Holomorphic Mappings, the Schwarz–Pick Lemma, and Curvature

DAVID GOLOFF & WING-KEUNG TO

0. Introduction

The main result of this paper bounds the dimension of the complex space of rank-k holomorphic mappings between two compact complex manifolds X and Y, which we denote by $\mathrm{Hol}_k(X,Y)$. The hypothesis of the main theorem involves curvature conditions on the image manifold Y. There are also two lemmas of independent interest. One shows that the evaluation mapping at any point $x \in X$, which we denote by $\mathrm{eval}(x)\colon \mathrm{Hol}_k(X,Y) \to Y$, does not reduce dimension. The other lemma is a variation on the Schwarz–Pick lemma. These results are part of the school of thought exemplified in [KSW], [K1], [NS], [No], [SY], and [La].

We list some results that motivate this work as follows.

0. (De Franchis' theorem) If X and Y are compact 1-dimensional complex manifolds (i.e. Riemann surfaces) and the genus of Y is at least 2, then $\dim \mathrm{Hol}_1(X,Y) = 0$.
1. [KSW] If X and Y are complex manifolds, X is compact, and the holomorphic tangent bundle $T(Y)$ has a Hermitian metric such that the curvature form $R(v,\bar{v})(\ ,\ )$ has at least $n-k$ negative eigenvalues for each nonzero vector $v \in T(Y)$, then $\dim \mathrm{Hol}_{k+1}(X,Y) = 0$.

   Here, $R(v,\bar{w})(s,\bar{t}) = h(D^2(s,\bar{t})v,\bar{w})$, where D is the connection associated to the metric h (see [GA] or [SS]).
2. [K2] Let M be a compact complex manifold whose first Chern class $c_1(M)$ is represented by a $(1,1)$-form

   $$\frac{i}{2\pi}\sum_{a,b} C_{a\bar{b}}\, dz_a \wedge d\bar{z}_b.$$

   If $\{C_{a\bar{b}}\}$ is negative semidefinite with maximal rank r, then

   $$\dim \mathrm{Aut}(M) \le \dim M - r,$$

   where $\mathrm{Aut}(M)$ denotes the automorphism group of M.
3. [NS] Let X be a compact complex manifold. If $\bigwedge^k T(Y)$ has a Hermitian metric such that $R(v,\bar{v})(\ ,\ )$ is negative definite for each nonzero $v \in \bigwedge^k T(Y)$, then $\dim \mathrm{Mero}_k(X,Y) = 0$. Here, "$\mathrm{Mero}_k$" stands for

Received August 27, 1991. Revision received September 20, 1994.
Michigan Math. J. 42 (1995).

meromorphic mappings of rank k. For the case when $k=1$, the condition becomes that Y has negative bisectional curvatures.

The authors would like to take this opportunity to express their sincere thanks to Bernard Shiffman for his generous help and encouragement during the preparation of this paper.

1. Statements of Results

We now state our main result as follows.

MAIN THEOREM. *Let X and Y be compact algebraic complex manifolds. If*

(1) $\bigwedge^k T(Y)$ *has a Hermitian metric* Ψ *such that, for each* $y \in Y$ *and each nonzero simple vector* v *in* $\bigwedge^k T_y(Y)$, *the Hermitian bilinear form* $R(v,\bar{v})(\ ,\ )$ *is negative definite on* $\operatorname{span}(v_1,\dots,v_k)$ *where* $v = v_1 \wedge \cdots \wedge v_k$, *and* $R(v,\bar{v})(\ ,\ )$ *has at least* s_k *negative eigenvalues on* $T_y(Y)$; *and*

(2) *Y has a Kähler metric with negative semidefinite Ricci curvature, then*

$$\dim \operatorname{Hol}_k(X,Y) \le \dim Y - s_k. \tag{1.1}$$

Here, if v is a "simple vector" this means that v is of the form $v = v_1 \wedge \cdots \wedge v_k$; that is, it need not be written as a sum of wedge products. We remark also that condition (1) implies that we necessarily have $s_k \ge k$.

In Section 3, we will consider the case when $Y = \Gamma \backslash D$, where D is a bounded symmetric domain and Γ is a torsion-free discrete subgroup of $\operatorname{Aut}(D)$. We will show that for such Y and for many values of k, the upper bound given in the Main Theorem improves that given by a result of Noguchi [No], and that in all cases our bound is always at least as good as that in [No]. Explicit determinations of these values of k will be carried out for those smooth quotients of classical bounded symmetric domains.

An immediate consequence of the Main Theorem is the following.

COROLLARY. *Let X and Y be compact algebraic complex manifolds, and let k be an integer such that* $1 \le k \le \dim Y$. *Assume that Y has a Kähler metric with negative semidefinite Ricci curvature. Assume further that* $T(Y)$ *has a Hermitian metric and that there exists a positive number B such that the following two conditions hold:*

(i) *the holomorphic sectional curvatures are strictly less than* $-B$;

(ii) *if* $k \ge 2$, *the holomorphic bisectional curvatures are less than or equal to* $B/(k-1)$.

Then $\dim \operatorname{Hol}_k(X,Y) \le \dim Y - k$.

(The case where $k = \dim Y$ follows from [KSW, Cor. 2].)

Note that this allows some positivity in the bisectional curvatures. This suggests that one could deform a compact Kähler manifold with negative holomorphic curvature and semidefinite holomorphic bisectional curvature, and retain some control even if the holomorphic bisectional curvatures become positive, as long as the Ricci curvature stays negative semidefinite.

