The PL Fibrators among Aspherical Geometric 3-Manifolds

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Results of [D6] indicate that manifolds in a surprisingly extensive collection act as PL fibrators. This paper adds evidence for a claim that most manifolds, in a psychological rather than a strictly mathematical sense, have this desirable attribute. Among closed, orientable 2-manifolds the exceptions are known to consist only of the 2-sphere and the torus. Among closed, orientable 3-manifolds, those having either hyperbolic, Sol, or $SL_2(R)$ geometric structure are all PL fibrators, as are those with infinite first homology which are connected sums of nonsimply connected manifolds, provided the fundamental groups are residually finite. A long advance in an effort to classify 3-manifolds with this feature, the main result here (Theorem 3.4) shows that an aspherical, virtually geometric 3-manifold is a PL fibrator if it is one in codimension 2. Similar in tone and next in importance, Theorem 2.10 shows that any closed manifold with $(k-1)$-connected compact universal cover is a codimension-k PL fibrator if it is one in codimension 2.

To explain what all this means, we begin by setting forth the notation and fundamental terminology to be employed throughout: M is a connected, orientable, PL $(n+k)$-manifold, B is a polyhedron, and $p: M \to B$ is a PL map such that each $p^{-1}b$ has the homotopy type of a closed, connected n-manifold. When N is a fixed orientable n-manifold, such a (PL) map $p: M \to B$ is said to be *N-like* if each $p^{-1}b$ collapses to an n-complex homotopy equivalent to N. (This PL tameness feature imposes significant homotopy-theoretic relationships, revealed in [D6, Lemma 2.4], between N and preimages of links in B.) One calls N a *codimension-k PL fibrator* if, for every orientable $(n+k)$-manifold M and N-like PL map $p: M \to B$, p is an approximate fibration. Finally, if N has this property for all $k > 0$, one simply calls N a *PL fibrator.*

Remarkably, at this stage of development only two types of manifolds are known not to be PL fibrators: those that already fail in codimension 2, and those that have a sphere as Cartesian factor. The codimension-2 situation, reviewed extensively in the introduction to [D5], is fairly well understood and is not treated here.

Earlier work showing certain manifolds to be PL fibrators typically require the fundamental groups to have no nontrivial, Abelian normal subgroups. That accounts for the richness of information available about connected

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sums (to which [D6] readily applies), a richness most evident in dimension 3, where an object expressed as a connected sum of at least two nonsimply connected, irreducible 3-manifolds is a PL fibrator if its fundamental group is residually finite and at least one of the factors in its (maximal) free product decomposition is infinite. However, while investigating aspherical 3-manifolds one frequently encounters fundamental groups that do contain Abelian normal subgroups, and the results of [D6] shed no light. This paper includes some techniques for circumventing the prohibition against these subgroups.

Generally, in the study of proper maps defined on manifolds such that all point preimages are closed manifolds, a central theme is to understand relationships among domain, image, and fiber(s). A satisfactory solution occurs when $p: M \to B$ is an approximate fibration, for then, just as with Hurewicz fibrations, there is a homotopy exact sequence developed by Coram-Duvall [CD1],

$$\cdots \to \pi_{i+1}(B) \to \pi_i(p^{-1}b) \to \pi_i(M) \to \pi_i(B) \to \cdots,$$

providing theoretically computable information about any one of these three objects when corresponding data about the other two are known. In order to exploit this easily, it becomes advantageous to develop natural conditions under which maps are automatically approximate fibrations. For any PL approximate fibration defined on a connected domain, the various point preimages have the same homotopy type. The natural condition treated here is the identification of manifolds N sustaining a simple converse—namely, every N-like PL map $p: M \to B$ is an approximate fibration—in which the hypothesis demands only that all point preimages are homotopy equivalent, in the aforementioned tame PL sense, to N.

It would have been reasonable to have defined "PL fibrator" in one of several alternate ways, each depending crucially on the fibers allowed with N-like maps $p: M \to B$. The most natural approaches, it seems, would require each $p^{-1}b$ to be either (1) homeomorphic to N or (2) homotopy equivalent to N. The somewhat artificial combination of the two chosen here deserves explanation. Although our preference is for (1), we hope eventually to use the PL setting for gaining insight about what occurs with the non-PL version—namely, for proper, continuous functions $M \to B$ having copies of N as point preimages. In that broader category S^n fails to be a fibrator in codimension $n+1$, the minimal possibility, shown by a map $f: S^n \times R^{n+1} \to R^{n+1}$ with $S^n \times \{0\}$ and $\{z\} \times (r \cdot S^n)$ (where $z \in S^n$ and $r > 0$) as its point preimages. Apparently f cannot be realized by a PL map, not if point preimages are genuine n-spheres, yet it can be with point preimages that collapse to copies of S^n [D4, Example 2.1]. This harmony of the latter class of maps with a foundational example accounts for the current definition.

