On Entire Rational Maps in Real Algebraic Geometry

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1. Introduction and Results

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be real algebraic sets. A map $F: X \to Y$ is said to be *entire rational* if there exist $f_i, g_i \in \mathbb{R}[x_1, \dots, x_n]$, $i = 1, \dots, m$, such that each g_i vanishes nowhere on X and

$$F = (f_1/g_1, \dots, f_m/g_m).$$

We say X and Y are *isomorphic* to each other if there are entire rational maps $F: X \to Y$ and $G: Y \to X$ such that $F \circ G = \mathrm{id}_X$ and $G \circ F = \mathrm{id}_Y$. Let $R(X, Y)$ denote the set of all the entire rational maps from X to Y. Although the set of polynomial maps between X and Y is not an isomorphism invariant of the pair (X, Y), $R(X, Y)$ is an isomorphism invariant of the pair (X, Y). In other words, $R(X, Y)$ is independent of the embeddings of the real algebraic sets into the affine spaces (cf. [4, Chap. 3]).

In general, very little is known about entire rational maps between X and Y. In [7], Loday showed that for $n > 1$, any polynomial map from T^n to S^n is null homotopic, where T^n is the n torus $S^1 \times \cdots \times S^1$ and S^n is the standard n-sphere in $\mathbb{R}^{n+1}$. Let k and n be positive integers where k is odd and $k < 2n$. In [3], Bochnak and Kucharz showed that any entire rational map from $X \times S^{2n-k}$ to S^{2n} is null homotopic, where X is any k-dimensional nonsingular real algebraic set. The proofs of these results use algebraic K-theory. For a nice account of similar results, and for the results dealing with approximations of smooth maps by entire rational maps, we refer the reader to [1; 2; 3; 4; 7]. In all these cited results the target space is mostly the standard n-sphere S^n. In this paper, by using different and rather elementary techniques, in some cases we will prove more general results. For instance, in the statement of the Bochnak–Kucharz result we will replace the target space S^{2n} with any nonsingular real algebraic set homeomorphic to S^{2n}.

Here, a *complexification* $X_{\mathbb{C}} \subseteq \mathbb{C}P^n$ of X will mean that X is embedded in some $\mathbb{R}P^n$ and $X_{\mathbb{C}} \subseteq \mathbb{C}P^n$ is the complexification of the pair $X \subseteq \mathbb{R}P^n$. Our main theorem follows.

THEOREM 1.1. *Let X and Y be compact connected nonsingular orientable real algebraic sets of the same dimension n. Then any entire rational map*

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$f: X \to Y$ induces the zero homomorphism $f_*: H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$ in the top homology provided that there exist nonsingular complexifications $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ of X and Y so that the homology class $[X]$ is torsion in $H_n(X_{\mathbb{C}};\mathbb{Z})$ and the homology class $[Y]$ is not torsion in $H_n(Y_{\mathbb{C}};\mathbb{Z})$.

The assumptions that the homology class $[X]$ is torsion in $H_n(X_{\mathbb{C}};\mathbb{Z})$ and the homology class $[Y]$ is not torsion in $H_n(Y_{\mathbb{C}};\mathbb{Z})$ may seem unusual. However, the propositions that follow will provide many examples of each kind.

PROPOSITION 1.2. *Let Y be any nonsingular compact orientable real algebraic set with nonzero Euler characteristic. Then the homology class $[Y]$ of Y in $H_n(Y_{\mathbb{C}};\mathbb{Z})$ is not torsion for any nonsingular complexification $Y_{\mathbb{C}}$ of Y.*

REMARK. Since any odd-dimensional closed orientable manifold has zero Euler characteristic, the algebraic set Y in Proposition 1.2 is necessarily of even dimension.

Using Lefschetz's theorem and the Euler characteristic formula for hypersurfaces (see e.g. [6, pp. 143, 152]), one can see that the middle homology group $H_{2n-1}(Q^{2n-1};\mathbb{Z})$ of an odd-dimensional quadric $Q^{2n-1} \subseteq \mathbb{C}P^{2n}$ is zero. Hence we have the following proposition.

PROPOSITION 1.3. *For any $n > 0$ and any compact nonsingular orientable real algebraic set X, the homology class $[S^{2n-1} \times X]$ is zero in its complexification, where S^{2n-1} is the standard $(2n-1)$-sphere.*

We have immediate corollaries of these results.

