The Dirichlet Problem for the Complex Monge–Ampère Operator: Perron Classes and Rotation-Invariant Measures

Urban Cegrell[1] & Sławomir Kołodziej[2]

0. Introduction

Let Ω be an open and bounded subset of $\mathbb{C}^n$. If $u_j \in C^2(\Omega)$, $1 \le j \le n$, then the Monge–Ampère operator $(dd^c)^n = dd^c u_1 \wedge \cdots \wedge dd^c u_n$ operates on $(u_1, \dots, u_n)$, where $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)$. This operator is of great importance in pluripotential theory. It was shown in [3] that $(dd^c)^n$ is well-defined and nonnegative on $\mathrm{PSH} \cap L^\infty_{\mathrm{loc}}$. In this paper, we will study the following Dirichlet problem.

Let Ω be an open, bounded, and strictly pseudoconvex subset of $\mathbb{C}^n$, let φ be in $C(\partial\Omega)$, and let μ be a positive measure on Ω. Consider the problem:

$$\begin{cases} u \in \mathrm{PSH} \cap L^\infty(\Omega), \\ (dd^c u)^n = \mu \ \text{ on } \Omega \\ \overline{\lim}_{z \to \xi} u(z) = \varphi(\xi) \ \ \forall \xi \in \partial\Omega. \end{cases} \tag{i}$$

There are measures for which (i) has no solution. For if (i) can be solved with μ, then μ cannot have mass on any pluripolar set. Thus, for example, if we take μ to be the Dirac measure for a point in Ω then (i) has no solution. On the other hand, if $\mu = f\,dV$ where $f \in C(\bar\Omega)$ and dV is Lebesgue measure, then it was proved in [2] that (i) has a unique solution for every $\varphi \in C(\partial\Omega)$. This was generalized in [5] to the case when $f \in L^\infty(\Omega, \mu)$ and in [7] to the case when $f \in L^2(\Omega, dV)$. The main result of this paper is the following. Let ν be any positive rotation invariant measure in the unit ball B for which there is a $u \in \mathrm{PSH} \cap L^\infty(B)$ with $(dd^c u)^n \ge \nu$. (These measures can be characterized; cf. [11].) Then, for every $f \in L^\infty(B, \nu)$ and for every $\varphi \in C(\partial B)$, there is a unique solution to (i) with $\mu = f\,d\nu$. For background and references see [1; 6; 10].

1. Perron Classes

To study the problem (i), we use the Perron method and therefore consider classes of subsolutions

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$$B(\varphi, \mu) = \{v \in \mathrm{PSH} \cap L^\infty(\Omega); (dd^c v)^n \geq \mu, \overline{\lim}_{z \to \xi} v(z) \leq \varphi(\xi)\ \forall \xi \in \partial\Omega\}$$

and their envelopes

$$u(\varphi, \mu)(z) = \sup\{v(z), v \in B(\varphi, \mu)\}.$$

The following theorem shows that if $B(\varphi, \mu) \neq \emptyset$ then $B(\varphi, \mu)$ has a lattice property such that $u(\varphi, \mu) \in B(\varphi, \mu)$. In particular, $u(\varphi, \mu) \in \mathrm{PSH} \cap L^\infty(\Omega)$.

THEOREM 1. *If $u, v \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$ and if μ is a positive measure on Ω such that $(dd^c u)^n \geq \mu$ and $(dd^c v)^n \geq \mu$, then*

$$(dd^c \max(u, v))^n \geq \mu.$$

The theorem can be deduced from [9, Prop. 6:11]; we give here a direct proof.

Proof. Since $(dd^c \max(u+\epsilon, v))^n \rightsquigarrow (dd^c \max(u, v))^n$ as $\epsilon \searrow 0$ where $\rightsquigarrow$ denotes weak convergence, and since $\mu(\{u+\epsilon = v\}) = 0$ outside a countable set of ϵs, we can assume that $\mu(\{u = v\}) = 0$. Let $\chi \in D(\Omega)$ and $\eta > 0$ be given. Without loss of generality, we assume that the moduli of u and v are smaller than 1 on the support of χ. Then, by [3], there is an open set O_η such that the Monge–Ampère capacity of O_η is less than η, and $u|_{CO_\eta}$ and $v|_{CO_\eta}$ are continuous. Extend these functions to continuous functions $\tilde{u}$ and $\tilde{v}$ on Ω.