We conclude these introductory remarks with a brief description of methodology. With an arbitrary PL map $p: M \to B$, significant benefits accrue upon investigating collapses $R: S' \to p^{-1}v$, where v denotes a vertex of B and S' the preimage of a star of v relative to B; p is an approximate fibration if R restricts to homotopy equivalences $p^{-1}c \to p^{-1}v$ for arbitrary points c in the corresponding link. When $p^{-1}c$ and $p^{-1}v$ are homotopically equivalent n-manifolds, this is often confirmed by verifying that $R|p^{-1}c$ induces H_n- and π_1-isomorphisms. (A certain hopfian property, imposed on fibers and defined in the next section, postulates that such isomorphisms ensure $R: p^{-1}c \to p^{-1}v$ is a homotopy equivalence.) The usual strategy is to assume that p restricts to approximate fibrations on $L' = p^{-1}(L)$, where L represents the link of an arbitrary vertex $v \in B$, to relate homology and homotopy data of fibers in L' with that of L' itself, and finally to compare the latter with the same data concerning S', for obviously the data of S' is isomorphic to that of $p^{-1}v$. The intricate step involves discerning relationships between L' and fibers in L'; see [D6] for a discussion of tools for this purpose.

1. Definitions

A *manifold* is understood to be connected, metric and boundaryless. When boundary occurs, the object will be called a *manifold with boundary.* Keep in mind the overriding assumption that all manifolds denoted M or N are orientable.

A proper map $p: M \to B$ between locally compact ANRs is called an *approximate fibration* if it has the following approximate homotopy lifting property: Given an open cover Ω of B, an arbitrary space X, and two maps f: $X \to M$ and $F: X \times I \to B$ such that $pf = F_0$, there exists a map $F': X \times I \to M$ such that $F_0' = f$ and pF' is Ω-close to F. The latter means: to each $z \times X \times I$ there corresponds $U_z \in \Omega$ such that $\{F(z), pF'(z)\} \subset U_z$.

A group Γ is *hopfian* if every epimorphism $\Psi: \Gamma \to \Gamma$ is an automorphism; it is *cohopfian* if every monomorphism $\Phi: \Gamma \to \Gamma$ is an automorphism. Two related concepts useful for sorting out fibrator properties are: Γ is *normally cohopfian* if every monomorphism $\Phi: \Gamma \to \Gamma$ with normal image is an automorphism, and Γ is *hyperhopfian* if every homomorphism $\psi: \Gamma \to \Gamma$ with $\psi(\Gamma)$ normal and $\Gamma/\psi(\Gamma)$ cyclic is an automorphism.

The (*absolute*) *degree* of a map $f: N \to N$, where N is a closed, connected, orientable n-manifold, is the nonnegative integer d such that the induced endomorphism of $H_n(N; Z) \cong Z$ amounts to multiplication by d, up to sign. A closed manifold N is *hopfian* if it is orientable and every degree-1 map $N \to N$ which induces a π_1-automorphism is a homotopy equivalence. The term aids in efficiently identifying approximate fibrations.

To streamline the applications envisioned, we write that a PL map p: $M \to B$ has Property $R^{\cong}$ if, for each $b \in B$, a retraction $R: U \to p^{-1}b$ defined on some open set $U \supset p^{-1}b$ induces π_1-isomorphisms $(R|\ )_\#: \pi_1(p^{-1}b') \to \pi_1(p^{-1}b)$ for all $b' \in B$ sufficiently close to b. If this property holds for one retraction R, then it holds for any retraction $U' \to p^{-1}b$ defined on another open set $U' \supset p^{-1}b$.

Let $f: X \to Y$ be a closed map and $m \in \{0, 1, 2, \ldots\}$. The symbol $\mathcal{H}^m[f]$ denotes the mth cohomology sheaf of f with integral coefficients.

A *homotopy* (*homology*) *n-sphere* is understood to be an n-manifold having the homotopy (homology) type of the n-sphere, S^n.

A 3-manifold is *virtually geometric* if it is finitely covered by a geometric one, meaning that the covering space is a connected sum of 3-manifolds which are either Haken or have some geometric structure. A 3-manifold N is *irreducible* if every PL 2-sphere in N bounds a 3-cell there, and N is *Haken* if it is irreducible and contains an incompressible (closed) surface. See [He] for a definition of "incompressible" and [S1] for more information about geometric structures. Finally, a 3-manifold is *virtually Haken* if it is finitely covered by a Haken 3-manifold.

2. Highly Connected Manifolds as PL Fibrators

We begin by spelling out an elementary connection between normally cohopfian groups and regular coverings.

PROPOSITION 2.1. *An aspherical, closed manifold N regularly covers itself, up to homotopy, if and only if $\pi_1(N)$ fails to be normally cohopfian.*

The proof is routine, for if $\Phi: \pi_1(N) \to \pi_1(N)$ is a monomorphism with normal image then the regular covering space of N corresponding to $\Phi(\pi_1(N))$ is homotopy equivalent to N.

LEMMA 2.2. *If X is a CW-complex such that $\pi_i(X) = 0$ for $1 < i \leq k$ and if the map $f: X \to X$ induces an isomorphism $\pi_1(X) \to \pi_1(X)$, then f also induces isomorphisms*

$$f_*: H_i(X) \to H_i(X) \quad \text{and} \quad f^*: H^i(X) \to H^i(X) \quad (i \leq k).$$

Proof. Build an Eilenberg-MacLane space $K = K(\pi_1(X), 1) \supset X$ by attaching cells of dimension at least $k+2$ to X. There is no obstruction to extending $f: X \to X$ to a map $F: K \to K$. Dimension restrictions pertaining to the attached cells cause the inclusion $X \to K$ to induce homotopy and homology isomorphisms for $i \leq k$. Since then F, like f, induces a fundamental group isomorphism, it is a homotopy equivalence; moreover, the consequence that $F_*: H_i(K) \to H_i(K)$ is an isomorphism shows $f_*: H_i(X) \to H_i(X)$ is one as well, provided $i \leq k$. The identical argument works for cohomology. □