The following lemma is a variation on the Schwarz–Pick lemma found in [La].

LEMMA 1. *Let Y be a compact complex manifold with a Hermitian metric Ψ on $\bigwedge^k T(Y)$ such that, for every nonzero simple vector $v \in \bigwedge^k T(Y)$, $R(v, \bar{v})(\ ,\ )$ is negative definite on $\operatorname{span}(v_1, \ldots, v_k)$ where $v = v_1 \wedge \cdots \wedge v_k$. Then there exists a positive number B such that, for any holomorphic mapping $f\colon \Delta^k \to Y$, we have $B(f^*\Psi) \le \Phi$ where Φ is the product of the Poincaré metrics and $f^*\Psi(w, \bar{w}) = \Psi(f_* w, \overline{f_* w})$ for $w \in \bigwedge^k T(\Delta^k)$.*

We also have the following.

LEMMA 2. *If Y has a Kähler metric with negative semidefinite Ricci curvature and X and Y are compact complex manifolds, then the evaluation mapping at any point x of X has 0-dimensional fibers. More specifically, for $\operatorname{eval}(x)\colon \operatorname{Hol}(X, Y) \to Y$ given by $\operatorname{eval}(x)(f) = f(x)$, we have that $\operatorname{eval}(x)^{-1}(y)$ is a finite set for any $y \in Y$. In particular, we have*

$$\dim \operatorname{Hol}(X, Y) = \dim \operatorname{Image}(\operatorname{eval}(x)).$$

2. Proofs

To prove the Main Theorem, we also need the following lemmas.

LEMMA 3 [KSW]. *Let M be a complex manifold, A an irreducible compact analytic subset in M, and φ a twice differentiable real function on M. Assume that the Levi form $L(\varphi)$ has at least p positive eigenvalues at each point of $A \setminus \varphi^{-1}(0)$ and that $\varphi \ge 0$. Then we have*

$$A \subseteq \varphi^{-1}(0) \quad \text{or} \quad \dim A \le \dim M - p.$$

LEMMA 4. *Let X and Y be compact algebraic complex manifolds. Let $\bar{H}$ denote the closure of a connected component H of $\operatorname{Hol}(X, Y)$ in $\operatorname{Sub}(X \times Y)$. Then:*

(a) *$\bar{H}$ and $\bar{H} \setminus H$ are algebraic subvarieties of $\operatorname{Sub}(X \times Y)$;*
(b) *for fixed $x \in X$ and $v_1, \ldots, v_k \in T_x(X)$, the mapping $s_x\colon H \to \bigwedge^k T(Y)$, defined by $s_x(f) = f_* v_1 \wedge \cdots \wedge f_* v_k$ for $f \in H$, is an algebraic mapping;*
(c) *s_x has a meromorphic extension to $\bar{H}$.*

Here $\operatorname{Sub}(V)$ is the Chow space of nonreduced subvarieties of V. It is well known that the components of the Chow space are compact algebraic varieties (see e.g. [Sh]). We regard $\operatorname{Hol}(X, Y)$ as a subspace of $\operatorname{Sub}(X \times Y)$ by identifying a holomorphic mapping with its graph.

LEMMA 5 [KSW]. *Let E be a Hermitian holomorphic vector bundle of rank r over an n-dimensional complex manifold Y, and let $\varphi: E \to \mathbb{R}^+ \cup \{0\}$ be given by $\varphi(v) = \|v\|^2$, where $\|v\|$ denotes the norm of v induced by the Hermitian metric of E. Suppose the curvature form $R(v_o, \bar{v}_o)(\ ,\ )$ has at least k negative eigenvalues at a vector $v_o \in E$. Then the Levi form $L(\varphi)$ has at least $r+k$ positive eigenvalues at v_o.*

We are now in a position to give the following.

Proof of the Main Theorem assuming the lemmas. Choose x in X such that there is a mapping f_o in $\mathrm{Hol}_k(X, Y)$ whose rank at x is at least k. Thus there is a simple k-vector $v_x = v_1 \wedge \cdots \wedge v_k \in \bigwedge^k T_x(X)$ such that

$$f_{o*}v_x = f_{o*}v_1 \wedge \cdots \wedge f_{o*}v_k \in \textstyle\bigwedge^k T(Y)$$

is not zero. Now define $s_x: \mathrm{Hol}_k(X, Y) \to \bigwedge^k T(Y)$ as in Lemma 4, that is,

$$s_x(f) = f_* v_x \quad \text{for } f \in \mathrm{Hol}_k(X, Y).$$

Now we fix the point $x \in X$ and $v_x \in \bigwedge^k T(X)$. Then s_x is a well-defined mapping. Note that Image(s_x) is contained in the set of simple vectors. We shall use s_x to analyze $\mathrm{Hol}_k(X, Y)$.

By Lemma 1, we obtain a bound on s_x in the following way. Fix an embedding $j: \Delta^k \to X$ such that $j(0) = x$ and $j_* w_o = \lambda v_x$, where $\lambda \in \mathbb{C} \backslash \{0\}$ and

$$w_o = \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_k}\bigg|_0 \in \textstyle\bigwedge^k T_0(\Delta^k).$$

By Lemma 1, for $f \in \mathrm{Hol}_k(X, Y)$ we have

$$B|f_* j_* w_o|_\Psi \le |w_o|_\Phi = 1.$$

By the definition of s_x, we thus have

$$|s_x(f)|_\Psi = |f_* v_x|_\Psi = \frac{1}{|\lambda|}|f_* j_* w_o|_\Psi \le \frac{1}{|\lambda| B},$$

which is an upper bound independent of f. Thus the image of s_x is a Ψ-bounded set.