Denote by u_ϵ and v_ϵ the usual regularizations of u and v. If $s > 0$ then by Heine–Borel there is an $\epsilon_s > 0$ such that

$$\{u_\epsilon|_{\operatorname{supp}\varphi \cap CO_\eta} < v|_{\operatorname{supp}\varphi \cap CO_\eta}\} \supset \{u|_{\operatorname{supp}\varphi \cap CO_\eta} + s < v|_{\operatorname{supp}\varphi \cap CO_\eta}\}$$

and

$$\{u|_{\operatorname{supp}\varphi \cap CO_\eta} > v_\epsilon|_{\operatorname{supp}\varphi \cap CO_\eta}\} \supset \{u|_{\operatorname{supp}\varphi \cap CO_\eta} > v|_{\operatorname{supp}\varphi \cap CO_\eta} + s\}\ \forall \epsilon < \epsilon_s.$$

Therefore

$$\begin{aligned}\int \chi(dd^c \max(u_\epsilon, v_\epsilon))^n &\geq \int_{\{u_\epsilon < v_\epsilon\}} \chi(dd^c v_\epsilon)^n + \int_{\{u_\epsilon > v_\epsilon\}} \chi(dd^c u_\epsilon)^n \\ &\geq \int_{O_\eta \cup \{u_\epsilon < v_\epsilon\}} \chi(dd^c v_\epsilon)^n + \int_{O_\eta \cup \{u_\epsilon > v_\epsilon\}} \chi(dd^c u_\epsilon)^n \\ &\quad - \int_{O_\eta} \chi(dd^c v_\epsilon)^n - \int_{O_\eta} \chi(dd^c u_\epsilon)^n \\ &\geq \int_{\{\tilde{u}+s<\tilde{v}\}} \chi(dd^c v_\epsilon)^n + \int_{\{\tilde{u}>\tilde{v}+s\}} \chi(dd^c u_\epsilon)^n - 2\eta\|\chi\|_{L^\infty}.\end{aligned}$$

Since $\{\tilde{u}+s < \tilde{v}\}$ and $\{\tilde{u} > \tilde{v}+s\}$ are open sets, we have

$$\begin{aligned}\lim_{\epsilon \to 0} \int \chi(dd^c \max(u_\epsilon, v_\epsilon))^n &\geq \int_{\{\tilde{u}+s<\tilde{v}\}} \chi(dd^c v)^n + \int_{\{\tilde{u}>\tilde{v}+s\}} \chi(dd^c u)^n - 2\eta\|\chi\|_{L^\infty} \\ &\geq \int_{\{\tilde{u}+s<\tilde{v}\}} \chi\, d\mu + \int_{\{\tilde{u}>\tilde{v}+s\}} \chi\, d\mu - 2\eta\|\chi\|_{L^\infty}.\end{aligned}$$

Letting $s \searrow 0$ we get

$$\int \chi(dd^c \max(u,v))^n \geq \int_{\{\tilde{u}<\tilde{v}\}} \chi\, d\mu + \int_{\{\tilde{u}>\tilde{v}\}} \chi\, d\mu - 2\eta \|\chi\|_{L^\infty}.$$

Since $\tilde{u} = u$ and $\tilde{v} = v$ outside Ω_η and since $\mu(\{u=v\}) = 0$, we see that

$$\int \chi(dd^c \max(u,v))^n \geq \int \chi\, d\mu - 3\eta \|\chi\|_{L^\infty},$$

which completes the proof. □

THEOREM 2. *If $B(\varphi,\mu) \neq \emptyset$ then $u(\varphi,\mu) \in B(\varphi,\mu)$.*

