Soul-Preserving Submersions

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In this note, we investigate the structure of Riemannian submersions π: $M \to N$ from open manifolds M with nonnegative sectional curvature K. By O'Neill's formula [5], N also has nonnegative curvature. In fact, most—but not all, see Question 2.4—open manifolds N with $K \geq 0$ arise in this fashion, by taking M to be a Riemannian product $M' \times P^k$, where P^k is diffeomorphic to $\mathbf{R}^k$.

The starting point is the observation that if N is open, then the pre-image of a soul is a totally convex submanifold M' of M. It follows that if π has compact fibers, then it is soul-preserving. Moreover, the structure of π is essentially determined by its restriction to the tangent and normal bundles of the soul Σ_M of M. This is used to derive a classification of the metric fibrations of $S^n \times \mathbf{R}^k$ with compact fibers, which turn out to be homogeneous —that is, generated by the action of a group of isometries—if $n \neq 15$. The curvature of the base is actually indicative of the metric structure of open manifolds with $K \geq 0$ in general: topologically, the base is a nontrivial vector bundle over the soul with positively curved fibers. This is true for any complete, noncompact M with $K \geq 0$: If every plane orthogonal to Σ_M has zero curvature, then M splits as a metric product $\Sigma_M \times P^l$, at least locally.

We mention two further applications. The first is that positively curved open manifolds admit no metric fibrations. The second is that 1-dimensional Riemannian fibrations of locally symmetric (open) spaces with $K \geq 0$ are homogeneous, unless perhaps the quotient space is trivial in the sense that it is isometric to a product of a compact manifold with Euclidean space.

1. Convexity and Submersions

The reader is referred to [1] for facts about open manifolds of nonnegative curvature that will be used freely. We adopt the notation of [4] for the basic geometric invariants of Riemannian foliations. Thus, the foliation $\mathfrak{F}$ on the (complete) manifold M determines an orthogonal splitting $TM = \Delta^h \oplus \Delta^v$ of the tangent bundle into so-called horizontal and vertical subbundles, where

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Δ^v is tangent to the leaves. The integrability tensor is the 2-form A on Δ^h with values in Δ^v given by

$$A_X Y = \tfrac{1}{2}[X, Y]^v = \overset{v}{\nabla}_X Y, \tag{1.1}$$

and the second fundamental form of $\mathfrak{F}$ is the 1-form S on Δ^h with values in the space of self-adjoint operators of Δ^v, given by

$$S_X T = -\overset{v}{\nabla}_T X. \tag{1.2}$$

If X is a *basic* field (i.e., the horizontal lift of a vector field on the local quotient), then $[X, T]$ is vertical for any vertical field T, so that by (1.1),

$$\overset{h}{\nabla}_T X = \overset{h}{\nabla}_X T = -A_X^* T, \tag{1.3}$$

where A_X^* is the pointwise adjoint of A_X.

Let c denote a horizontal geodesic. The horizontal lifts of the local projections of c yield diffeomorphisms h^t from neighborhoods of $c(0)$ in the leaf to corresponding neighborhoods of $c(t)$, called "holonomy displacements" after [4]. For vertical $u \in M_{c(0)}$, $h_*^t u = J(t)$, where J is a nowhere-zero vertical Jacobi field along c with $J(0) = u$. By (1.2) and (1.3), we have

$$J' = -A_{\dot c}^* J - S_{\dot c} J. \tag{1.4}$$

As an easy application, we have the following generalization of [4, (2.8)] to the nonconstant curvature case.