By Lemma 4, s_x extends meromorphically to the compactification $\bar{H}$ of a component of $\mathrm{Hol}_k(X, Y)$. Since s_x is bounded, the image of the meromorphic extension stays away from the infinity points of the compactification of $\bigwedge^k T(Y)$; more precisely, the graph of the meromorphic extension of s_x to $\bar{H}$ is a compact subvariety of $\bar{H} \times \bigwedge^k T(Y)$. The projection of this graph to $\bigwedge^k T(Y)$ is a compact subvariety A coinciding with the closure of Image(s_x). Thus A is contained in the set of simple vectors.

From condition (1) of the Main Theorem, $R(v, \bar{v})$ has at least s_k negative eigenvalues for each simple v. Let $r = \mathrm{rank}(\bigwedge^k T(Y))$. By Lemma 5, $L(\varphi)$ has at least $r + s_k$ positive eigenvalues at each nonzero simple v. Then, by Lemma 3, we have

$$\dim A \le \dim(\textstyle\bigwedge^k T(Y)) - (r + s_k) = (r + \dim Y) - (r + s_k) = \dim Y - s_k.$$

Let $\pi: \bigwedge^k T(Y) \to Y$ denote the projection map onto Y, that is, $\pi(v) = y$ for $v \in \bigwedge^k T_y(Y)$. Since s_x is defined as some lifting of eval(x) to $\bigwedge^k T(Y)$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hol}_k(X,Y) & \xrightarrow{s_x} & \bigwedge^k T(Y) \\ {\scriptstyle \mathrm{eval}(x)} \downarrow & \swarrow {\scriptstyle \pi} & \\ Y. & & \end{array}$$

It follows from the above commutative diagram that

$$\operatorname{rank}(s_x) \geq \operatorname{rank}(\operatorname{eval}(x)).$$

Hence, by Lemma 2, we have

$$\dim(\mathrm{Hol}_k(X,Y)) = \dim(\mathrm{Image}(s_x)) = \dim A \leq \dim Y - s_k.$$

This finishes the proof of the Main Theorem assuming Lemmas 1–5. □

We also give the following.

Proof of Corollary. Let $v \in \bigwedge^k T(Y)$ be a simple, nonzero unit vector with respect to the Hermitian metric of $\bigwedge^k T(Y)$ as given in the Corollary. Let $v = v_1 \wedge \cdots \wedge v_k$ with the $\{v_i\}$ orthonormal. Then

$$R(v,\bar{v})(v_k,\overline{v_k}) = R(v_k,\overline{v_k})(v_k,\overline{v_k}) + \sum_{j \neq k} R(v_j,\overline{v_j})(v_k,\overline{v_k}).$$

Adding the bounds for each term given by the assumptions of the Corollary shows that condition (1) of the Main Theorem is satisfied with $s_k = k$, and this finishes the proof of the Corollary. □

To complete our arguments, we prove Lemmas 1–5 in the remaining part of this section.

Proof of Lemma 1. The usual sign convention for holomorphic curvature (used to state the results of this paper) is designed to force agreement between negative Gaussian curvature in two real dimensions and negative holomorphic curvature in one complex dimension. Since the argument in the proof of Lemma 1 is more easily understood by applying the language of volumes and metrics to curvature operators, we will work with the opposite sign convention while denoting curvature by the same symbol R.

Given a holomorphic map $f: \Delta^k \to Y$ and a point $x \in \Delta^k$, either $f^*\Psi|_x = 0$ or f is locally an injection at x. In the first case, the lemma holds automatically. In the second case, without loss of generality we may assume that f is an injection, since our argument will be applied at each point of Δ^k. Thus we may consider f as an inclusion of Δ^k in Y, and $\bigwedge^k T(\Delta^k)$ becomes a Hermitian holomorphic vector subbundle of $f^* \bigwedge^k T(Y)$ with the induced metric simply given by $f^*\Psi$.

With our present sign convention, curvature increases in subbundles (see e.g. [GH]). Thus the curvature of Ψ restricted to Δ^k is less than or equal to the curvature of $f^*\Psi$. In the first case, we restrict the curvature operator. In the second case, we consider the curvature of the restricted metric. In either case, we obtain a curvature operator on the line bundle $\bigwedge^k T(\Delta^k)$. Now $v = v_1 \wedge \cdots \wedge v_k$ generates the fibers of $\bigwedge^k T(\Delta^k)$ if $\{v_i\}_{1\le i\le k}$ is a frame for $T(\Delta^k)$. Let $v = v_1 \wedge \cdots \wedge v_k$ be a simple $f^*\Psi$-unit vector in $\bigwedge^k T(\Delta^k)$. The fact that curvature increases in subbundles can be written as

$$f^*R^\Psi \le R^{f^*\Psi},$$

where the superscript indicates the metric to which each curvature operator is associated. Also, f^*R^Ψ and $R^{f^*\Psi}$ refer to the $(1,1)$-forms $f^*R^\Psi(v,\bar v)(\,,\,)$ and $R^{f^*\Psi}(v,\bar v)(\,,\,)$, respectively. Recall that a $(1,1)$-form is said to be positive if the associated pairing is positive definite. Under the present sign convention, the hypothesis of Lemma 1 implies that f^*R^Ψ is positive. Thus we have

$$0 < f^*R^\Psi \le R^{f^*\Psi}.$$

Now we can imitate the proof of Theorem 4.1 in [La]. It is well known that the wedge product is order-preserving on positive forms (see e.g. [La, p. 101]). Thus

$$(f^*R^\Psi)^k = \underbrace{f^*R^\Psi \wedge \cdots \wedge f^*R^\Psi}_{k \text{ times}}$$

is a positive volume form on Δ^k, and by the preceding inequality we have $(f^*R^\Psi)^k \le (R^{f^*\Psi})^k$, which will be used shortly.