Proof. An application of Choquet's lemma shows that there exist $v_j \in B(\varphi,\mu)$ for $j \in \mathbb{N}$ such that $(\sup_{j\in\mathbb{N}} v_j)^* = u^*(\varphi,\mu)$, where g^* denotes the smallest upper semicontinuous majorant of g. Thus $u^*(\varphi,\mu)$ is plurisubharmonic on Ω, and since $v_j \leq u(\varphi,0)$, where $u(\varphi,0)$ is continuous on $\bar{\Omega}$ and equals φ on $\partial\Omega$ (cf. [2]), it follows that $\overline{\lim}_{z\to\xi} u^*(z) \leq \varphi(\xi)$, for all $\xi \in \partial\Omega$. Furthermore, $\max_{1\leq j\leq p}(v_j) \nearrow u^*(\varphi,\mu)$ a.e. as $p \to +\infty$ and $(dd^c \max_{1\leq j\leq p} v_j)^n \geq \mu$ by Theorem 1. Since the Monge-Ampère operator is continuous on uniformly bounded and monotone sequences (cf. [3]), we have $(dd^c u^*(\varphi,\mu))^n \geq \mu$ and so $u^*(\varphi,\mu) \in B(\varphi,\mu)$ and the proof is complete. □

REMARK 1. In [8] it was proved that for dV the Lebesgue measure on $\mathbb{C}^n$, $f \in L^1(\Omega)$, and $B(\varphi, f\,dV) \neq \emptyset$, if $\mu = f\,dV$ then $u(\varphi,\mu)$ solves (i).

3. Some Properties of $u(\varphi,\mu)$

From now on, let μ be a positive measure with compact support in Ω. We can then replace $\overline{\lim}$ by lim in the last condition of (i).

We thus have a sequence of nonnegative compactly supported continuous functions φ_j such that $\varphi_j dV \rightsquigarrow \mu$ as $i \to +\infty$. Therefore, $(dd^c u(\varphi, \varphi_j dV))^n = \varphi_j dV \rightsquigarrow \mu$ as $j \to +\infty$, but the sequence need not be locally uniformly bounded. However, if $B(0,\mu) \neq \emptyset$ then $(\varphi_j)_{j=1}^\infty$ can be chosen so that $u(0, \varphi_j dV)$ is locally bounded.

For if $u \in B(0,\mu)$, let $C^\infty \ni u_j \searrow u$. Then $(dd^c u_j)^n \rightsquigarrow (dd^c u)^n \geq \mu$. Hence if θ is any nonnegative continuous function, then $(dd^c u(0, \theta(dd^c u_j))^n)^n \rightsquigarrow \theta(dd^c u)^n$ and $0 \geq u(0, \theta(dd^c u_j)^n) \geq (\sup\theta)^{1/n} u(0, (dd^c u_j)^n) \geq (\sup\theta)^{1/n} u$.

REMARK 2. It is natural to ask: If $\mu_j \rightsquigarrow \mu$ and $-1 \leq u(0,\mu_j) \leq 0$, does there exist a subsequence (j_k) such that $u(0,\mu_{j_k}) \to u(0,\mu)$ in the sense of distributions? The answer is No. For by [4] there exists a sequence of continuous and plurisubharmonic functions φ_j, $j \in \mathbb{N}$, defined on the unit ball B so that $-1 \leq \varphi_j \leq -\frac{1}{2}$ and $\varphi_j \to \varphi_0$ as $j \to +\infty$ in $L^p(\Omega)$ for $1 \leq p < +\infty$ but $(dd^c\varphi_j)^n \not\rightsquigarrow (dd^c\varphi_0)^n \neq 0$. If we choose A large enough, the functions $\psi_j(z) = \max(A \log|z|, \varphi_j(z))$, $j \in \mathbb{N}$, are plurisubharmonic and continuous on B.