PROPOSITION 2.3. *Suppose N^n is a hopfian n-manifold and m, k are integers, $1 < m \leq k$, such that $\pi_i(N^n) = 0$ for $1 < i \leq m$ and $H_i(N^n) = 0$ for $m < i \leq k$; suppose further that $p: M^{n+k} \to B$ is an N^n-like PL map. Then p is an approximate fibration if and only if p has Property $R^{\cong}$.*

Proof. If $R: p^{-1}c \to p^{-1}b$ induces an isomorphism at the fundamental group level, then it also does so for ith cohomology groups, $0 \leq i \leq k$, by Lemma 2.2 and a standard universal coefficient theorem. Hence, the ith cohomology sheaf, $\mathcal{H}^i[p]$, is locally constant in the same range. According to [DS, Thm.

2.6], $\mathcal{H}^n[p]$ is locally constant. In particular, $R: p^{-1}c \to p^{-1}b$ is a degree-1 map inducing a π_1-isomorphism; by the hypothesized hopfian feature of these point inverses, this map is a homotopy equivalence. The Coram–Duvall characterization of approximate fibrations in terms of movability properties [CD2] completes the proof. □

The preceding and its application to follow are similar to fibrator results of [D6] involving homology conditions alone.

Before stating the application, we name a group-theoretic condition which will appear as an hypothesis in many of the results of this paper. We call a group G *sparsely Abelian* if it contains no Abelian normal subgroup $A \neq 1$ such that G/A itself is isomorphic to a normal subgroup of G.

THEOREM 2.4. *Suppose N^n is a hopfian n-manifold having an m-connected universal cover. Suppose $\pi_1(N^n)$ is a sparsely Abelian, normally cohopfian group, and suppose $H_i(N^n) = 0$ for $m < i \leq k$. Then N^n is a codimension-k PL fibrator if and only if it is a codimension-2 PL fibrator.*

Proof. Explanation of the reverse implication proceeds by induction on $k \geq 2$. Assume N^n to be a codimension-$(j-1)$ PL fibrator $(2 \leq j-1 < k)$, and consider an N^n-like PL map $p: M^{n+j} \to B$. Lemma 4.2 (see also the remark following its proof) of [D6] ensures that p has Property $\mathrm{R}^{\cong}$, and Proposition 2.3 does the rest. □

As a consequence, we immediately obtain a result also proved as Theorem 8.1 of [D6].

COROLLARY 2.5. *If a closed, aspherical n-manifold N^n is a codimension-2 fibrator, and if N^n has a sparsely Abelian, normally cohopfian fundamental group, then N^n is a PL fibrator.*

COROLLARY 2.6. *If N_1, N_2 are closed, nonsimply connected n-manifolds whose universal covers are m-connected for some $m \leq n-2$, and if $N_1 \# N_2$ is a hopfian n-manifold with hopfian fundamental group $\pi \neq Z_2 * Z_2$, then $N_1 \# N_2$ is a codimension-m PL fibrator. Furthermore, if $m = n-2$ and $\beta_1(N_1) = \beta_1(N_2) = 0$, then $N_1 \# N_2$ is a codimension-$(n-1)$ PL fibrator.*

Proof. By [D5, Cor. 4.12 & Thm. 5.4], $N_1 \# N_2$ is a codimension-2 fibrator, and by [D6, Cor. 4.3] $\pi_1(N_1 \# N_2)$ is normally cohopfian and sparsely Abelian. Hence, Theorem 2.4 promises that $N_1 \# N_2$ is a codimension-$(m \leq n-2)$ PL fibrator, which it also does for codimension $n-1$ when $\beta_1(N_i) = 0$, $i \in \{1, 2\}$, since that yields $H_{n-1}(N_1 \# N_2) = 0$. □

REMARK. In the $m = n-2$ setting of Corollary 2.6, work of Swarup [Sw] guarantees that $N_1 \# N_2$ is a hopfian manifold whenever it has hopfian fundamental group.

The remainder of this section is devoted to developing variations on Theorem 2.4 with no cohopficity assumption on fundamental groups.

THEOREM 2.7. *Suppose the hopfian manifold N^n is a codimension-$2m$ PL fibrator, $m \geq 2$. Suppose that $\pi_i(N^n) = 0$ for $1 < i \leq 2m+1$ and that $\pi_1(N^n)$ is sparsely Abelian. Finally, suppose that N^n admits no map $f: N^n \to N^n$ such that $f_\#(\pi_1(N^n))$ has index 2 in $\pi_1(N^n)$. Then N^n is a codimension-$(2m+1)$ PL fibrator.*

Proof. The sort of analysis repeatedly done in [D6] confirms that every link $L \subset B$ satisfies $\pi_i(L) = 0$ for $1 < i < 2m$. Here are some details. Consider an N^n-like PL map $p: M^{2m+1} \to B$ and an arbitrary link L of a vertex $v \in B$. By [D4, Lemma 3.1], $L' = p^{-1}L$ is a PL $2m$-manifold, so the hypothesis on N^n being a codimension-$2m$ PL fibrator implies $p|L'$ is an approximate fibration. As

$$\pi_i(L') \cong \pi_i(S') \cong \pi_i(p^{-1}v) \cong 0 \quad (1 < i+1 < 2m)$$

[D6, Lemma 2.4], the homotopy exact sequence of $p|L'$ immediately gives $\pi_i(L) \cong 0$ for $2 < i < 2m$. The sparsely Abelian restriction fits in merely for the $i = 2$ case, causing the homomorphism $\pi_2(L) \to \pi_1(\text{fiber})$ at the lower end of the same exact sequence to be trivial and then implying $\pi_2(L) \cong 0$ as well.