Because $f^*\Psi$ is a metric on $\bigwedge^k T(\Delta^k)$, it, as well as $(f^*R^\Psi)^k$, is a volume form on Δ^k. Since the ratio of two nonzero vectors in a 1-dimensional vector space gives rise to a real number, the ratio

$$\mu_f(x) = \frac{(1/k!)(f^*R^\Psi)^k(x)}{f^*\Psi(x)}, \quad x \in \Delta^k,$$

is a positive real-valued function on Δ^k. Let $S(\bigwedge^k T(Y))$ denote the sphere bundle of $\bigwedge^k T(Y)$ (i.e., the bundle of unit vectors in $\bigwedge^k T(Y)$), and let $\nu\colon S(\bigwedge^k T(Y)) \to \mathbb{R}^+$ denote the positive continuous function given by $\nu(t) = (R^\Psi(t,\bar t))^k(t,\bar t)$ for $t \in S(\bigwedge^k T(Y))$. Since Y is compact, $S(\bigwedge^k T(Y))$ is also compact, and thus the function ν is bounded from below by a positive number B; that is, $\nu(t) \ge B > 0$ for all $t \in S(\bigwedge^k T(Y))$. For a point $x \in \Delta^k$, let $w = cv = (cv_1) \wedge v_2 \wedge \cdots \wedge v_k$, where v is a unit simple vector of $\bigwedge^k T_x(\Delta^k)$ and $v_i \in T_x(\Delta^k)$ for $1 \le i \le k$. Then

$$\begin{aligned}\mu_f(x) = \frac{(f^*R^\Psi(v,\bar v))^k(w,\bar w)}{f^*\Psi(w,\bar w)} &= \frac{|c|^2(f^*R^\Psi(v,\bar v))^k(v,\bar v)}{|c|^2 f^*\Psi(v,\bar v)}\\ &= \frac{\det\{R^\Psi(f_*v,\overline{f_*v})(f_*v_i,\overline{f_*v_j})\}}{\Psi(f_*v,\overline{f_*v})}\\ &= (R^\Psi(f_*v,\overline{f_*v}))^k(f_*v,\overline{f_*v})\\ &= \nu(f_*v) \ge B,\end{aligned}$$

which is independent of f. Note that $f_* v \in S(\bigwedge^k T(Y))$ since v is a $f^*\Psi$-unit vector in $\bigwedge^k T(\Delta^k)$. From the definition of μ_f and the above inequality, we have

$$(k!)Bf^*\Psi \leq (f^*R^{\Psi})^k.$$

Let us now combine the two inequalities obtained thus far to yield

$$(k!)Bf^*\Psi \leq (f^*R^{\Psi})^k \leq (R^{f^*\Psi})^k.$$

We now relate all of this to Φ, the volume form on Δ^k associated to the product of the Poincaré metrics on the unit disk. Since $f^*\Psi$ and Φ are both Hermitian metrics on the line bundle $\bigwedge^k T(\Delta^k)$, there exists a function $u\colon \Delta^k \to \mathbb{R}$ such that $u > 0$ and $f^*\Psi = u\Phi$. Let Φ_t be the volume form on the polydisk $(\Delta_t)^k$ associated to the product of the Poincaré metrics on the Δ_ts, where Δ_t denotes the disk of radius t. Then $\Phi_t \to \Phi$ as $t \to 1$. Let

$$f^*\Psi = h(z)dz_1 \wedge \cdots \wedge dz_k \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_k.$$

Analogous to the definition of u, we define a function u_t on $(\Delta_t)^k$ for each $t < 1$ given by

$$f^*\Psi = u_t\Phi_t \text{ on } (\Delta_t)^k.$$

It is easy to see that h is bounded on $(\Delta_t)^k$ for each $t < 1$ and

$$u_t(z) = \prod_{1 \leq j \leq k} \frac{(t^2 - |z_j|^2)^2}{2t^2} h(z) \quad \text{for } z \in (\Delta_t)^k,$$

which readily imply that $u_t(z) \to 0$ as $z \to \partial((\Delta_t)^k)$, and thus u_t attains a maximum at some point in $(\Delta_t)^k$. Also, it is easy to see that $u_t \to u$ as $t \to 1$, and thus it suffices to show that $u_t \leq 1/B$ for each $t < 1$. For a fixed $t < 1$, let $z_o \in (\Delta_t)^k$ be a point where u_t attains its maximum. If $u_t(z_o) = 0$, we automatically have $u_t \leq 1/B$. Thus without loss of generality we may assume that $u_t(z_o) > 0$. We now employ the following second derivative test [La, p. 101]:

$$u_t \text{ has a maximum at } y \text{ only if } (dd^c \log u_t)(y) \leq 0.$$

By definition, we have $R^{f^*\Psi} = R^{\Phi_t} + dd^c \log u_t$. So by the above second derivative test, we have $R^{f^*\Psi}(z_o) \leq R^{\Phi_t}(z_o)$. Since the wedge product is order-preserving on positive forms,

$$(R^{f^*\Psi})^k(z_o) \leq (R^{\Phi_t})^k(z_o).$$

We now use this, along with the above sequence of inequalities, to obtain

$$\begin{aligned}(k!)Bf^*\Psi(z_o) &\leq (f^*R^{\Psi})^k(z_o)\\ &\leq (R^{f^*\Psi})^k(z_o)\\ &\leq (R^{\Phi_t})^k(z_o)\\ &= (k!)\,\Phi_t(z_o)\\ &= (k!)\frac{1}{u_t(z_o)} f^*\Psi(z_o).\end{aligned}$$