Then

$$\psi_j(z) = u(0, (dd^c \max(A \log|z|, \varphi_j))^n)$$

and $(dd^c\psi_j)^n \rightsquigarrow \mu$ as $j \to +\infty$, where $\mu \neq (dd^c \max(A \log|z|, \varphi_0))^n$. But

$$\psi_j(z) = u(0, (dd^c \max(A \log|z|, \varphi_j(z)^n) \to \max(A \log|z|, \varphi_0(z))$$

as $j \to +\infty$ in $L^p(B)$. Note that since $\psi_j = A \log|z|$ near ∂B we have

$$\int_B (dd^c\psi_j)^n = \int_B (dd^c \max(A \log|z|, \varphi_0))^n$$

$$= \int_B (dd^c \max(A \log|z|, -1))^n \quad \forall j \in \mathbb{N}.$$

If $\psi_j \to u(0, \mu)$ in the sense of distributions, it would follow that $u(0, \mu) = \max(A \log|z|, \varphi_0)$ so in particular $(dd^c(A \log|z|, \varphi_0))^n \geq \mu$. But since these two measures have the same mass they would be equal, implying a contradiction.

4. The Case $\Omega = B$, the Unit Ball in $\mathbb{C}^n$

PROPOSITION 1. *Let $(u_j)_{j=1}^{\infty}$ be a uniformly bounded sequence of plurisubharmonic functions on B with the property that for each j, for each r in $[0,1]$ and for each $\epsilon > 0$, there is a $\delta > 0$ such that if $|z-w| < \delta$ and $|z| = |w| = r$ then $|u_j(z) - u_j(w)| < \epsilon$. Then there exists a subsequence $(j_k)_{k=1}^{\infty}$ such that u_{j_k} converges uniformly on B.*

Proof. Let $(r_j)_{j=1}^{\infty}$ be dense in $[0,1]$ and let $(\xi_j^s)_{s=1}^{\infty}$ be dense in $\{z \in B; |z| = r_j\}$. Since $(u_t)_{t=1}^{\infty}$ is a uniformly bounded sequence, we can choose a subsequence $(j_k)_{k=1}^{\infty}$ such that $\lim_{k \to +\infty} u_{j_k}(\xi_j^s)$ exist for all $(\xi_j^s)_{j,s=1}^{\infty}$. Note that this means that u_{j_k} converges uniformly on each $\{z \in B;\ |z| = r_j\}$, $j \in \mathbb{N}$. To avoid too many indices, denote this convergent subsequence again by $(u_t)_{t=1}^{\infty}$. Define $V = (\overline{\lim}_{j \to +\infty} u_j)^*$; then $V \in \mathrm{PSH}(B)$ and we claim that $u_t \to V$ uniformly on B.

To get a contradiction, suppose there is a $\sigma > 0$ and a sequence $z^p \in B$ such that $|u_{j_p}(z^p) - V(z^p)| > \sigma$ for all p. We can assume $z^p \to z^0 \in B$. If $u_{j_p}(z^p) < V(z^p) - \sigma$ then from some p_0 on we have $u_{j_p}(z^p) < V(z^0) - \sigma/2$. It is clear that $\overline{\lim}_{p \to +\infty} u_{j_p} \leq V$ everywhere and that $\lim_{p \to +\infty} u_{j_p}(z) = \underline{\lim}_{p \to +\infty} u_j(z)$ for all z with $|z| = r_j$. Since the $2n-1$ Hausdorff measure of $\{z \in B; \overline{\lim}_{j \to +\infty} u_j < V\}$ is zero,

$$\int_{|\xi|=1} \overline{\lim_{j \to +\infty}}\ u_j(r\xi)\, d\sigma(\xi) = \int_{|\xi|=1} V(r\xi)\, d\sigma(\xi) \quad \text{for every } r,\ 0 < r < 1,$$

where $d\sigma$ is the Lebesgue measure on the unit sphere. In particular,

$$\int_{|\xi|=1} \overline{\lim}\, u_{j_p}(r_j\xi)\, d\sigma(\xi) = \int_{|\xi|=1} V(r_j\xi)\, d\sigma(\xi)$$

and so, by Fatou's lemma, $(\overline{\lim}_{p \to +\infty} u_{j_p})^*(z) = V(z)$ everywhere. Choose now $\delta > 0$ so that if $|z| = |w|$ and $|z-w| < \delta$ then $|u_j(z) - u_j(w)| < \sigma/4$.

