\text{core}(I) = \left(\sum_{j=1}^n a_j Y_{ij} \mid i = 1, \dots, d \right) R[Y_{11}, Y_{12}, \dots, Y_{dn}]_{mR[Y]} \cap R.$$

However, note that Rees–Sally’s formulation of the core does not make the actual computations of the core any easier, as one must introduce many new variables.

Another consequence of Lemma 3.8 is that when a is part of a minimal reduction of I then the core of $I/(a)$ equals the image of $\text{core}(I)$ in $R/(a)$. Furthermore, we have the following proposition.

(3.10) PROPOSITION. *If (a, b) is a reduction of I , then*

$$b \cdot I_{n-2}(A)T = I \cdot I_{n-2}(A)T,$$

where $T = R/(a)$. If C is any other ideal in R that satisfies $bCT = ICT$ for all such b , then C is contained in the integral closure of $I_{n-2}(A)T$.

Proof. The first statement follows from (2.4).

For the rest, note that $IC \subseteq \bigcap_b (a, b) = (a) + II_{n-2}(A)$. Hence, modulo (a) , $ICT \subseteq II_{n-2}(A)T$. As IT contains a nonzero divisor, this inclusion implies (using the determinant trick) that CT is contained in the integral closure of $I_{n-2}(A)T$. \square

We combine this with Lemma 3.2 from [RS] in the case when a is square-free. Let C be the ideal of all elements of R which multiply the integral closure of $R/(a)$ into $R/(a)$ (i.e., the conductor). Lemma 3.2 in [RS] says that C has the property as in the proposition. Thus C is contained in the

integral closure of the image in $R/(a)$ of $I_{n-2}(A)$ for any m -primary integrally closed ideal I in R for which a is part of a minimal reduction. In particular, C is contained in $m^{\text{ord}(a)-1}$.

With the main theorem we are also able to calculate the multiplicity of an integrally closed ideal I using elementary means. Recall the definition of multiplicity of an m -primary ideal in a Noetherian local ring of dimension d given in Section 2. In a Cohen–Macaulay ring (which regular local rings certainly are) the multiplicity of I also equals the length of $R/(x_1, \dots, x_d)$, where (x_1, \dots, x_d) is a reduction of I . However, the calculation of multiplicities following either of these two definitions can be difficult. There is a simple method for calculating the multiplicity of integrally closed ideals in a 2-dimensional regular local ring:

$$\text{multiplicity of } (I) = \text{length}(R/I^2) - 2 \text{length}(R/I).$$

We given another simple method as follows.

(3.11) PROPOSITION. *If I is integrally closed and m -primary in a 2-dimensional regular local ring (R, m) with infinite residue field, then*

$$\text{multiplicity of } (I) = \text{length}(R/\text{core}(I)) - 2 \text{length}(R/I_{n-2}(A)).$$

Proof. Let (x_1, x_2) be a reduction of I . Then the multiplicity of I equals the length of $R/(x_1, x_2)$. We use that

$$\begin{aligned} \text{length}(R/(x_1, x_2)) &= \text{length}(R/(x_1, x_2)I_{n-2}(A)) \\ &\quad - \text{length}((x_1, x_2)/(x_1, x_2)I_{n-2}(A)). \end{aligned}$$

Now, $(x_1, x_2)/(x_1, x_2)I_{n-2}(A)$ is isomorphic to $R/I_{n-2}(A) \oplus R/I_{n-2}(A)$, and the product $(x_1, x_2)I_{n-2}(A)$ equals $\text{core}(I)$. \square

Theorem 3.9 also allows us to answer some of our questions about the core.

(3.12) COROLLARY. *The core of an integrally closed ideal is integrally closed.*

Proof. When I is integrally closed, $I_{n-2}(A)$ is also integrally closed by Proposition 3.3. As the product of two integrally closed ideals is integrally closed (see (2.4)(b)), the result follows. \square

(3.13) COROLLARY. *If R and S are 2-dimensional regular local rings with infinite residue fields and S is a faithfully flat R -algebra, then $\text{core}(I)S = \text{core}(IS)$ whenever both I and IS are integrally closed. In particular, this holds if S is the completion of R .*

In the next section we shall answer more questions, but we first want to introduce a simpler notation. So far we have written about $I_{n-2}(A)$ for any ideal I and any $n \times (n-1)$ presenting matrix A of I . In [L4, Prop. 3.3] it is proved that the adjoint of an integrally closed ideal I in R is equal to $(x_1, x_2) : I$ for any minimal reduction x_1, x_2 of I (see [L4] for the definition of the adjoint). By our results, it follows that the adjoint of I is exactly

$I_{n-2}(A)$. Thus, from now on we will denote $I_{n-2}(A)$ as $\text{adj}(I)$, the *adjoint* of I . We refer the reader to [L3; L4] for the rich theory of adjoints developed by Lipman.