Comparing the left side with the right side yields $u_t \le u_t(z_o) \le 1/B$. Thus we have finished the proof of Lemma 1. □

Proof of Lemma 2. Let x_o be a fixed point in X and let $h: \Delta \to \mathrm{Hol}(X, Y)$ be a holomorphic map such that $h(\Delta)$ is contained in a fiber of $\mathrm{eval}(x_o)$. To show that h is a constant map, we let $F: X \times \Delta \to Y$ be defined by setting $F(x, t) = \mathrm{eval}(x)(h(t))$ (i.e. $F(x, t) = h(t)(x)$) for $x \in X$ and $t \in \Delta$. By an argument in [KSW, pp. 292–293], we have that $K(t) = \|F_{*(x,t)}(\partial/\partial t)\|^2$ does not depend on x. Since $F(x_o, t)$ is a constant independent of t, it follows that $K(t) \equiv 0$ and therefore $F(x, 0) = F(x, t)$ for any $x \in X$ and $t \in \Delta$. Therefore $h(0) = h(t)$ for all $t \in \Delta$. This finishes the proof of Lemma 2. □

Lemma 3 and Lemma 5 are proved in [KSW], so it remains to prove Lemma 4.

Proof of Lemma 4(a). Let S be the component of $\mathrm{Sub}(X \times Y)$ containing H. Recall that H is injected into S by associating each function with its graph. Let $\bar{R} = \{(x, y, s) \in X \times Y \times S: (x, y) \in s\}$ and $\bar{G} = \mathrm{Grass}(n, T(X \times Y))$ be the Grassmannian bundle over $X \times Y$ with fibers $\bar{G}(x, y) = \mathrm{Grass}(n, T_{(x,y)}(X \times Y))$, where $n = \dim X$. Let G be the canonical vector bundle over $\bar{G}$, and let R be the closure of the set $\{(x, y, s, v) \in \bar{R} \times G: v \in T_{(x,y)}(s)$ and $(x, y) \in s^{\mathrm{reg}}\}$ in $\bar{R} \times G$, where s^{reg} denotes the smooth part of s. Define the mapping τ: $R \to T(X)$ by $\tau(x, y, s, v) = v_X$, where v_X denotes the projection of v to $T(X)$. Thus τ^{-1}(zero-section) projected to S yields those subvarieties with vertical tangents.

Now we show that R is a compact variety. For any vector bundle E, we denote by $\mathbb{P}E$ its associated projectivized bundle obtained by taking the projective quotient of the fibers of E. We consider

$$R' = \{(x, y, s, \bar{v}) \in R \times \mathbb{P}G: \bar{v} \in \mathbb{P}T(s)_{(x,y)}\}.$$

The projection maps are easily seen to be proper maps. Let B denote the image of τ^{-1} (zero-section) in R', and let p_s denote the projection of R' onto S. By the proper mapping theorem, $p_s(B)$ is a (proper) subvariety of S. Let $S_o = S \setminus p_s(B)$. Clearly $H \subset S_o$. A set $s \in S_o$ is a λ-sheeted cover over X. Since S_o is irreducible, λ is independent of the choice of $s \in S_o$. By choosing $s \in H$, we see that $\lambda = 1$, and thus $H = S_o$. Hence H is a Zariski open subset of S and Lemma 4(a) follows immediately. □

Proof of Lemma 4(b). Let $T'(H) = \{(v, f) \in T(X \times Y) \times H: v \in T(G_f)\}$, where G_f is the graph of f and $T'(H)$ is the relative tangent bundle of the family of subvarieties given by H. Relative tangent bundles exist in the algebraic category. Let $\overline{T(Y)}$ denote the projective closure of $T(Y)$. Let $\{w_i\}_{1 \le i \le k}$ be k linearly independent tangent vectors in $T_x(X)$ for some $x \in X$. We denote

$$V_i = (\{w_i\} \times \overline{T(Y)} \times \bar{H}) \cap \overline{T'(H)},$$

where $\bar{H}$ is the closure of H in S. The projection $p_{\bar{H}}: V_i \to \bar{H}$ is one-to-one since $f \to (w_i, f_*(w_i), f)$ is the inverse. Also, the projections $p_{\bar{H}}$ and $p_{\overline{T(Y)}}$ are easily seen to be proper mappings. This implies that $\{(p_{\bar{H}}(u), p) : u \in V_i\}$ is an algebraic set in $\bar{H} \times \overline{T(Y)}$. Note that it is also the graph of the map s_i: $H \to T(Y)$ defined by letting $s_i(f) = f_*(w_i)$. Hence s_i is an algebraic mapping, which implies that $s_x = s_1 \wedge \cdots \wedge s_k$ is also an algebraic mapping. This finishes the proof of Lemma 4(b). □

Proof of Lemma 4(c). Lemma 4(c) follows easily from Chow's theorem and the fact that algebraic mappings do not have essential singularities. □

3. Applications

In this section, we apply our Main Theorem to the case when $Y = \Gamma\backslash D$, where D is a bounded symmetric domain and Γ is a torsion-free discrete subgroup of $\operatorname{Aut}(D)$. We will show that for those Y which are quotients of the classical bounded symmetric domains and for many values of k, the bound in (1.1) improves that given by a result of Noguchi [No], and in all cases our bound is at least as good as that in [No]. To start with, we first recall Noguchi's results which are relevant here.