Denote by $W=\{z\in B; |z|=|z^0|, |z-z^0|<\delta\}^0$ the interior of the polynomial convex hull of $\{z\in B; |z|=|z^0|, |z-z^0|<\delta\}$. If $z\in W$, then $z\in\{w\in B; |w|=|z^p|, |w-z^p|<\delta\}^0$ for all sufficiently large p.

Therefore, $u_{j_p}(z)-u_{j_p}(z^p)<\sigma/4$ for all sufficiently large p. Hence $u_{j_p}(z)<V(z^0)-\sigma/4$ and $(\overline{\lim}\, u_{j_p})^*(z)\le V(z^0)-\sigma/4$ for all $z\in W$; so $V(z)<V(z^0)-\sigma/4$ for all $z\in W$. But since V is plurisubharmonic $V(z^0)=\lim_{r\nearrow 1}V(rz^0)$, which gives a contradiction since $W\supset\{rz^0; r_0<r<1\}$ for some $r_0<1$.

Thus, if u_j does not converge uniformly to V, we must have $V(z^p)+\sigma<u_{j_p}(z^p)$, $p\in\mathbb{N}$. It follows from what we have done so far that $V(z^p)\to V(z^0)$ as $p\to+\infty$. Choose k so large that $(\sup_{j\ge k}u_j)^*(z^0)<V(z^0)+\sigma/2$. Since $(\sup_{j\ge k}u_j)^*$ is upper semicontinuous, there is a $\theta>0$ such that if $|z-z^0|<\theta$ then $(\sup_{j\ge k}u_j)^*(z)<V(z^0)+\sigma/2$. In particular, if $j_p>k$ and $|z^0-z^p|<\theta$, then $u_{j_p}(z^p)<V(z^0)+\sigma/2$ which is impossible since $\underline{\lim}_{p\to+\infty}u_{j_p}(z^p)\ge\underline{\lim}_{p\to+\infty}V(z^p)+\sigma=V(z^0)+\sigma$. This completes the proof of the proposition. □

THEOREM 3. *Suppose $B(0,\mu)\neq\emptyset$, where μ is a positive measure in B that is invariant under rotations. For every $\varphi\in C(\partial B)$ and every $0\le f\in L^\infty(d\mu)$, $u(\varphi, fd\mu)$ solves* (i). *Furthermore, if $0\le f\in C(B)$ then $u(\varphi, fd\mu)\in C(\bar B)$.*

Proof. Suppose first that μ has compact support in B. If μ is invariant under rotations, so is $u(0,\mu)$. To prove Theorem 3 we prove that if μ is as above with $(dd^cu(\varphi,\mu))^n=\mu$ and if $0\le f\in C(\bar B)$, then $u(\varphi, fd\mu)\in C(\bar B)$ and solves (i). The proof of Theorem 3 can then be completed by copying the proof of Lemma 2 in [5].

Since $B(0,\mu)\neq\emptyset$, $(dd^cu(0,\mu))^n=\mu$ by [11]. We can then approximate $u(0,\mu)$ from above by smooth radially symmetric plurisubharmonic functions v_j, so that $u(0,\mu)\le v_j\le u(0,\phi_j dV)$ where $0\le\phi_j\in C_0^\infty(B)$ are rotation invariant with $\phi_j dV \rightsquigarrow d\mu$ as $j\to+\infty$. We claim that $u(\varphi, f\phi_j dV)$ tends uniformly to $u(\varphi, fd\mu)$ and so, by the convergence theorem, $(dd^cu(\varphi, f\phi_j dV))^n\rightsquigarrow(dd^cu(\varphi, fd\mu))^n=fd\mu$. Let T be a complex rotation. Then, by the transformation rule for a complex Monge–Ampère operator,

$$(dd^c[u(\nu, f\phi_j dV)\circ T])^n=f\circ T\phi_j dV$$

so that $u(\nu, f\phi_j dV)\circ T=u(\nu\circ T, f\circ T\phi_j dV)$ for each $\nu\in C(\partial B)$.