We may therefore rephrase Theorem 3.9 in terms of adjoint ideals.

(3.14) THEOREM. *Let (R, m) be a 2-dimensional regular local ring with infinite residue field. Let I be an integrally closed m -primary ideal. Then $\text{core}(I) = I \cdot \text{adj}(I) = \text{adj}(I^2)$.*

Proof. The first equality is simply a change of notation. The second equality follows from the fact that (x_1^2, x_2^2) is a reduction of I^2 and from Corollary 3.7. \square

The definition of the adjoint in [L4] makes it clear that $\text{adj}(J) \subseteq \text{adj}(I)$ if J and I are integrally closed and J is contained in I . Thus we have our next proposition.

(3.15) PROPOSITION. *Let R be a 2-dimensional regular local ring with infinite residue field. If J and I are integrally closed ideals and J is contained in I , then $\text{core}(J) \subseteq \text{core}(I)$.*

This result is interesting insofar as the inclusions on cores do not hold in general. For example, let I be an integrally closed m -primary ideal of order greater than 1 and let $J = (a, b)$ be a minimal reduction of I . Then $\text{core}(J) = J$ and $\text{core}(I) \subseteq m^{\text{ord}(I)-1}I$, which does not contain an ideal of order equal to $\text{ord}(I)$.

It is worthwhile to rephrase some of our results in terms of the Fitting ideals of I . Recall that if M is a finitely presented module with an $n \times m$ presenting matrix A (so that under our conventions, M is the cokernel of the map from R^m to R^n determined by A), then the i th Fitting ideal of M , denoted $\mathfrak{F}_i(M)$, is the ideal generated by the $(n-i)$ -size minors of A . By convention, the ideal generated by the k -size minors of A is equal to R if $k \leq 0$, and is 0 for $k > \min\{m, n\}$. The Fitting ideals are independent of the presenting matrix. Our results may be summarized as follows.

(3.16) PROPOSITION. *Let (R, m) be a 2-dimensional regular local ring with infinite residue field. Let I be an integrally closed m -primary ideal. Then:*

- (a) $\mathfrak{F}_1(I) = I$, $\mathfrak{F}_2(I) = \text{adj}(I)$, $\mathfrak{F}_1(I)\mathfrak{F}_2(I) = \text{core}(I)$;
- (b) $\mathfrak{F}_{j+1}(I) = \mathfrak{F}_j(\text{adj}(I))$ for $j \geq 1$; and
- (c) $\mathfrak{F}_{j+1}(I) = \text{adj}(\mathfrak{F}_j(I))$ for $j \geq 1$.

Proof. Part (a) is immediate from the definition of the Fitting ideals and the results above; only (b) and (c) require comment. By Corollary 3.4, if A is a presenting matrix of I and B is defined as in (1), we obtain that $I_j(A) = I_j(B)$ for all $j \leq n-2$, where A is an $n \times (n-1)$ matrix. Moreover, Lemma 3.1 proves that B is a presenting matrix for $\text{adj}(I)$. Hence $\mathfrak{F}_j(\text{adj}(I)) = I_{(n-1)-j}(B) = I_{n-(j+1)}(A) = \mathfrak{F}_{j+1}(I)$, proving (b). Part (c) follows by repeated use of (b);

it follows easily from (b) that $\mathfrak{F}_j(I) = \text{adj}(\text{adj}(\dots(I))\dots)$, where we take the adjoint $j-1$ times. (By Corollary 3.4, all of these Fitting ideals are integrally closed.) Taking adjoints once more on each side yields that $\text{adj}(\mathfrak{F}_j(I)) = \text{adj}(\text{adj}(\dots(I))\dots)$, where now the adj is taken j times, which again by (b) is equal to $\mathfrak{F}_{j+1}(I)$. \square

4. The Arithmetic of Cores and Adjoints

Throughout this section, R is a 2-dimensional regular local ring with maximal ideal m and with an infinite residue field.