THEOREM [No]. *Let N be a Zariski open subset of a compact Kähler manifold $\bar{N}$ such that $\partial N = \bar{N} - N$ is a hypersurface with only simple normal crossings, let D be a bounded symmetric domain, and let Γ be a torsion-free discrete subgroup of* $\operatorname{Aut}(D)$ *such that* $\operatorname{vol}(\Gamma\backslash D) < \infty$. *Denote the maximum dimension of proper boundary components of D by $l(D)$. Then*

(i) [No, Thm. (3.3)(ii)] $\operatorname{Hol}_k(N, \Gamma\backslash D)$ *is finite for $k > l(D)$, which implies*

$$\dim \operatorname{Hol}_k(N, \Gamma\backslash D) = 0 \quad \textit{for } k > l(D); \tag{3.1}$$

(ii) [No, Thm. (4.7)(iii)]

$$\dim \operatorname{Hol}_k(N, \Gamma\backslash D) \leq l(D) \quad \textit{for } k > 0. \tag{3.2}$$

We remark that we are primarily interested in the special case of the above theorem when X and Y are both compact, and such assumptions are implicit in the ensuing discussion.

For $x \in \Gamma\backslash D$, we denote

$$l_x(\Gamma\backslash D) = \min\{\mu : \Xi^\perp = \{0\} \text{ for any vector subspace } \Xi \subset T_x(\Gamma\backslash D) \text{ with } \dim \Xi > \mu\},$$

where $\Xi^\perp = \{w \in T_x(\Gamma\backslash D) : R(v, \bar{v})(w, \bar{w}) = 0 \text{ for all } v \in \Xi\}$. By homogeneity of D, $l_x(\Gamma\backslash D)$ is independent of the choice of x, and we write $l(\Gamma\backslash D) = l_x(\Gamma\backslash D)$ for any $x \in \Gamma\backslash D$. We also denote

$$\lambda(\Gamma\backslash D) = \max\{\dim(\mathbb{C}v)^\perp : 0 \neq v \in T(\Gamma\backslash D)\},$$

where $\mathbb{C}v$ denotes the complex linear span of v. First we have the following lemma.

LEMMA 6. $l(D) = l(\Gamma\backslash D) = \lambda(\Gamma\backslash D)$.

Proof. The first equality is a result of Noguchi and Sunada [NS, Thm. (3.4)]. To prove the second equality, we first show that $\lambda(\Gamma\backslash D) \leq l(\Gamma\backslash D)$. For any nonzero $v \in T(\Gamma\backslash D)$ and $w \in (\mathbb{C}v)^{\perp}$, we have $R(w,\bar{w})(v,\bar{v}) = R(v,\bar{v})(w,\bar{w}) = 0$, which implies that $0 \neq v \in ((\mathbb{C}v)^{\perp})^{\perp}$. By definition of $l(\Gamma\backslash D)$ we thus have $\dim(\mathbb{C}v)^{\perp} \leq l(\Gamma\backslash D)$, which, upon taking the maximum over all v, implies that $\lambda(\Gamma\backslash D) \leq l(\Gamma\backslash D)$.

Next we shall show that $l(\Gamma\backslash D) \leq \lambda(\Gamma\backslash D)$. For any $x \in \Gamma\backslash D$ and vector subspace $\Xi \subset T_x(\Gamma\backslash D)$ with $\dim \Xi > \lambda(\Gamma\backslash D)$, it follows from dimensional considerations that $\Xi \not\subset (\mathbb{C}v)^{\perp}$ for any nonzero $v \in T_x(\Gamma\backslash D)$, which implies that $\Xi^{\perp} = \{0\}$. Then it follows from the definition of $l(\Gamma\backslash D)$ that $l(\Gamma\backslash D) \leq \lambda(\Gamma\backslash D)$, and we have finished the proof of Lemma 6. □

Let s_k be defined for $\Gamma\backslash D$ as given in the Main Theorem, where $1 \leq k \leq \dim(\Gamma\backslash D)$; that is, s_k is the minimum number of negative eigenvalues of the Hermitian bilinear forms $R(v,\bar{v})(\ ,\ )$ over nonzero simple vectors $v \in \bigwedge^k T(\Gamma\backslash D)$.

PROPOSITION 1.

(i) $s_k = \dim(\Gamma\backslash D)$ *for* $k > l(D)$.
(ii) $\dim(\Gamma\backslash D) - s_1 = \lambda(\Gamma\backslash D)$.
(iii) $s_k \leq s_{k+1}$ *for all* $k \geq 1$.
(iv) $\dim(\Gamma\backslash D) - s_k \leq l(D)$ *for all* $k \geq 1$.
(v) $\dim(\Gamma\backslash D) - s_k =$ *maximum number of zero eigenvalues of* $R(v,\bar{v})(\ ,\ )$ *over nonzero simple* $v \in \bigwedge^k T(\Gamma\backslash D)$.

