On the Boundary Behavior of Singular Inner Functions

EVA DECKER

1. Introduction

A holomorphic function f in the open unit disc $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ is called an *inner function* if $|f(z)| \le 1$ for $z \in \mathbf{D}$ and if f has radial limits $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ of modulus 1 at almost every point $e^{i\theta}$ of the unit circle $\mathbf{T} = \partial\mathbf{D}$. If (a_k) is a sequence of complex numbers in $\mathbf{D}$ that satisfies the Blaschke condition $\sum_k (1 - |a_k|) < \infty$, then the *Blaschke product*

$$B(z) = \prod_k \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \bar{a}_k z}, \quad z \in \mathbf{D},$$

is an inner function whose zeros are exactly at the points (a_k). The boundary behavior of Blaschke products was investigated in various contexts. For a survey on this subject we refer to Colwell [7, pp. 13–44, 83]. In particular, Belna, Carroll, Colwell, and Piranian have shown in [2] and [3] that the radial boundary behavior of Blaschke products can be prescribed on any countable subset E of $\mathbf{T}$. In [11] and [12] Nicolau extended some of their results to even more general subsets E of $\mathbf{T}$ of Lebesgue measure zero.

Contrary to the situation for Blaschke products, related questions on the boundary behavior of the second basic type of inner functions, the *singular* inner functions, remained open. If f is an inner function that does not vanish in $\mathbf{D}$, then $f = cS_\mu$, where c is a unimodular constant and

$$S_\mu(z) = \exp\left(-\int_{\mathbf{T}} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t)\right), \quad z \in \mathbf{D},$$

is given by a (uniquely determined) positive finite Borel measure μ on T which is singular with respect to linear Lebesgue measure. Therefore S_μ is called a *singular inner function* with associated measure μ (cf. [13, pp. 32–33). For simplicity we shall often identify $e^{it} \in \mathbf{T}$ with $t \in (-\pi, \pi]$.

It is the aim of this paper to establish the existence of singular inner functions having a prescribed radial boundary behavior. Here, in contrast to Blaschke products, we shall be concerned with *two* different kinds of nonvanishing inner functions. For *discrete* singular inner functions S_σ, the generating

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measure σ consists entirely of point masses—that is, $\sigma = \sum_k s_k \delta_{t_k}$ where δ_{t_k} denotes the Dirac measure that assigns unit mass to the point $e^{it_k} \in \mathbf{T}$. On the other hand, a continuous singular measure ν satisfies $\nu(\{e^{it}\}) = 0$ for all $e^{it} \in \mathbf{T}$ and induces a so-called *continuous* singular inner function S_ν. In general, a singular inner function will be of mixed type $S_\mu = S_\sigma S_\nu$.

A major tool in our investigations will be the concept of *angular derivatives.* Let f be holomorphic on $\mathbf{D}$ with $|f(z)| \le 1$ for $z \in \mathbf{D}$ and let $e^{i\theta}$ be a boundary point. Then f is said to have a *finite angular derivative* at $e^{i\theta}$ if there exists $\zeta \in \mathbf{T}$ such that the limit of the difference quotient

$$f'(e^{i\theta}) = \lim \frac{\zeta - f(z)}{e^{i\theta} - z}$$

exists and is finite as $z \to e^{i\theta}$ nontangentially through $\mathbf{D}$. Obviously, this is only possible if f possesses the radial limit $f^*(e^{i\theta}) = \zeta$. The basic results on angular derivatives were developed by Carathéodory [4, no. 295–300] and Herzig [10]; see also [14, §4.3]. In particular, $f'(e^{i\theta}) \neq 0$ whenever $f \neq$ const. (cf. [14, Prop. 4.13]). For singular inner functions there is a close relationship between the mass distribution of the measure μ and the angular derivative $S'_\mu(e^{i\theta})$ of the induced function S_μ which is due to M. Riesz [16]: The singular inner function S_μ has a finite angular derivative at $e^{i\theta} \in \mathbf{T}$ if and only if $\int_{\mathbf{T}} 1/|t-\theta|^2 \, d\mu(t) < \infty$ and in the latter case

$$|S'_\mu(e^{i\theta})| = \int_{\mathbf{T}} \frac{2}{|e^{it} - e^{i\theta}|^2} \, d\mu(e^{it}) \quad \text{and} \quad S'_\mu(e^{i\theta}) = \frac{S^*_\mu(e^{i\theta})}{e^{i\theta}} |S'_\mu(e^{i\theta})|. \tag{1}$$