By the comparison principle,

$$u(\varphi, f\phi_j dV)\circ T+u(-\varphi\circ T+\varphi, |f\circ T-f|\phi_j dV)\le u(\varphi, f\phi_j dV)$$

and

$$u(\varphi, f\phi_j dV)+u(\varphi\circ T-\varphi, |f\circ T-f|\phi_j dV)\le u(\varphi, f\phi_j dV)\circ T,$$

so

$$\begin{aligned}
&\|u(\varphi, f\phi_j dV)\circ T-u(\varphi, f\phi_j dV)\|\\
&\quad\le 2[\sup|\varphi\circ T-\varphi|-u(0,|f\circ T-f|\phi_j dV)]\\
&\quad\le 2[\sup_{\partial B}|\varphi\circ T-\varphi|-(\sup_B|f\circ T-f|)^{1/n}u(0,\phi_j dV)]\\
&\quad\le 2[\sup_{\partial B}|\varphi\circ T-\varphi|-\sup_B|f\circ T-f|^{1/n}u(0,\mu)].
\end{aligned}$$

By the uniform continuity of φ and f on ∂B and $\operatorname{supp}\mu$, respectively, the right-hand side is small, independently of j, when T is close to the identity transformation. By Proposition 1, we can select a uniformly convergent subsequence $(u(\varphi, f\phi_{j_k}dV))_{k=1}^{\infty}$. By uniform convergence $(dd^c u(\varphi, f\phi_{j_k}dV))^n \rightsquigarrow (dd^c \lim_k u(\varphi, f\phi_{j_k}dV))^n$. But $(dd^c u(\varphi, f\phi_{j_k}dV))^n \rightsquigarrow f d\mu$, which completes the proof of Theorem 3 in the case when μ has compact support in B. Now, let $(\chi_m(r))_{m=1}^{\infty}$ be an increasing sequence of nonnegative continuous functions on the real axis such that each function is zero near 1 and such that $\chi_m(r) \nearrow 1$ as $m \to \infty$ for all $r \in [0,1[$. For each m, we solve (i) with $f\chi_m d\mu$ and find $u(\varphi, \chi_m f d\mu)$ as above. Since $(\chi_m)_{m=1}^{\infty}$ is increasing, $u(\varphi, \chi_m f d\mu)$ is decreasing and

$$u(\varphi, \chi_m f d\mu) \geq \|f\|_{L^{\infty}(\mu)}^{1/n} u(0,\mu) + u(\varphi, 0).$$

Since $\lim_{z\to\xi} u(0,\mu)(\xi) = 0$ for all $\xi \in \partial B$ (cf. [11, §II]), $\lim_{n\to\infty} u(\varphi, \chi_m f d\mu)$ solves (i). □

EXAMPLE. Let $0 < r < 1$ be fixed and let $d\sigma$ be the normalized Lebesgue measure on $\{z \in \mathbb{C}^n; |z| = r\}$. Then

$$\psi(z) = \frac{1}{(2\pi)^n} \max[\log|z|, \log r] \in \mathrm{PSH} \cap L^{\infty}(B),$$

$(dd^c\psi)^n = d\sigma$, $B(0, d\sigma) \neq \emptyset$, and Theorem 3 applies. Thus, if $\varphi \in C(\partial\Omega)$ and $0 \leq f \in L^{\infty}(d\sigma)$, then $u(\varphi, f d\sigma)$ is plurisubharmonic on B, continuous on ∂B, and solves (i).

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U. Cegrell
Department of Mathematics
Umeå University
S-901 87 Umeå
Sweden

S. Kołodziej
Department of Mathematics
Jagiellonian University
Reymonta 4
30-059 Krakow
Poland