LEMMA 1.5. *Let Y be a horizontally parallel vector field along a horizontal geodesic c. Then*

$$(A_{\dot c} Y)'^v = 2 S_{\dot c} A_{\dot c} Y + R^v(Y, \dot c)\dot c.$$

Proof. Let J be a holonomy Jacobi field as in (1.4). Then

$$\begin{aligned}
\langle R(Y,\dot c)\dot c, J\rangle &= \langle R(J,\dot c)\dot c, Y\rangle = -\langle J'', Y\rangle \\
&= \langle (S_{\dot c} J)', Y\rangle + \langle (A_{\dot c}^* J)', Y\rangle \\
&= -\langle S_{\dot c} J, Y'\rangle + \langle A_{\dot c}^* J, Y\rangle' \\
&= -\langle S_{\dot c} J, A_{\dot c} Y\rangle + \langle J, A_{\dot c} Y\rangle' \\
&= -\langle J, S_{\dot c} A_{\dot c} Y\rangle + \langle J', A_{\dot c} Y\rangle + \langle J, (A_{\dot c} Y)'\rangle \\
&= \langle (A_{\dot c} Y)' - 2 S_{\dot c} A_{\dot c} Y, J\rangle.
\end{aligned}$$

The lemma follows, since for any t_0 there exists a vertical basis of holonomy fields that is orthonormal at t_0. □

A similar argument shows that, more generally, if U and V are horizontally parallel along c, then

$$(A_U V)'^v = S_{\dot c} A_U V + S_U A_{\dot c} V - S_V A_{\dot c} U - R^v(U, V)\dot c. \tag{1.6}$$

The details are left to the reader (alternatively, this can be deduced by inserting appropriate curvature terms in Lemmas 2.3 and 2.5 of [4]).

Throughout the remainder of the paper, M will denote an open (i.e., a noncompact, connected, and complete) manifold with nonnegative sectional curvature, $\pi: M \to N$ a Riemannian submersion, so that $K_N \geq 0$. We are mainly interested in the case when N is noncompact. Under these hypotheses, we have the following theorem.

THEOREM 1.7. *Let $\pi: M \to N$ be a Riemannian submersion between open manifolds with $K \geq 0$. If Σ_N denotes a soul of N, then $\pi^{-1}(\Sigma_N)$ is a totally convex submanifold of M.*

Proof. In the special case where M is flat Euclidean space, this was proved in [2]. The argument actually goes through with no major changes in the general case, so we merely sketch an outline. Given $p \in N$ and a ray c starting at p, any horizontal lift $\tilde{c}$ of c is again a ray in M; furthermore, for $t > 0$, $\pi(B_t(\tilde{c}(t)) = B_t(c(t))$ because π does not increase distances. The basic soul construction in N can thus be "lifted" to M. More precisely, for $\tilde{p} \in \pi^{-1}(p)$, set

$$C_{\tilde{p}} = M \setminus \bigcup_{t>0} B_t(\tilde{c}(t)), \qquad \tilde{C} = \bigcap_{\tilde{p} \in \pi^{-1}(p)} C_{\tilde{p}}, \qquad C = N \setminus \bigcup_{t>0} B_t(c(t)).$$

It is straightforward to check that $\pi^{-1}(C) = \tilde{C}$. This remains true if one intersects over all rays emanating from p. One thus obtains a compact totally convex set D in N, and a closed totally convex set $\tilde{D}$ in M, with $\pi^{-1}(D) = \tilde{D}$; π preserves both interior and boundary of these sets. Once again, basic distance properties of π ensure that $\pi^{-1}(D^r) = \tilde{D}^r$, where the superscript denotes those points at distance $\geq r$ from the boundary. The statement now clearly follows. □

If π has compact fiber, then $\pi^{-1}(\Sigma_N)$ is a compact totally convex submanifold without boundary of M. In general, compact totally convex submanifolds need not be souls—for example, when M is a paraboloid. In our case, however, although $\pi^{-1}(\Sigma_N)$ is not strictly speaking obtained via the soul construction of [1], there exists a filtration of M by compact totally convex sets such that $\pi^{-1}(\Sigma_N)$ is obtained by applying the soul construction to them. By a striking result of Perelman [6], there exists a distance nonincreasing retraction $\rho: M \to \pi^{-1}(\Sigma_N)$ which is a Riemannian submersion given by metric projection in a neighborhood of $\pi^{-1}(\Sigma_N)$. In particular, $\pi^{-1}(\Sigma_N)$ is isometric to any soul of M, and we shall therefore also call it a soul. Observe, though, that the above retraction still exists if $\pi^{-1}(\Sigma_N)$ is noncompact. This implies that if E is a parallel vector field along a geodesic of $\pi^{-1}(\Sigma_N)$ with $E(0)$ orthogonal to $\pi^{-1}(\Sigma_N)$, then the rectangle