COROLLARY 1.4. *Let n be a positive even integer, and let X be any n-dimensionial compact connected orientable nonsingular real algebraic set with a nonsingular complexification $X_{\mathbb{C}}$ such that the homology class $[X]$ is torsion in $H_n(X_{\mathbb{C}};\mathbb{Z})$. Assume also that Y is a nonsingular real algebraic set homeomorphic to S^n. Then any entire rational map $f: X \to Y$ is null homotopic.*

Proof. By Theorem 1.1 and Proposition 1.2, the degree of any entire rational map $f: X \to Y$ is zero. Since Y is homeomorphic to S^n, any degree-zero map $f: X \to Y$ is null homotopic. □

COROLLARY 1.5. *Let k and n be positive integers where k is odd and $k < 2n$. Also, let X be a compact connected nonsingular orientable real algebraic set of dimension k and let Y be any nonsingular real algebraic set homeomorphic to S^{2n}. Then any entire rational map from $X \times S^{2n-k}$ to Y is null homotopic.*

Proof. By Proposition 1.3, the homology class $[X \times S^{2n-k}]$ is zero in its complexification. Now we are done, by Corollary 1.4. □

REMARK. The main difference between Corollary 1.5 and the results of Bochnak and Kucharz (mentioned in the introduction) is that our statement holds independent of the algebraic structure Y of S^{2n}. Bochnak and Kucharz prove their results mostly for the case where Y is the standard $2n$-sphere S^{2n}, and their proofs make use of the standard algebraic structure of S^{2n}.

2. Proofs

First we need a technical lemma.

LEMMA 2.1. *Let X and Y be compact nonsingular orientable real algebraic sets of dimension n, with nonsingular complexifications $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$, such that the homology class $[X]$ is torsion in $H_n(X_{\mathbb{C}};\mathbb{Z})$. Let $f: X \to Y$ be an entire rational map. Then, by blowing up $X_{\mathbb{C}}$ along some smooth centers defined over the reals and away from X, we can assume that the complexification map $f_{\mathbb{C}}: X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is well-defined on all of $X_{\mathbb{C}}$ and that the homology class $[X]$ is still torsion in $H_n(X_{\mathbb{C}};\mathbb{Z})$.*

Proof. Let $\Delta \subseteq X_{\mathbb{C}}$ be the indeterminacy set of the map $f_{\mathbb{C}}: X_{\mathbb{C}} \to Y_{\mathbb{C}} \subseteq \mathbb{C}P^N$. Note that $\Delta \cap X = \emptyset$. By Hironaka's theorem, we can desingularize the subvariety Δ by blowing up smooth centers contained in the Sing(Δ). Let $L \subseteq$ Sing(Δ) be such a smooth center, and let $\pi: \tilde{X}_{\mathbb{C}} \to X_{\mathbb{C}}$ be the blow-up of $X_{\mathbb{C}}$ along L. Let $E = \pi^{-1}(L)$ be the exceptional divisor in $\tilde{X}_{\mathbb{C}}$, and let $\tilde{\Delta}$ be the strict transform of Δ. Since E is smooth and of codimension 1, the map $f_{\mathbb{C}} \circ \pi: \tilde{X}_{\mathbb{C}} \to \mathbb{C}P^N$ extends over $E - \tilde{\Delta}$. Continuing this process, we make the indeterminacy set $\tilde{\Delta}$ of $f_{\mathbb{C}} \circ \pi: \tilde{X}_{\mathbb{C}} \to \mathbb{C}P^N$ smooth. Finally, by blowing up the smooth center $\tilde{\Delta}$, we eliminate the indeterminacy set. Note that, since X is away from Δ, this blowing up process does not spoil X. Moreover, since everything is defined over the reals, these blow-up centers are also defined over the reals and therefore $\tilde{X}_{\mathbb{C}}$ is another nonsingular complexification of the real algebraic set X.