In addition, $\pi_{2m}(L') \neq 0$ by [D6, Lemma 2.12], and so the exact sequence

$$0 \cong \pi_{2m}(N^n) \to \pi_{2m}(L') \to \pi_{2m}(L)$$

gives $\pi_{2m}(L) \neq 0$. Now from the sequence

$$0 \to \pi_1(N^n) \cong \pi_1(p^{-1}c) \to \pi_1(L') \to \pi_1(L) \to 0, \quad c \in L,$$

we deduce that $\pi_1(L)$ is finite, because the universal cover of L is S^{2m} (it is a $(2m-1)$-connected manifold of dimension $2m$ with nontrivial homotopy in the top dimension). Euler characteristic considerations ensure that $\pi_1(L)$ has order at most 2; equivalently, the image of $\pi_1(N^n)$ in $\pi_1(L')$ has index at most 2. The order-2 case is ruled out by hypothesis and because $R_\#: \pi_1(S') \to \pi_1(p^{-1}v)$ is an isomorphism. This means that $\pi_1(N^n) \to \pi_1(L')$ is an isomorphism, which implies the same of the composition

$$\pi_1(p^{-1}c) \to \pi_1(L') \cong \pi_1(S') \to \pi_1(p^{-1}v),$$

and indicates that p has Property $R^{\cong}$. Application of Proposition 2.3 completes the argument. □

COROLLARY 2.8. *Suppose the closed, aspherical manifold N^n is a codimension-$(2m \geq 4)$ PL fibrator, and suppose $\pi_1(N^n)$ is a sparsely Abelian, hopfian group. Suppose also that every epimorphism of $\pi_1(N^n)$ onto an index-2 subgroup is a monomorphism. Then N^n is a codimension-$(2m+1)$ PL fibrator.*

Proof. Should there exist a map $f: N^n \to N^n$ such that

$$[\pi_1(N^n): f_\#(\pi_1(N^n))] = 2,$$

then by hypothesis N^n would 2-1 cover itself (up to homotopy), which would preclude its being a codimension-2 fibrator. □

The subsequent lemma supplies algebraic information essential to the main result of this section, Theorem 2.10. Related versions for free Abelian groups appear in Section 4.

LEMMA 2.9. *If Π is a finite Abelian group and Σ is a closed k-manifold such that $\pi_1(\Sigma) = 0$, $\pi_2(\Sigma) = \Pi$, and $\pi_i(\Sigma) = 0$ for $2 < i < k$, then Π is trivial.*

Proof. The $k = 3$ case is standard; the $k = 4$ case is also easy, since duality implies that $H_2(\Sigma) \cong \pi_2(\Sigma)$ is free.

Suppose $k \geq 5$ and $\Pi \neq 1$. Since $K(\Pi, 2)$ can be obtained by attaching cells of dimension at least $k+1$ to Σ, $H_i(\Sigma) \cong H_i(K(\Pi, 2))$ for $i < k$, just as in Lemma 2.2. In particular, $H_{2m}(\Sigma) \neq 0$ for $2m < k$, since $H_{2t}(K(\Pi, 2)) \neq 0$ for all $t \geq 0$ [EM]. When k is even, $H^{k-1}(\Sigma) \cong H_1(\Sigma) \cong 0$ but $H_{k-2}(\Sigma)$ is nontrivial and finite, which is impossible; when k is odd, $H_{k-1}(\Sigma)$ is nontrivial and finite, but $0 \cong H^1(\Sigma) \cong H_{k-1}(\Sigma)$. Since neither possibility can occur, Π must be trivial. □

The next result represents a restricted extension of [D6, Thm. 7.2] to codimensions greater than 5.

THEOREM 2.10. *Suppose N^n has a closed $(k-1)$-connected universal cover $(k > 2)$. Then N^n is a codimension-k PL fibrator if and only if it is a codimension-2 fibrator.*

Proof. The cases $k = 3, 4$ are treated in [D6]. Assume that N^n is a PL codimension-j fibrator, $4 \leq j < k$, and consider a PL map $p\colon M^{n+j+1} \to B$. Identify a link $L \subset B$ of a vertex v in B. With $L' = p^{-1}L$, since $\pi_i(L') \cong \pi_i(N^n)$ for $i < j$, the homotopy exact sequence for the fibration $p|L'$ yields $\pi_i(L) = 0$ for $2 < i < j$; moreover, the lower portion of the sequence

$$0 \to \pi_2(L) \to \pi_1(N^n) \to \pi_1(L') \to \pi_1(L) \to 0$$

reveals that $\pi_2(L)$ and $\pi_1(L)$ are finite, the first because it is isomorphic to a subgroup of the finite group $\pi_1(N^n)$, and the second because it is an image of $\pi_1(L') \cong \pi_1(N^n)$. When we pass to the universal cover L^* of L, Lemma 2.9 assures that $\pi_2(L) \cong \pi_2(L^*)$ is trivial. Hence, the monomorphism

$$\pi_1(N^n) \to \pi_1(L') \cong \pi_1(N^n)$$

is an isomorphism, so $\pi_1(L) = 1$. In other words, L is a homotopy j-sphere. That p is an approximate fibration follows directly from [D4, Thm. 5.6]. □