$$(t, s) \mapsto \exp(tE(s))$$

is flat and totally geodesic. There are two more consequences that will be used later. First, parallel translation along geodesics of $\pi^{-1}(\Sigma_N)$ preserves ray directions. Second, there are as many rays emanating pointwise from Σ_N as from its pre-image: we have already noticed that horizontal lifts of rays are again rays; conversely, if c is a ray from $\pi^{-1}(\Sigma_N)$, then we claim that $\pi\circ c$ must be a ray from Σ_N. Otherwise, there would exist, for some $t>0$, a geodesic γ from $\pi\circ c(t)$ to $\pi\circ c(0)$ of length $<t$. If $\tilde{\gamma}$ is the horizontal lift of γ starting at $c(t)$, then the endpoint of $\tilde{\gamma}$ lies in $\pi^{-1}(\Sigma_N)$. But this endpoint cannot be p (since c is a ray), nor can it be different from p, because the above retraction ρ is characterized by $\rho(c(t))=c(0)$ for any geodesic c emanating from and perpendicular to $\pi^{-1}(\Sigma_N)$.

2. Global Applications

An immediate consequence of the proof of Theorem 1.7 is the following.

THEOREM 2.1. *An open manifold M of positive curvature admits no Riemannian fibrations.*

Proof. Suppose $\pi: M\to N$ is a Riemannian submersion, and consider a soul $\{p\}$ of N. By the final remarks of Section 1, any plane spanned by a vector tangent to $\pi^{-1}(p)$ and a vector orthogonal to it has zero curvature. Thus, the pre-image of p is 0-dimensional, and π is a covering map. By [1], π is an isometry. □

Perhaps the simplest nontrivial example of an open manifold with nonnegative curvature is $M^4=S^3\times_{S^1}\mathbf{R}^2$, where S^1 acts diagonally, on S^3 via the Hopf fibration, and on $\mathbf{R}^2$ by rotations. Since this action is by isometries, the natural projection $\pi: S^3\times\mathbf{R}^2\to M^4$ is a Riemannian submersion. Theorem 1.7 says that π restricts to a submersion between souls. This is easily verified directly in this case, since the restriction is just the Hopf fibration $S^3(1)\to S^2(\frac{1}{2})$. One thus has the commutative diagram

$$\begin{array}{ccc} S^3\times\mathbf{R}^2 & \xrightarrow{\pi} & M^4 \\ \downarrow & & \downarrow \\ S^3 & \xrightarrow{\pi} & S^2(\frac{1}{2}), \end{array}$$

where the vertical maps, given by metric projections, are Riemannian submersions. What is noteworthy here is that the restriction to the soul actually determines all of π, as the next theorem shows.

THEOREM 2.2. *Let $\pi: S^n\times\mathbf{R}^k\to M$ be a Riemannian submersion with compact fibers from a metric product of some Euclidean sphere with Euclidean space. Then π is homogeneous if $n\neq 15$. More precisely, M is isometric to $S^n\times_G\mathbf{R}^k$, where $G=\mathrm{SO}(2)$ or S^3 acts diagonally, via a Hopf fibration on S^n, and by an orthogonal representation on $\mathbf{R}^k$.*