Now we need to show that the homology class $[X]$ is still torsion in $H_n(\tilde{X}_{\mathbb{C}};\mathbb{Z})$, where $\pi: \tilde{X}_{\mathbb{C}} \to X_{\mathbb{C}}$ is the blow-up of $X_{\mathbb{C}}$ along some smooth center L. Consider the exact homology sequence of the pair $X \subseteq X_{\mathbb{C}}$:

$$\cdots \longrightarrow H_{n+1}(X_{\mathbb{C}}, X;\mathbb{Z}) \xrightarrow{\partial} H_n(X;\mathbb{Z}) \xrightarrow{i_*} H_n(X_{\mathbb{C}};\mathbb{Z}) \longrightarrow \cdots.$$

Since $[X] \in H_n(X_{\mathbb{C}};\mathbb{Z})$ is torsion, there is a class $\alpha \in H_{n+1}(X_{\mathbb{C}}, X;\mathbb{Z})$ such that $\delta(\alpha) = k[X]$ for some nonzero integer k. By the Steenrod representability theorem, there exists a compact orientable smooth $(n+1)$-dimensional manifold W with boundary ∂W and a smooth map $T: (W, \partial W) \to (X_{\mathbb{C}}, X)$ such that $T_*([W]) = l\alpha \in H_{n+1}(X_{\mathbb{C}}, X;\mathbb{Z})$ for some odd integer l (cf. [5, Cor. 15.3, p. 49]). Now make $T: (W, \partial W) \to (X_{\mathbb{C}}, X)$ transversal to L without changing T on ∂W. Then $K = T^{-1}(L)$ is a submanifold of W not intersecting the boundary ∂W. Moreover, the normal bundle N of K in W has a complex structure. Let $p: \tilde{W} \to W$ be the complex blow-up of W along

K. Studying the Mayer–Vietoris sequences for the pairs $(U, W-U)$ and $(\tilde{U}, \tilde{W}-\tilde{U})$, where U is a tubular neighborhood of K in W and $\tilde{U}=p^{-1}(U)$, one can easily check that $\tilde{W}$ is orientable. The map $T\colon (W, \partial W) \to (X_{\mathbb{C}}, X)$ induces a map on the blow-ups $\tilde{T}\colon \tilde{W} \to \tilde{X}_{\mathbb{C}}$. Now, since $T_*([\partial W]) = kl[X] \in H_n(X_{\mathbb{C}}; \mathbb{Z})$ and $\tilde{T}_*([\partial \tilde{W}]) = T_*([\partial W])$, we see that

$$\tilde{T}_*([\partial \tilde{W}]) = kl[X] \in H_n(\tilde{X}_{\mathbb{C}}; \mathbb{Z})$$

and therefore the homology class $[X]$ is still torsion in $H_n(\tilde{X}_{\mathbb{C}}; \mathbb{Z})$. Replacing $X_{\mathbb{C}}$ by $\tilde{X}_{\mathbb{C}}$ finishes the proof. □

Proof of Theorem 1.1. By the preceding lemma, we can assume that the map $f_{\mathbb{C}}\colon X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is defined on all of $X_{\mathbb{C}}$. Let $i\colon X \to X_{\mathbb{C}}$ and $j\colon Y \to Y_{\mathbb{C}}$ denote the inclusion maps of X and Y into $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$, respectively. Consider the following commutative diagram:

$$\begin{array}{ccc} H_n(X;\mathbb{Z}) & \xrightarrow{f_*} & H_n(Y;\mathbb{Z}) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle j_*} \\ H_n(X_{\mathbb{C}};\mathbb{Z}) & \xrightarrow[(f_{\mathbb{C}})_*]{} & H_n(Y_{\mathbb{C}};\mathbb{Z}) \end{array}$$

By the hypotheses, $(f_{\mathbb{C}} \circ i)_*([X])$ is a torsion element in $H_n(Y_{\mathbb{C}}; \mathbb{Z})$. However, since $H_n(Y; \mathbb{Z}) = \mathbb{Z}$ and $[Y]$ is not a torsion element in $H_n(Y_{\mathbb{C}}; \mathbb{Z})$, the homomorphism $j_*\colon H_n(Y; \mathbb{Z}) \to H_n(Y_{\mathbb{C}}; \mathbb{Z})$ is an injection and therefore the class $f_*([X]) \in H_n(Y; \mathbb{Z})$ should be zero. □

Proof of Proposition 1.2. To see this, first note that the normal bundle N of Y in $Y_{\mathbb{C}}$ and the tangent bundle $T(Y)$ are isomorphic via multiplication by $i = \sqrt{-1}$, and therefore the self-intersection number $[Y]\cdot[Y] = \pm\chi(Y)$ of Y in $Y_{\mathbb{C}}$ is not zero. Hence the homology class $[Y]$ in $H_n(Y_{\mathbb{C}}; \mathbb{Z})$ is not zero. Note that this argument also implies that $[Y]$ is not a torsion element in $H_n(Y_{\mathbb{C}}; \mathbb{Z})$. □

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