For an alternative proof of this fact we refer to [1, Thm. 2]; see also the connection with Theorem 5 in [8]. Moreover, by considering the argument of the difference quotient, it is easily seen that a finite angular derivative implies that f is in a certain sense conformal at $e^{i\theta}$: f carries any curve in $\mathbf{D}$ that ends nontangentially at $e^{i\theta}$ into a curve terminating at $\zeta = f^*(e^{i\theta})$ in such a way that the angle with $\mathbf{T}$ compared with the original one remains invariant (see [14, Prop. 4.10]). In particular, f maps each Stolz angle at $e^{i\theta}$ into a suitable Stolz angle at ζ; hence, following the notion in [2], $e^{i\theta}$ is a so-called *strong Fatou-ζ-point* of f. These geometric consequences will play a dominant role in our further investigations when studying angular limits of composite functions $g \circ f$.

The aim of Section 2 is the construction of singular inner functions having prescribed radial limits $S^*_\mu(e^{i\theta_j}) = a_j$ $(j = 1, 2, \ldots)$ on a given countable set $E = \{e^{i\theta_j} : j = 1, 2, \ldots\} \subseteq \mathbf{T}$ (Theorem 1). Our solution S_μ actually will be constructed in the smaller class of *purely discrete* singular inner functions. The main problem will be the solution of the special case where all the boundary values a_j are of modulus 1 (Lemma 6). Here, in addition, the measure μ can be chosen such that S_μ assumes its boundary values with *finite* angular derivative at each point $e^{i\theta_j} \in E$ and extends analytically whenever $e^{i\theta_j}$ is an isolated point of E. These nice geometric properties of the solving singular inner function will play an essential role in Section 4. In consideration of the angular derivative, the countability of E is best possible (Remark 7). These positive results on singular inner functions generalize Theorem 3 of Cargo in [5] and Theorem 9 in [8] and show the correspondence to the results of Belna, Carroll, and Piranian [2] on Blaschke products.

In Section 3 we adapt our methods to the more general problem of prescribing radial cluster sets instead of radial limits in countably many points of $\mathbf{T}$. Theorem 9 is a straightforward combination of the boundary interpolation problem of Section 2 with the constructive results in [8, §3]. This main result establishes a direct analogue to the case of Blaschke products that was investigated by Belna, Colwell, and Piranian in [3] and by Nicolau in [11, §II.4].

In Section 4 we shall be concerned with boundary value problems for the class of singular inner functions that are generated by purely *continuous* singular measures. In general, the construction of continuous singular measures is more complicated than that of discrete measures. In contrast, we shall show in Corollary 13 that the existence of continuous singular inner functions having prescribed radial limits can immediately be derived from the above investigations on the discrete case. Here the proof is mainly based on a composition argument involving angular derivatives. Finally, some open questions are discussed.

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2. Prescription of Countably Many Boundary Values

THEOREM 1. *Given any countable subset $E=\{e^{i\theta_j}: j=1,2,\dots\}$ of the unit circle $\mathbf{T}$ and any sequence (a_j) of complex numbers with $|a_j|\le 1$, there exists a discrete singular inner function S_μ having prescribed radial limits*

$$S_\mu^*(e^{i\theta_j})=a_j \quad (j=1,2,\dots). \tag{2}$$

In addition, the discrete measure μ can be chosen such that S_μ has finite angular derivative in $e^{i\theta_j}$ whenever $|a_j|=1$.

The proof of Theorem 1 will be divided into several steps. We shall first restrict ourselves to the special case where all boundary values are of modulus 1, $a_j=e^{i\omega_j}$; thus (2) becomes

$$S_\mu^*(e^{i\theta_j})=e^{i\omega_j} \quad (j=1,2,\dots).$$

Here μ will be constructed as the weak* limit of discrete measures of the form

$$\mu_n=\sum_{k=1}^{n} s_k^{(n)}\delta_{t_k^{(n)}},$$

which interpolate the first n boundary values,

$$S_{\mu_n}(e^{i\theta_j})=e^{i\omega_j} \quad (j=1,\dots,n). \tag{3}$$

(This notation of the radial limits indicates that S_{μ_n} actually extends analytically to $e^{i\theta_j}$; cf. [9, Thm. II.6.2].) This weak*-process $\mu_n \to^* \mu$ is equivalent to the convergence of $S_{\mu_n}(z) \to S_\mu(z)$ for $z \in \mathbf{D}$ $(n \to \infty)$. In order to ensure an additional convergence of the radial limit function $S^*_{\mu_n}(e^{i\theta_j}) \to S^*_\mu(e^{i\theta_j})$ as $n \to \infty$ at every point $e^{i\theta_j} \in E$, further restrictions on μ_n are necessary. We shall succeed by controlling the angular derivatives of the approximating sequence (S_{μ_n}) in the sense of the following lemma.