Proof. By Theorem 1.7, π restricts to $S^n\to\Sigma$, where Σ is a soul of M. For $n\neq 15$, this restriction is a Hopf fibration according to [4], and thus homogeneous under the action of $G=\mathrm{SO}(2)$ or S^3 on S^n. We assume $G=S^3$, so that $n=4l+3$. The other case is similar, but easier. X and Y will denote basic fields on S^n, as well as their natural lifts to $S^n\times\mathbf{R}^k$ along the soul $S^n\times 0$; U and V will denote sections of the normal bundle of the soul, which are basic along fibers in $\pi^{-1}(\Sigma)$. If I, J, K denote the canonical almost complex structures on $\mathbf{R}^{4(l+1)}$ with N the unit normal field along $S^{4l+3}\subset\mathbf{R}^{4l+4}$, then IN, JN, and KN are Killing fields on the sphere that form an orthonormal basis for the fibers of $S^{4l+3}\to\mathbb{H}P^l$. The corresponding Lie algebra coincides with the Lie algebra $\mathfrak{g}$ of integrability fields spanned by all A_XY, X and Y basic, since $A_XIX=-IN$, and similarly for J, K.

By O'Neill's formula [5], the curvature tensors of the total space and the base are related by

$$\begin{aligned}\pi_*R(X,Y)U &= R_M(\pi_*X,\pi_*Y)\pi_*U\\ &\quad+\pi_*(A_X^*A_YU-A_Y^*A_XU-2A_U^*A_XY). \qquad (2.3)\end{aligned}$$

Since the curvature of a plane spanned by two vectors, one tangent and the other orthogonal to a soul, is zero, (2.3) becomes

$$R_M(\pi_*X,\pi_*Y)\pi_*U=2\pi_*A_U^*A_XY.$$

In particular, for $T=A_XY$, A_U^*T is basic. It follows that if c is an integral curve of T then $U\circ c(t)=P_t\circ u(t)$, where P_t is parallel translation along c and u is the curve in the tangent space

$$E=(c(0)\times\mathbf{R}^k)_{(c(0),0)}\subset(S^{4l+3}\times\mathbf{R}^k)_{(c(0),0)}$$

given by

$$u(t)=e^{tA_T}u(0).$$

Here, A_T is the skew-adjoint operator $A_Tu:=A_u^*T_{c(0)}$. Identify E with $\mathbf{R}^k$. The map $T\mapsto A_T$ from $\mathfrak{g}$ to the Lie algebra of SO(k) is a Lie algebra homomorphism, since

$$\overset{h}{\nabla}_{T_i}\overset{h}{\nabla}_{T_j}U-\overset{h}{\nabla}_{T_j}\overset{h}{\nabla}_{T_i}U-\overset{h}{\nabla}_{[T_i,T_j]}U=R(T_i,T_j)U=0.$$

It follows that if $h\colon S^3\to\mathrm{SO}(k)$ denotes the induced Lie group homomorphism, then the fiber through $(p,u)\in S^{4l+3}\times\mathbf{R}^k$ is $\{g(p),h(g)u\mid g\in S^3\}$ (here, $p:=c(0)$). It remains to check that the fiber through (q,u) can be described in the same way if $q\neq p$. Evidently, we may assume that p and q can be joined by a horizontal geodesic. Since the Hopf fibration has totally geodesic fibers, (1.6) implies that

$$(A_XY\circ c)'^v=(A_UV\circ c)'^v=0$$

for parallel U, V, and for horizontally parallel X, Y along c. Thus, $A_U^*A_XY$ is parallel along c, and the theorem follows. □

It is tempting to conjecture that any open manifold with $K\geq 0$ can be realized as the result of a submersion from a metric product, as in the case of $S^3\times\mathbf{R}^2\to M^4$ above. One must be a little more precise, of course, since any

M can (for example) be viewed as the projection of $M \times \mathbf{R}$ onto the first factor. In light of Theorem 1.7, the most plausible version is as follows.

QUESTION 2.4. Given any open M with $K \geq 0$ and soul Σ, does there exist a metric product $M_0 \times P^k$ of nonnegative curvature, and a Riemannian submersion $\pi: M_0 \times P^k \to M$ with $\pi^{-1}(\Sigma) = M_0 \times 0$? (Here, P^k denotes $\mathbf{R}^k$ together with some nonnegatively curved metric.)