Proof. Statement (i) follows immediately from a result of Noguchi and Sunada [NS, Lemma (3.3)] that the curvature tensor of $\bigwedge^k T(\Gamma\backslash D)$ is negative definite for $k > l(D)$. To prove (ii), we observe that $\Gamma\backslash D$ is of seminegative bisectional curvature, and that all vectors in $T(\Gamma\backslash D)$ are simple, which imply that $\dim(\Gamma\backslash D) - s_1$ is the maximum number of zero eigenvalues of the Hermitian bilinear forms $R(v,\bar{v})(\ ,\ )$ over nonzero vectors $v \in T(\Gamma\backslash D)$; that is, $\dim(\Gamma\backslash D) - s_1 = \lambda(\Gamma\backslash D)$. Next we proceed to prove (iii). For a nonzero simple vector $v \in \bigwedge^{k+1} T(\Gamma\backslash D)$ of unit norm, $v = v_1 \wedge \cdots \wedge v_{k+1}$ for some orthonormal vectors $v_1, \ldots, v_{k+1} \in T(\Gamma\backslash D)$. Then, for $w \in T(\Gamma\backslash D)$,

$$\begin{aligned} R(v,\bar{v})(w,\bar{w}) &= \sum_{i=1}^{k+1} R(v_i,\overline{v_i})(w,\bar{w}) \\ &= \sum_{i=1}^{k} R(v_i,\overline{v_i})(w,\bar{w}) + R(v_{k+1},\overline{v_{k+1}})(w,\bar{w}) \\ &\leq R(v',\overline{v'})(w,\bar{w}), \end{aligned}$$

where $v' = v_1 \wedge \cdots \wedge v_k \in \bigwedge^k T(\Gamma\backslash D)$. This implies that the number of negative eigenvalues of $R(v,\bar{v})(\ ,\ )$ is greater than or equal to that of $R(v',\overline{v'})(\ ,\ )$, which leads to (iii) upon taking the minimum over v and v'. Statement (iv) follows easily from Lemma 6 and statements (ii) and (iii) of Proposition 1. Finally statement (v) follows as in (ii) from the definition of s_k and the fact that D is of seminegative bisectional curvature. □

REMARK. Statements (i) and (iv) of Proposition 1 imply that, in the case when $Y=\Gamma\backslash D$, the upper bound of $\dim \mathrm{Hol}_k(X,\Gamma\backslash D)$ given by the Main Theorem in (1.1) is always at least as good as that given by Noguchi [No] in (3.1) and (3.2).

Finally we will show that, for the four types of classical bounded symmetric domains D and $2\le k\le l(D)$ (as long as $l(D)>1$), the upper bound of $\dim \mathrm{Hol}_k(X,\Gamma\backslash D)$ in (1.1) is actually better than that of Noguchi [No] given in (3.2). The classical bounded symmetric domains of type I, II, III, and IV are denoted by $D^{\mathrm{I}}_{p,q}$, D^{II}_n, D^{III}_n, and D^{IV}_n respectively, and we refer the reader to [Mo, Chap. 4] for background materials on them.

Type I: $D^{\mathrm{I}}_{p,q}:=\{Z\in M(p,q;\mathbb{C}): I_q-\bar{Z}^tZ>0\}$. Here $M(p,q;\mathbb{C})$ denotes the space of $p\times q$ complex matrices, and I_q denotes the $q\times q$ identity matrix. The holomorphic tangent space of $D^{\mathrm{I}}_{p,q}$ at the origin o is identified naturally with $M(p,q;\mathbb{C})$, that is, $T^{1,0}_o(D^{\mathrm{I}}_{p,q})\cong M(p,q;\mathbb{C})$. For $X,Y\in M(p,q;\mathbb{C})$, the curvature tensor of the Bergman metric of $D^{\mathrm{I}}_{p,q}$ gives

$$R(X,\bar{X})(Y,\bar{Y})=-\|X\bar{Y}^t\|^2-\|X^t\bar{Y}\|^2 \tag{3.3}$$

(cf. [Mo, p. 84]). From (3.3) or [Mo, Chap. 4, (3.2), Prop. 1], we have

$$\dim(\mathbb{C}X)^{\perp}=(p-r(X))(q-r(X)), \tag{3.4}$$

where $r(X)$ denotes the rank of X. Thus $\dim(\mathbb{C}X)^{\perp}$ is maximum if and only if $r(X)=1$, and we have

$$\lambda(\Gamma\backslash D^{\mathrm{I}}_{p,q})=(p-1)(q-1) \quad \text{for any } \Gamma. \tag{3.5}$$

For $k\ge 2$ and nonzero $X=X_1\wedge\cdots\wedge X_k\in\bigwedge^k T_o(D^{\mathrm{I}}_{p,q})$ of unit norm, where $X_1,\dots,X_k\in M(p,q;\mathbb{C})$, we consider two cases.

Case (a): $\mathrm{span}(X_1,\dots,X_k)$ contains a matrix $X'\in M(p,q;\mathbb{C})$ with $r(X')\ge 2$. In this case we may assume that X' is of unit length, and we can extend it to an orthonormal basis $\{X'_1=X',X'_2,\dots,X'_k\}$ of $\mathrm{span}(X_1,\dots,X_k)$. Then, for $Y\in M(p,q;\mathbb{C})$,

$$\begin{aligned} R(X,\bar{X})(Y,\bar{Y}) &= \sum_{i=1}^{k} R(X'_i,\overline{X'_i})(Y,\bar{Y}) \\ &\le R(X'_1,\overline{X'_1})(Y,\bar{Y}). \end{aligned} \tag{3.6}$$

This implies that

$$\begin{aligned} &\text{number of zero eigenvalues of } R(X,\bar{X})(\ ,\ ) \\ &\quad\le \dim(\mathbb{C}X'_1)^{\perp} \\ &\quad= (p-r(X'_1))(q-r(X'_1)) \quad \text{(by (3.4))} \\ &\quad\le (p-2)(q-2). \end{aligned} \tag{3.7}$$