COROLLARY 2.11. *P^{2n+1} is a codimension-$(2n+1)$ PL fibrator.*

Proof. P^{2n+1} is a codimension-2 fibrator [D1, Thm. 6.1]. □

COROLLARY 2.12. *If Γ is a finite, hyperhopfian group of orientation-preserving homeomorphisms acting freely on a $(k-1)$-connected closed n-manifold, then the orbit space is a codimension-k PL fibrator.*

Proof. The orbit space is a codimension-2 fibrator, as its fundamental group Γ is hyperhopfian [D5]. □

COROLLARY 2.13. *If N^n has a closed, $(k-1)$-connected universal cover and is a codimension-2 PL fibrator, and if $p: M^{n+k+1} \to B$ is an N^n-like PL map, then B is a $(k+1)$-manifold.*

Proof. The argument given for Theorem 2.10 also establishes that all links $L \subset B$ are homotopy k-spheres. □

THEOREM 2.14. *Suppose that N^n is a closed n-manifold with a $(k+1)$-connected universal cover, and that N^n is a codimension-k PL fibrator. Suppose $\pi_1(N^n)$ is sparsely Abelian. Finally, suppose that $p: M^{n+k+1} \to B$ is a PL N^n-like map and that L is a link of a vertex in B. Then $\pi_1(L)$ is a group that acts freely on some homotopy k-sphere.*

Proof. From the homotopy exact sequence of the approximate fibration $p|L'$, standard analysis again yields $\pi_i(L) = 0$ for $1 < i < k$. Furthermore, $\pi_k(L) \neq 0$ [D6, Lemma 2.12], implying finiteness of $\pi_1(L)$ (otherwise the universal cover L^* would be contractible, since it would satisfy $H_k(L^*) = 0$ as well as $\pi_i(L^*) = 0$, $i < k$). Therefore, the universal cover of L is a homotopy k-sphere. □

COROLLARY 2.15. *Suppose the closed aspherical n-manifold N^n is a PL fibrator in codimension k but not codimension $k+1$, and suppose $\pi_1(N^n)$ is sparsely Abelian. Then there exists a group $\Gamma \neq 1$ which acts freely on both a homotopy k-sphere and on N^n such that the orbit space N^n/Γ is homotopy equivalent to N^n.*

Finally, we close this section with a converse to Corollary 2.15. The orbit space construction of [D1], repeated as Example 2B of [D6], shows how to produce the required N^n-like map.

PROPOSITION 2.16. *If N^n is a closed n-manifold and Γ is a finite group that acts (PL) freely, preserving orientations, on both S^k and on N^n such that the orbit space N^n/Γ is homotopy equivalent to N^n, then N^n fails to be a codimension-$(k+1)$ (PL) fibrator.*

3. Aspherical 3-Manifolds as Fibrators

LEMMA 3.1. *Let N^3 be any closed, aspherical 3-manifold which does not have Nil geometric structure but which is a codimension-2 fibrator. Then $\pi_1(N^3)$ is sparsely Abelian.*

Proof. Let A be an abelian normal subgroup of $\pi_1(N^3)$ with $\pi_1(N^3)/A$ also isomorphic to a normal subgroup of $\pi_1(N^3)$. As it is torsion free (see [He, Cor. 9.9]) and finitely generated [EJ, Cor. 3.3], A is free on either three, two, one, or zero generators [He, Thm. 9.13]. We eliminate all but the last possibility, thereby confirming that $A = 1$. Since interest here pertains to the homotopy type of N^3, without loss of generality the assumption of asphericity can be boosted to include irreducibility.

The three-generator case is ruled out: N^3 itself cannot be $S^1 \times S^1 \times S^1$, which fails to be a codimension-2 fibrator, making $A = Z \oplus Z \oplus Z$ a finite-index, proper subgroup of $\pi_1(N^3)$ and giving rise to the finite subgroup $\pi_1(N^3)/A$ of $\pi_1(N^3)$, violating torsion freeness.

The two-generator case is a quick consequence of earlier work. By a result originally due to Hempel–Jaco [HJ] (see also [He, Thm. 11.1]), N^3 is a torus bundle over S^1. Depending on the monodromy, N^3 has geometric structure either that of E^3, Nil or Sol [S1, Thm. 5.5]. Nil structure is not an issue at this point, by hypothesis; with Euclidean structure, N^3 fails to be a codimension-2 fibrator [D3], and with Sol structure the conclusion appears in [D6, Cor. 8.5].

In the single-generator case, we have exact sequences

$$1 \to Z \to \pi_1(N^3) \to Q \to 1,$$

$$1 \to Q \to \pi_1(N^3) \to Q' \to 1,$$

where conceivably Q, Q' are distinct groups. The key step involves confirming that Q' has infinite order.