Notice that M_0 need not, in general, be a soul. Consider, for instance, the Riemannian product $M_0 = N \times \mathbf{R}$, where N is a compact manifold with Killing field X. If D is the standard coordinate vector field on $\mathbf{R}$, and ∂_θ the polar coordinate vector field on $\mathbf{R}^2$, then (X, D, ∂_θ) generates a Riemannian fibration $\pi: M_0 \times \mathbf{R}^2 \to M$, and $M_0 = \pi^{-1}(\Sigma_M)$ is noncompact. In [9] it was shown that the standard metric on $M^4 = S^3 \times_{S^1} \mathbf{R}^2$ could be deformed in some compact neighborhood C away from the soul, so that the resulting metric still has nonnegative curvature, but those fibers over the soul that intersect C are no longer rotationally symmetric. We use this fact to give a negative answer to Question 2.4 (cf. [9], where a weaker assertion is deduced). So suppose there exists a Riemannian submersion from some product $M_0 \times P^2 \to M$, where M now denotes $S^3 \times_{S^1} \mathbf{R}^2$ with the canonical metric deformed on the compact set C. Let u be a vector orthogonal to M_0 at some p, and denote by J the almost complex structure on the normal bundle of M_0 in $M_0 \times P^2$. By (2.3), $A_u Ju \neq 0$. In fact,

$$\langle A_u Ju, A_x y \rangle = -\omega(\pi_* x, \pi_* y),$$

where ω is the volume form of $\Sigma_M = S^2(1/2)$. Let ∂_θ denote the polar coordinate vector field on P^2, and set $T_p := A_u Ju$. It is straightforward to check with (1.4) that the vertical space at $(p, \exp(tu))$ is spanned by

$$(T_p, -|A_u Ju|^2 \partial_\theta |_{\exp(tu)})$$

together with a (possibly trivial) subspace of $(M_0)_p \times 0_{\exp(tu)}$. Thus, if $\bar{\partial}_\theta$ is the corresponding polar coordinate vector field on the fibers of $M \to \Sigma_M$, then

$$|\bar{\partial}_\theta|^2 = |(0, \partial_\theta)^h|^2 = \frac{|\partial_\theta|^2}{1 + |A_u Ju|^2 |\partial_\theta|^2}. \tag{2.5}$$

Since C does not intersect the soul, all fibers of $M \to \Sigma_M$ have curvature 3 (= curvature of the standard metric) along the zero section. Thus,

$$|A_u Ju|^2 = \tfrac{1}{3}(3 - K_{P^2}(0))$$

is constant along M_0. But $\bar{\partial}_\theta$, when restricted to any fiber that does not intersect C, is Killing. It follows from (2.5) that ∂_θ is Killing on P^2. Again by (2.5), $\bar{\partial}_\theta$ is then Killing on M (cf. [9, Lemma 1.7]). This is impossible, so M cannot be constructed as the result of a submersion from a metric product.

The fact that the fibers of $S^3 \times_{S^1} \mathbf{R}^2 \to S^2$ have positive curvature along the zero section is rather suggestive: any trivial bundle admits, of course, a nonnegatively curved metric with flat fibers. This is no longer true if the bundle is nontrivial.

PROPOSITION 2.6. *Let M be a simply connected open manifold of nonnegative curvature with soul Σ. If every 2-plane orthogonal to Σ has zero curvature, then M is isometric to a Riemannian product $\Sigma \times P^k$.*

REMARK. By a standard argument, a local splitting still holds when M is not simply connected.