Case (b): span$(X_1, \ldots, X_k)$ contains only matrices of rank ≤ 1. The isotropy subgroup of $\mathrm{Aut}(D^{\mathrm{I}}_{p,q})$ at o is isomorphic to $U(p) \times U(q)$, and each $U \times V \in U(p) \times U(q)$ acts isometrically on $M(p,q;\mathbb{C}) \cong T_o(D^{\mathrm{I}}_{p,q})$ via the linear transformation

$$X \to UXV \quad \text{for } X \in M(p,q;\mathbb{C}), \tag{3.8}$$

leaving the curvature tensor invariant (cf. e.g. [Mo, Chap. 4, (2.2), Lemma 1]). For $1 \leq i \leq p$ and $1 \leq j \leq q$, we denote by E_{ij} the $p \times q$ matrix whose (i,j)th entry is 1 and whose other entries are all 0. It is easy to see that for suitable $U \times V \in U(p) \times U(q)$, the linear transformation in (3.8) maps span(X_1, X_2) bijectively onto span(E_{11}, E_{12}) or span(E_{11}, E_{21}). In the first case, we have, as in case (a),

$$\begin{aligned}\text{number of zero eigenvalues of } R(X,\bar{X})(\ ,\ ) &\leq \dim(\mathbb{C}E_{11})^{\perp} \cap (\mathbb{C}E_{12})^{\perp} \\ &= (p-1)(q-2),\end{aligned} \tag{3.9}$$

since one can easily see from (3.3) that $(\mathbb{C}E_{11})^{\perp} \cap (\mathbb{C}E_{12})^{\perp}$ consists of $p \times q$ matrices whose entries in the first row and the first two columns are all zero. In the second case, when $U \times V$ maps span(X_1, X_2) to span(E_{11}, E_{21}), an analysis similar to the first case gives

$$\text{number of zero eigenvalues of } R(X,\bar{X})(\ ,\ ) \leq (p-2)(q-1). \tag{3.10}$$

Combining (3.7), (3.9), and (3.10), one sees that in all cases

$$\begin{aligned}&\text{number of zero eigenvalues of } R(X,\bar{X})(\ ,\ ) \\ &\leq \max\{(p-1)(q-2), (p-2)(q-1)\} \\ &< (p-1)(q-1),\end{aligned} \tag{3.11}$$

which, together with Lemma 6, Proposition 1 (i), (v), and (3.5), implies that

$$\dim(\Gamma \backslash D^{\mathrm{I}}_{p,q}) - s_k < l(D^{\mathrm{I}}_{p,q}) \quad \text{for } 2 \leq k \leq (p-1)(q-1),\ p,q \geq 2. \tag{3.12}$$

We now give the comparisons between $\dim(\Gamma \backslash D) - s_k$ and $l(D)$ for the other three types of classical bounded symmetric domains without giving details of computations, which are similar to those for $D^{\mathrm{I}}_{p,q}$.

Type II: $D^{\mathrm{II}}_n := \{Z \in D^{\mathrm{I}}_{n,n} : Z^t = -Z\}$. In this case, $l(D^{\mathrm{II}}_n) = \frac{1}{2}(n-2)(n-3)$ and $\dim(\Gamma \backslash D^{\mathrm{II}}_n) - s_2 = \frac{1}{2}(n-3)(n-4)$. For $2 \leq k \leq \frac{1}{2}(n-2)(n-3)$ with $n > 3$, we have

$$\begin{aligned}\dim(\Gamma \backslash D^{\mathrm{II}}_n) - s_k \leq \dim(\Gamma \backslash D^{\mathrm{II}}_n) - s_2 &= \tfrac{1}{2}(n-3)(n-4) \\ &< \tfrac{1}{2}(n-2)(n-3) = l(D^{\mathrm{II}}_n).\end{aligned}$$

Type III: $D^{\mathrm{III}}_n := \{Z \in D^{\mathrm{I}}_{n,n} : Z^t = Z\}$. In this case, $l(D^{\mathrm{III}}_n) = \frac{1}{2}n(n-1)$ and $\dim(\Gamma \backslash D^{\mathrm{III}}_n) - s_2 = \frac{1}{2}(n-1)(n-2)$. For $2 \leq k \leq \frac{1}{2}n(n-1)$ with $n \geq 3$, we have

$$\begin{aligned}\dim(\Gamma \backslash D^{\mathrm{III}}_n) - s_k \leq \dim(\Gamma \backslash D^{\mathrm{III}}_n) - s_2 &= \tfrac{1}{2}(n-1)(n-2) \\ &< \tfrac{1}{2}n(n-1) = l(D^{\mathrm{III}}_n).\end{aligned}$$

Type IV:

$$D_n^{\mathrm{IV}} := \{Z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \|Z\|^2 < 2 \quad \text{and} \quad \|Z\|^2 < 1 + |\tfrac{1}{2} \textstyle\sum_{1 \le i \le n} z_i^2|^2\}.$$

In this case, $l(D_n^{\mathrm{IV}}) = 1$ (cf. [Mo, Chap. 4, (3.2), Prop. 4]), and thus by Proposition 1, $\dim(\Gamma \backslash D_n^{\mathrm{IV}}) - s_k = l(D_n^{\mathrm{IV}})$ for all k.

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D. Goloff
Department of Mathematical
and Physical Sciences
Northwestern State University
of Louisiana
Natchitoches, LA 71457

W.-K. To
Department of Mathematics
National University of Singapore
10 Kent Ridge Crescent
Singapore 0511