Based on the first sequence, recent work of Casson–Jungreis [CJ] and Gabai [Ga] implies that N^3 is a Seifert fiber space. Of the six possible geometric structures, N^3 cannot have that of Nil, put aside by hypothesis, nor that of S^3 or $S^2 \times R$, due to asphericity. Checking that Q' has infinite order when N^3 has Euclidean structure is left to the reader. In the remaining cases $\pi_1(N^3)$ contains a finite-index subgroup G which is the group of an S^1-bundle over a closed, orientable, hyperbolic 2-manifold S, and G meets the named infinite cyclic group Z in the subgroup C of G determined by the circle in the bundle structure. Nontriviality of $Z \cap C$ follows, because $\pi_1(S) \cong G/C$ contains no infinite cyclic normal subgroup, after which equality of C and Z follows easily since their respective images in Q and $\pi_1(S)$ must be torsion-free. Now $\text{order}(Q') = [\pi_1(N^3) : Q] = \infty$ since Q contains a finite-index subgroup isomorphic to $G/C \cong \pi_1(S)$, but no finite-index subgroup of $\pi_1(N^3)$ can be the fundamental group of an aspherical 2-manifold.

Applying [He, Thm. 11.1] to the second sequence, we find Q to be the fundamental group of a closed 2-manifold F. Thus, N^3 is both an S^1-bundle over F [He, Thm. 11.10] and a surface bundle over a 1-dimensional orbifold [He, Thm. 11.1]. It cannot carry either E^3 or $H^2 \times R$ structure, for those that are S^1-bundles over F are necessarily product bundles and, hence, obviously not codimension-2 fibrators. Finally, N^3 cannot possess $\text{SL}_2(R)$ structure, the remaining possibility, for no such manifold is a surface bundle over a 1-dimensional orbifold [S1, p. 465]. □

LEMMA 3.2. *If the closed 3-manifold N^3 has the geometric structure of $H^2 \times R$ and is a codimension-2 PL fibrator, then $\pi_1(N^3)$ is normally cohopfian.*

Proof. Consider a regular covering $\theta: N^3 \to N^3$ of degree $d = m \cdot l$, where m is the degree of the associated cover of regular fibers and l is the degree of the induced cover $X \to X$ of orbifolds. Then $\chi(X) = l \cdot \chi(X)$ [S1, Thm. 3.6], which implies $l = 1$, since $\chi(X) \neq 0$. Being completely determined by the self-covering of regular fibers, θ is a regular cyclic cover, forcing $d = m = 1$, for anything larger would contradict N^3 being a codimension-2 PL fibrator. □

LEMMA 3.3. *If the closed 3-manifold N^3 has the geometric structure of E^3 and is a codimension-2 fibrator, then $\pi_1(N^3)$ is normally cohopfian.*

Proof. Only one example N^3 with Euclidean geometric structure fails to be a codimension-2 fibrator [D2]. Its orbifold is P^2 with 2 cone points of order 2, and it is the unique closed, orientable Euclidean 3-manifold that does not fiber over S^1. It is shown in [D2] that every 2-sheeted cover of N^3 does fiber over S^1. Supposing $\pi_1(N^3)$ fails to be normally cohopfian, we consider a regular covering $\theta: N^3 \to N^3$ of least degree $d > 1$. Identify the subgroup G of $\pi_1(N^3)$ generated by $\theta_\#(\pi_1(N^3))$ and all elements determined by covering translations that induce the trivial action at the orbifold level. One can see that the only nontrivial orbifold self-cover has order 2, so here $[\pi_1(N^3) : G] = 2$. Form the covering space $\tilde{N}$ corresponding to G. Now note that θ factors through regular, nontrivial coverings $N^3 \to \tilde{N} \to N^3$, where by minimality $\tilde{N}$ fibers over S^1, an impossibility, as it would force N^3 to do the same. □

Were this paper organized in strictly linear fashion, next would come an investigation of Nil manifolds. We isolate that rather technical topic in Section 4 and proceed instead to a derivation of the main result.

THEOREM 3.4. *Let N^3 be a closed, aspherical 3-manifold which is virtually geometric. Then N^3 is a PL fibrator if and only if it is a codimension-2 fibrator.*

Proof. Again assume N^3 is irreducible. Corollary 8.5 of [D6] confirms that all 3-manifolds having hyperbolic, Sol, or $\mathrm{SL}_2(R)$ geometric structure are PL fibrators. The other possible geometric structures each include examples that fail to be codimension-2 PL fibrators. Nevertheless, for those with either Euclidean or $H^2 \times R$ structure, the result follows from Corollary 2.6, Lemma 3.1, and either Lemma 3.2 or Lemma 3.3; for those with Nil structure, it follows from Proposition 4.6 in the next section. Finally, for 3-manifolds

N^3 that are virtually geometric but support no geometric structure, the conclusion follows from Lemma 3.1 and Corollary 2.5, since then $\pi_1(N^3)$ actually is cohopfian [GW, Thm. 1.1], not just normally so, which would be sufficient. (The work in [GW] depends significantly on [WW], where the cohopficity conclusion for the groups of Haken 3-manifolds supporting no geometric structure is derived.) □

4. Nil 3-Manifolds as Fibrators

Nil 3-manifolds shatter the pattern found in Section 3 for other Seifert fibered spaces: they can be codimension-2 PL fibrators yet fail to possess normally cohopfian fundamental groups. Nevertheless, Proposition 4.6 certifies, as promised earlier, that Nil 3-manifolds fulfill Theorem 3.4.