(4.1) LEMMA. *Let (x_1, x_2) be an m -primary ideal which is a reduction of the integrally closed ideal I . Then $(x_1, x_2)^n : I = (x_1, x_2)^{n-1} \text{adj}(I)$.*

Proof. Note that $(x_1, x_2)^{n-1} \text{adj}(I)I = (x_1, x_2)^n \text{adj}(I)$ by Corollary 3.6, so the right-hand side is always contained in the left-hand side.

Proposition 3.5 gives the lemma for $n = 1$. Now let $n > 1$ and assume that the result is true for $n - 1$. Then any $\alpha \in (x_1, x_2)^n : I$ is contained in $(x_1, x_2)^{n-2} \text{adj}(I)$. Write $\alpha = \sum r_i x_1^i x_2^{n-2-i}$ for some $r_i \in \text{adj}(I)$. By assumption, $\alpha x_2 = \sum s_i x_1^i x_2^{n-1-i}$ for some $s_i \in R$. Hence $\alpha x_2 = \sum r_i x_1^i x_2^{n-1-i} = \sum s_i x_1^i x_2^{n-1-i}$ says that $r_i \in (x_1, x_2)$ for all i . Thus we may write $\alpha = \sum t_i x_1^i x_2^{n-1-i}$ for some $t_i \in R$. Now let c be any element of I . Then $\alpha c = \sum s_i x_1^i x_2^{n-1-i}$ for some $s_i \in R$. As x_1, x_2 form a regular sequence, $\alpha c = \sum t_i c x_1^i x_2^{n-1-i} = \sum s_i x_1^i x_2^{n-1-i}$ says that $t_i c \in (x_1, x_2)$. Since this holds for all $c \in I$, all the t_i lie in $\text{adj}(I)$ and hence α lies in $(x_1, x_2)^{n-1} \text{adj}(I)$. \square

The next results lead toward the determination of the core of the core and the core of higher powers of the ideals.

(4.2) PROPOSITION. *If I is integrally closed, then $\text{adj}(I^n) = I^{n-1} \text{adj}(I)$.*

Proof. Let (x_1, x_2) be a reduction of I . Then (x_1^n, x_2^n) is a reduction of the integrally closed ideal I^n , so $\text{adj}(I^n) = (x_1^n, x_2^n) : I^n$ by Proposition 3.5. By (2.4), $I^n = (x_1, x_2)^{n-1} I$; thus $\text{adj}(I^n) = (x_1^n, x_2^n) : (x_1, x_2)^{n-1} I$, which equals $((x_1^n, x_2^n) : (x_1, x_2)^{n-1}) : I = (x_1, x_2)^n : I$. Hence, by Lemma 4.1, $\text{adj}(I^n) = (x_1, x_2)^{n-1} \text{adj}(I)$, and we are done by Corollary 3.6. \square

(4.3) REMARKS. Proposition 4.2 gives a proof in dimension 2 of a conjecture of Lipman [L4, 1.6]. The conjecture states that if R is a regular local ring of any dimension and I is an ideal of R , then $\text{adj}(I^n) = I \text{adj}(I^{n-1})$ if $n \geq l$, where l is the analytic spread of I . Lipman proves this in the 2-dimensional case [L4, 2.3]; however, his proof uses some machinery including duality and a vanishing theorem. Proposition 4.2 furnishes an elementary proof.

(4.4) PROPOSITION. *If I is integrally closed, then $\text{core}(I^n) = I^{2n-1} \text{adj}(I) = I^{2n-2} \text{core}(I)$.*

Proof. We know that $\text{core}(I^n) = I^n \text{adj}(I^n)$, and we finish by using Proposition 4.2. \square

Now we want to understand $\text{core}(\text{core}(I))$ and higher iterates, too. We first need a lemma.