Proof of Proposition 2.6. We show that, under the above hypothesis, the normal bundle of Σ is flat. The conclusion then follows from [7]. By [6], the metric projection onto the soul is a Riemannian submersion on a small enough neighborhood of Σ. If ∂_r denotes the gradient of the distance function from Σ on this neighborhood, then it is straightforward to check that, for basic X,

$$\nabla_X \partial_r = \nabla_{\partial_r} X = 0. \tag{2.7}$$

Moreover, if Y also is basic and γ is an integral curve of ∂_r with $\gamma(0) \in \Sigma$, then $A_X Y \circ \gamma$ is the Jacobi field J with $J(0) = 0$ and $J'(0) = -\frac{1}{2} R(x, y)\dot{\gamma}(0)$, where $x = X|_{\gamma(0)}$, and similarly for y (cf. [8]). Since $R(X, \partial_r)Y = -\nabla_{\partial_r} \nabla_X Y$ by (2.7), we deduce that for any x, y tangent to Σ and u orthogonal to Σ,

$$R(x, y)u = 2R(x, u)y. \tag{2.8}$$

Let v be orthogonal to Σ. Consider $e = \alpha x + \beta u$ and $f = \gamma y + \delta v$ with $\alpha, \beta, \gamma, \delta \in \mathbf{R}$. By repeated use of (2.8), and recalling that planes spanned by a vector tangent to Σ and a vector orthogonal to it have zero curvature, we obtain

$$\begin{aligned}\langle R(e, f), f, e\rangle &= (\alpha\gamma)^2 \langle R(x, y)y, x\rangle \\ &\quad + 3\alpha\beta\gamma\delta \langle R(x, y)v, u\rangle + (\beta\delta)^2 \langle R(u, v)v, u\rangle.\end{aligned}$$

This expression is nonnegative for all scalars only if

$$\langle R(x, y)y, x\rangle \langle R(u, v)v, u\rangle - \tfrac{9}{4} \langle R(x, y)v, u\rangle^2 \geq 0.$$

Since $\langle R(u, v)v, u\rangle = 0$ by assumption, so does $\langle R(x, y)v, u\rangle$. □

In light of Theorem 2.2, one may conjecture that Riemannian fibrations of open locally symmetric spaces with $K \geq 0$ are always homogeneous. The result for $S^n \times \mathbf{R}^k$, however, relied heavily on the classification of fibrations on spheres, and there is (as yet) no comparable result for compact locally symmetric spaces, not even in the case of 1-dimensional fibers. The following fact is therefore noteworthy.

THEOREM 2.9. *Let $\pi: M \to N$ be a Riemannian fibration with 1-dimensional fibers of an open, nonnegatively curved locally symmetric space M. Then π is homogeneous, or N splits (locally) isometrically as $\Sigma_N \times \mathbf{R}^l$.*

Proof. We may assume without loss of generality that M is simply connected, so that $M = \Sigma_M \times \mathbf{R}^k$ (see [1]), and that the fibers of π are connected, so that N also is simply connected by the long exact homotopy sequence for fibrations. With notation as in Theorem 2.2,

$$\langle A_U V, A_X Y\rangle = -\tfrac{1}{2}\langle R^N(\pi_* X, \pi_* Y)\pi_* U, \pi_* V\rangle \tag{2.10}$$

is constant along each fiber in $\pi^{-1}(\Sigma_N)$. But so is $|A_U V|$, and hence also $|A_X Y|$, unless perhaps $A_U V = 0$ for all U, V along some fiber F. If this is the case, consider a horizontal geodesic c from F in $\pi^{-1}(\Sigma_N)$, and extend U, V to horizontally parallel fields along c. Then

$$(A_U V)'^{v} = S_{\dot c} A_U V \tag{2.11}$$

by (1.6). Thus, $A_U V = 0$ for all U, V orthogonal to $\pi^{-1}(\Sigma_N)$. By (2.10) and [7] (or alternatively by Proposition 2.6 and O'Neill's formula), N splits as $\Sigma_N \times P^l$. Using (1.4), it is easy to see that the vertical space along $t \mapsto \exp(tU)$ is tangent to $\pi^{-1}(\Sigma_N)$ (in the decomposition $M = \pi^{-1}(\Sigma_N) \times \mathbf{R}^l$) for all t. This implies that P^l is flat, so N is isometric to $\Sigma_N \times \mathbf{R}^l$.