EXAMPLE. A closed 3-manifold N^3 which is a codimension-2 PL fibrator but $\pi_1(N^3)$ is not normally cohopfian. Consider the S^1-bundle over the Klein bottle such that

$$\pi_1(N^3) = \langle a, b, k \mid a^{-1}ka = k^{-1} = b^{-1}kb,\ k^2 = a^2b^2\rangle.$$

It is easy to check that a, b^2, k^2 generate a normal subgroup of $\pi_1(N^3)$ isomorphic to $\pi_1(N^3)$, and therefore the associated covering space is another copy of N^3. It turns out that the group of deck transformations is $Z_2 \oplus Z_2$.

In the same manner one finds that circle bundles over the torus regularly cover themselves, but they are less interesting because they never serve as codimension-2 fibrators [D3].

Direct sums such as those in the example above typically arise as subgroups of the deck transformations associated with coverings stemming from the failure of normal cohopficity (cf. Lemma 4.4), and ultimately play an instrumental role in the key result.

LEMMA 4.1. *No $2m$-manifold T satisfies all of the following homotopy data: $\pi_1(T)$ is infinite; $\pi_2(T)$ is free Abelian of rank r, $0 < r < \infty$; and $\pi_i(T) = 0$ for $2 < i < 2m$.*

Proof. Assume the contrary. Then construct an Eilenberg–MacLane space $K(\pi_2(T), 2)$ by attaching cells of dimension greater than $2m$ to T', the universal cover of T. From this perspective,

$$H_{2m}(K(\pi_2(T), 2)) = 0,$$

being the image of $H_{2m}(T') = 0$. On the other hand, by applying the Kunneth formula to the product of r copies of CP^∞, a standard model for $K(\pi_2(T), 2)$, one finds that $H_i(K(\pi_2(T), 2))$ must be nontrivial (precisely) when i is even. The contrary assumption is absurd. □

REMARK. With no fundamental group restriction whatsoever, Lemma 4.1 holds for all noncompact, even-dimensional manifolds.

LEMMA 4.2. *No k-manifold T, $k \geq 3$, satisfies all of the following homotopy data: $\pi_1(T)$ is infinite; $\pi_2(T)$ is free Abelian of rank r, $1 < r < \infty$; and $\pi_i(T) = 0$ for $2 < i < k$. Moreover, there is no such T in the $r = 1$ case unless $\pi_1(T)$ contains an infinite cyclic subgroup of finite index.*

Proof. Suppose otherwise. In light of Lemma 4.1, k must be odd. Standard homology computations for the r-fold Cartesian product of CP^∞ with itself, the model for $K(\pi_2(T), 2)$, shows $H_{k-1}(K(\pi_2(T), 2))$ to be free Abelian of rank $r' \geq r$. Again the universal cover T' of T is noncompact, and $H_{k-1}(T') \cong H_{k-1}(K(\pi_2(T), 2))$ as in Lemma 2.2. Hence, $H_c^1(T') \cong H_{k-1}(T')$ is free Abelian of rank $r' \geq r$. It follows from work of Epstein [Ep, Thm. 1] that T' has $r'+1 > r$ ends. But this is impossible for $r > 1$: Epstein also proved that an infinite-sheeted regular covering has either 1, 2, or infinitely many ends [Ep, Thm. 10]. It is also impossible for $r = 1$ under the supplemental π_1-restriction: When an infinite-sheeted regular covering of a finite complex has two ends, then the group of deck transformations contains an infinite cyclic, finite-index subgroup [Ep, Thm. 12]. □

LEMMA 4.3. *If the Nil 3-manifold N^3 is a circle bundle over a torus and J is an infinite cyclic, normal subgroup of $\pi_1(N^3)$, then J is contained in the subgroup C of $\pi_1(N^3)$ determined by the circle in the specified (Seifert) bundle structure.*

Proof. Here $\pi_1(N^3)$ has the presentation

$$\langle a, b, k\colon a^{-1}k^{-1}ak = 1 = b^{-1}k^{-1}bk,\ a^{-1}b^{-1}ab = k^e\ (e \neq 0)\rangle,$$

and the element k generates C. Each element of $\pi_1(N^3)$ can be uniquely expressed in the form $k^l a^m b^n$. If the generator γ of J did not belong to C, then γ could be written as $\gamma = k^p a^r b^s$ with either $r \neq 0$ or $s \neq 0$; say $r \neq 0$ for definiteness. Routine checking will verify that the subgroup generated by γ fails to be normal—indeed, direct computaton yields that $b^{-1}\gamma b = k^{p+er}a^r b^s$ while $\gamma^n = k^t a^{nr} b^{ns}$ (where t depends on $n \in Z$), so $\gamma^n \neq b^{-1}\gamma b$. □

LEMMA 4.4. *Let N^3 be a Nil 3-manifold and a codimension-2 fibrator. Then there exists no pair of exact sequences*

$$1 \to Z \to \pi_1(N^3) \to Q \to 1 \quad \textit{and} \quad 1 \to Q \to \pi_1(N^3) \to Q' \to 1.$$

Proof. Two separate cases must be treated, namely, (1) N^3 is a circle bundle over a Klein bottle and (2) the Seifert structure on N^3 includes an irregular fiber.

Suppose otherwise, and name the infinite cyclic normal subgroup C of $\pi_1(N^3)$ determined by a regular fiber. Passing to a finite-index subgroup of $\pi_1(N^3)$ corresponding to a cover by a circle bundle over $S^1 \times S^1$ and applying Lemma 4.3, we obtain $C \cap Z \neq \{0\}$. That means $C \subset Z$, as the image of C in $Q \subset \pi_1(N^3)$ cannot give rise to torsion elements.