(4.5) LEMMA. *For an integrally closed ideal I , $\text{adj}(\text{core}(I)) = (\text{adj}(I))^2$.*

Proof. Let (x_1, x_2) be a reduction of I . Then $\text{core}(I) = (x_1, x_2) \text{adj}(I)$. If $n = \mu(I)$, let A be a $n \times (n-1)$ presenting matrix of I whose first two rows correspond to x_1 and x_2 , respectively. Let the ij th entry be a_{ij} , let B be the submatrix composed of the last $n-2$ rows, and let R_i be the i th row. Let t_i be the signed $n-2$ minor of B after deleting the i th column. Thus $\text{adj}(I) = (t_1, \dots, t_{n-1})$, $x_1 = \sum_i a_{2i} t_i$, and $x_2 = -\sum_i a_{1i} t_i$. Since $\mu(\text{core}(I)) = \text{ord}(\text{core}(I)) + 1 = \text{ord}(I) + \text{ord}(\text{adj}(I)) + 1 = 2n - 2$, it is not hard to show that the $t_i x_1$ and the $t_i x_2$ minimally generate $\text{core}(I)$. We want to find the relations on $t_1 x_1, \dots, t_{n-1} x_1, t_1 x_2, \dots, t_{n-1} x_2$. As B^{tr} is a presenting matrix of t_1, \dots, t_{n-1} (see the proof of 3.1), the columns of

$$C = \left[\begin{array}{c|c|c} B^{\text{tr}} & 0 & R_2^{\text{tr}} \\ \hline 0 & B^{\text{tr}} & R_1^{\text{tr}} \end{array} \right]$$

are some of the relations on the given generators of $\text{core}(I)$. We now prove that they generate all the relations.

Let $\sum r_i t_i x_1 + \sum s_i t_i x_2 = 0$. Then there exists $\beta \in R$ such that $\sum r_i t_i = \beta x_2$ and $\sum s_i t_i = -\beta x_1$. As $x_2 = -\sum a_{1i} t_i$, the $(n-1)$ -tuple $(r_1 + \beta a_{11}, \dots, r_{n-1} + \beta a_{1, n-1})$ is a relation on the t_i . Similarly, $(s_1 + \beta a_{21}, \dots, s_{n-1} + \beta a_{2, n-1})$ is a relation on the t_i . Thus our relation $(r_1, \dots, r_{n-1}, s_1, \dots, s_{n-1})$ lies in the span of the column space of C . This proves that C is a presenting matrix of $\text{core}(I)$.

By Proposition 3.5, $\text{adj}(\text{core}(I)) = I_{2n-4}(C)$. Thus $\text{adj}(\text{core}(I)) = (\text{adj}(I))^2$. \square

(4.6) REMARK. Proposition 4.4 follows quite easily from the work of Lipman [L4, 3.1.2], where he proves that if I is integrally closed then $\text{adj}(I)$ is the unique integrally closed ideal with point basis $\{\max\{0, \text{ord}_S(I^S) - 1\}\}_{R \subseteq S}$. (See (2.5) for remarks concerning the point basis.) It is worth noting that this characterization for the adjoint of I also follows easily from our work. One uses that $\text{adj}(I)$ is the second Fitting ideal of I , and the order of this Fitting ideal is 1 less than the order of I by Lemma 3.2. These properties are preserved under quadratic transformations. Similarly, it also follows that the adjoint commutes with transforms, which is Corollary 3.1.3 in [L4]. (Here we use the factorization theorem of Zariski, which states that any 2-dimensional regular local ring birationally dominating R can be reached by a finite number of quadratic transformations. See [Sa] for a generalization.)

Now we introduce some notation: $\text{core}^1(I) = \text{core}(I)$ and, for $n > 1$, $\text{core}^n(I) = \text{core}^{n-1}(\text{core}(I))$.

(4.7) PROPOSITION. *For an integrally closed ideal I , $\text{core}^n(I) = I(\text{adj}(I))^{2^n - 1}$. In particular, $\text{core}(\text{core}(I)) = I(\text{adj}(I))^3$.*

Proof. If $n = 1$, this is just Theorem 3.9. Now let $n > 1$ and assume that the proposition holds for $n - 1$. Then

$$\begin{aligned} \text{core}^n(I) &= \text{core}^{n-1}(\text{core}(I)) \\ &= \text{core}(I)(\text{adj}(\text{core}(I)))^{2^{n-1}-1} \quad (\text{by induction}) \\ &= I \text{adj}(I)((\text{adj}(I))^2)^{2^{n-1}-1} \quad (\text{by Lemma 4.5}) \\ &= I(\text{adj}(I))^{2^n-1}. \quad \square \end{aligned}$$

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