Assume then that $|A_X Y|$ is constant along any fiber in $\pi^{-1}(\Sigma_N)$. Let κ denote the mean curvature form of π, that is, the horizontal 1-form given by

$$\kappa(X) = \langle S_X T, T\rangle = \langle \nabla_T T, X\rangle,$$

where T is a unit vector field spanning the fiber. We first show that κ is closed. It is straightforward to compute that, in general,

$$d\kappa(X, Y) = -2 \operatorname{div} A_X Y,$$

which in our case is zero, since A has constant norm. Next, by Lemma 1.5, if Y is horizontally parallel (of unit length) along the horizontal geodesic c in $\pi^{-1}(\Sigma_N)$, then

$$\begin{aligned} |A_{\dot c} Y|^{2\prime} &= 4\langle S_{\dot c} A_{\dot c} Y, A_{\dot c} Y\rangle + 2\langle R(Y, \dot c)\dot c, A_{\dot c} Y\rangle \\ &= 4\langle S_{\dot c} T, T\rangle |A_{\dot c} Y|^2 + 2\langle R(Y, \dot c)\dot c, Y'\rangle \\ &= 4\langle S_{\dot c} T, T\rangle |A_{\dot c} Y|^2 + K'_{\dot c, Y}, \end{aligned}$$

since M is locally symmetric. Differentiating in the fiber direction yields $T\langle S_X T, T\rangle = 0$, so that κ is basic. This in turn implies that $d\kappa(X, T) = 0$. Thus, κ is closed. The same argument given in [3] for the constant curvature case now implies that π is homogeneous along the pre-image of Σ_N. In fact, a Killing field Z generating the vertical space can be explicitly constructed as follows: Let f be a function on $\pi^{-1}(\Sigma_N)$ such that $\kappa = df$, and set $L = e^{-f}$. Thus, $XL = -L\kappa(X)$. It is straightforward to check that $Z := LT$ is a Killing field on $\pi^{-1}(\Sigma_N)$. Moreover, $A_U^* Z$ is basic. As for spheres, it remains to check that for a horizontal geodesic c and parallel fields U, V along c orthogonal to $\pi^{-1}(\Sigma_N)$, $\langle A_U^* Z, V\rangle = \langle Z\circ c, A_U V\rangle$ is constant. But $Z\circ c$ is Jacobi, so

$$\begin{aligned}\langle Z\circ c, A_U V\rangle' &= \langle (Z\circ c)', A_U V\rangle + \langle Z\circ c, (A_U V)'\rangle \\ &= \langle -S_{\dot c} Z, A_U V\rangle + \langle Z\circ c, S_{\dot c} A_U V\rangle \\ &= 0,\end{aligned}$$

where we have used (1.4) and (2.11) in the second line, and the fact that $S_{\dot c}$ is self-adjoint in the last line. □

References

[1] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature,* Ann. of Math. (2) 96 (1972), 413–443.

[2] I.-P. Chu, *Riemannian fibrations of Euclidean spaces,* thesis, SUNY, Stony Brook, 1981.

[3] D. Gromoll and K. Grove, *One-dimensional metric foliations in constant curvature spaces,* Differential geometry and complex analysis, pp. 165–168, Springer, Berlin, 1985.

[4] ———, *The low-dimensional metric foliations of Euclidean spheres,* J. Differential Geom. 28 (1988), 143–156.

[5] B. O'Neill, *The fundamental equations of a submersion,* Michigan Math. J. 13 (1966), 459–469.

[6] G. Perelman, *Proof of the soul conjecture of Cheeger and Gromoll,* preprint.

[7] M. Strake, *A splitting theorem for open nonnegatively curved manifolds,* Manuscripta Math. 61 (1988), 315–325.

[8] M. Strake and G. Walschap, *Σ-flat manifolds and Riemannian submersions,* Manuscripta Math. 64 (1989), 213–226.

[9] G. Walschap, *Nonnegatively curved manifolds with souls of codimension 2,* J. Differential Geom. 27 (1988), 525–537.

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