First consider (1) N^3 an S^1 bundle over the Klein bottle. Let $q: N^3 \to \mathrm{KB}$ denote this Seifert projection. Asphericity of KB implies Z equals the kernel of $q_\#$ (the induced homomorphism on π_1). Hence, Q is the fundamental group of the Klein bottle and therefore Q' is infinite. However, the Klein bottle group cannot be an infinite-index normal subgroup of a closed orientable 3-manifold, as it would be isomorphic to a finite-index subgroup of some 2-sided, necessarily orientable surface [He, Thm. 11.1].

Next suppose (2) that the Seifert fibering of N^3 includes irregular fibers. The given epimorphism $\pi_1(N^3) \to Q$ induces an epimorphism from the orbifold fundamental group, $\pi_1(N^3)/C$, to Q. But $\pi_1(N^3)/C$ has a finite-index subgroup generated by torsion elements, all of which die in Q, so the image of the orbifold group itself is trivial. This implies $\pi_1(N^3) \cong Z$, an obvious contradiction. □

Although Nil manifolds which are S^1-bundles over a torus fail to be codimension-2 fibrators, they are relatively exceptional in that, unlike virtually all the other known examples, they do not regularly, cyclically cover themselves. The lemma below strengthens the limitations on the allowable deck transformations.

Lemma 4.5. *If the closed 3-manifold N^3 has Nil geometric structure, and if $\Gamma \neq 1$ is a group acting freely on N^3 such that N^3/Γ is homotopy equivalent to N^3, then Γ acts freely on no homology sphere.*

Proof. According to [S2], the orbit space N^3/Γ is actually homeomorphic to N^3. Since every Nil manifold satisfies unique Seifert data, in the given covering $N^3 \to N^3/\Gamma \cong N^3$, if m denotes the degree with which regular fibers of the domain cover those of the orbit space and l denotes the degree of the induced orbifold covering, then $m = l$ [S1, Thm. 3.6]. As in Lemma 3.3, the covering of regular fiberings is cyclic. Consider any element $\alpha \in \Gamma$ corresponding to a standard generator of the orbifold group. It follows that there exists $\beta \in \Gamma$ corresponding to a multiple of the fiber, with order $\beta =$ order $\alpha = t > 1$. Inspection of the Seifert bundle data implies that either α, β commute or $\alpha^{-1}\beta\alpha = \beta^{-1}$ (standard orbifold generators commute or anticommute with the generator from a regular fiber). In the anticommutative case t is divisible by 2, so $\alpha^{t/2}, \beta^{t/2}$ commute. Inevitably Γ contains a subgroup of the form $Z_d \oplus Z_d$. Smith theory (see [Br, p. 81]) assures that no such group acts freely on a homology sphere. □

Proposition 4.6. *Every codimension-2 PL fibrator with the geometric structure of a Nil 3-manifold is a PL fibrator.*

Proof. The argument proceeds by induction. Assume that N^3 is such a codimension-k PL fibrator, and consider an N^3-like PL map $p: M^{4+k} \to B$. As usual, for a link L of an arbitrary vertex $v \in B$, $p \mid L'$ is an approximate fibration and $\pi_i(L) \cong 0$, $2 < i < k$ (cf. the proof of Theorem 2.7).

Now we prove that $\pi_2(L) \cong 0$. The exact sequence shows $\pi_2(L)$ isomorphic to a normal subgroup A of $\pi_1(N^3)$. Consider the various possibilities for $\pi_2(L) \cong A$. As in Lemma 3.1, A is free on $r \leq 3$ generators, and the analysis given there disposes of $r = 3$. The argument of 3.1 also gives

$$\text{order}(\pi_1(L)) = [\pi_1(N^3) : A] = \infty$$

when $r = 2$, and Lemma 4.2 consequently prohibits this possibility. Finally, Lemma 4.4 prohibits the $r = 1$ case. Hence, $\pi_2(L)$ is trivial.

The final point involves confirming that p has Property $R^{\cong}$. Should $R \mid p^{-1}c$: $p^{-1}c \to p^{-1}v$ fail to induce a π_1-isomorphism ($c \in L$), the lower end of the homotopy exact sequence of $p \mid L'$,

$$0 \to \pi_1(p^{-1}c) \to \pi_1(L') \to \pi_1(L) \to 0,$$

would couple with the isomorphism $\pi_1(L') \to \pi_1(S') \to \pi_1(p^{-1}v)$ to imply that $\pi_1(L)$ acts freely on a finite-sheeted cover of N^3 (corresponding to $p^{-1}c$) with orbit space homotopy equivalent to N^3 (corresponding to $p^{-1}v$). On the other hand, the universal cover of L would be a homotopy k-sphere on which $\pi_1(L)$ acts freely, in violation of Lemma 4.5. Thus, Property $R^{\cong}$ holds, Theorem 2.4 ensures that p is an approximate fibration, and N^3 is a codimension-$(k+1)$ PL fibrator, as required. □

CONCLUDING REMARKS. The 3-sphere is the only known exception to the statement: A closed 3-manifold N^3 is a PL fibrator if and only if it is a codimension-2 fibrator. Are there any others? The most prominent matters are to decide this for 3-manifolds covered by S^3 and for the ones arising as connected sums of manifolds with nontrivial, finite fundamental groups (other than exactly two summands, both with $\pi_1 = Z_2$).

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