LEMMA 2. *Let $e^{i\theta} \in \mathbf{T}$ and let (f_n) be a sequence of holomorphic self-maps of $\mathbf{D}$ whose angular derivatives at $e^{i\theta}$ are bounded by a constant C; that is,*

$$|f_n'(e^{i\theta})| \le C < \infty \quad (n = 1, 2, \ldots). \tag{4}$$

If $f_n(z) \to f(z)$ as $n \to \infty$ for $z \in \mathbf{D}$, then the limit function f possesses a radial limit of modulus 1 at $e^{i\theta}$ with

$$f_n^*(e^{i\theta}) \to f^*(e^{i\theta}) \quad \text{as } n \to \infty \tag{5}$$

and

$$|f'(e^{i\theta})| \le \liminf_{n \to \infty} |f_n'(e^{i\theta})| \le C. \tag{6}$$

Proof. Without loss of generality, we may assume that f is not a constant. Hypothesis (4) implies the existence of radial limits $f_n^*(e^{i\theta}) = \zeta_n \in \mathbf{T}$. Let (ζ_{n_k}) be an arbitrary convergent subsequence and $\zeta \in \mathbf{T}$ its limit as $k \to \infty$. According to the Julia–Carathéodory theorem (cf. [10] and [4], no. 295–300]) we have

$$\frac{|\zeta_{n_k} - f_{n_k}(z)|^2}{1 - |f_{n_k}(z)|^2} \le C \frac{|e^{i\theta} - z|^2}{1 - |z|^2} \quad \text{for all } z \in \mathbf{D}.$$

Hence, if $k \to \infty$,

$$\frac{|\zeta - f(z)|^2}{1 - |f(z)|^2} \le C \frac{|e^{i\theta} - z|^2}{1 - |z|^2} \quad \text{for all } z \in \mathbf{D}.$$

Conversely, the Julia–Carathéodory theorem gives

$$|f'(e^{i\theta})| \le C \quad \text{and} \quad f^*(e^{i\theta}) = \zeta.$$

Since these considerations are valid for any convergent subsequence of (ζ_n), the existence of the limit $\lim_{n \to \infty} f_n^*(e^{i\theta}) = \lim_{n \to \infty} \zeta_n = f^*(e^{i\theta})$ is established. The stronger inequality (6) easily follows by considering appropriate subsequences of (f_n). □

REMARK 3. Ahern and Clark [1, Thm. I(iii)] ensure (6) under the weaker hypothesis $\liminf_{n \to \infty} |f_n'(e^{i\theta})| < \infty$, which obviously can also easily be derived from Lemma 2. However, it should be noted that with the weaker hypothesis the convergence of the radial limits (5) in general does not hold. For example, consider the singular inner functions $f_n = S_{\mu_n}$ where $\mu_{2k-1} = (1/k^2)(\delta_{-1/k} + \delta_{1/k})$ and $\mu_{2k} = (1/k)\delta_0$. Then $f_n \to f \equiv 1$ on $\mathbf{D}$ and, by (1), $\liminf_{n \to \infty} |S_{\mu_n}'(1)| = 0$, but $S^*_{\mu_{2k}}(1) = 0 \not\to 1 = f^*(1)$.

The construction of (μ_n) satisfying (3) will be done via an inductive process. In order to ensure weak*-convergence of these measures, the main problem in the step from $n-1$ to n will be adequate control of the change of its point masses and discontinuities. This is the purpose of Lemma 4. For motivation, notice that the prescription of finitely many boundary values of modulus 1 can be considered as an interpolation problem for the argument function. Thus (3), with $n-1$ instead of n, becomes

$$\arg S_{\mu_{n-1}}(e^{i\theta_j}) = \sum_{k=1}^{n-1} s_k^{(n-1)} \cot \frac{t_k^{(n-1)}-\theta_j}{2} \equiv \omega_j \bmod 2\pi \quad (j=1,\dots,n-1) \tag{7}$$

(see also relation (5) in [8]). Here the discontinuities $(t_1^{(n-1)},\dots,t_{n-1}^{(n-1)})$ may be regarded as solutions of a system of *nonlinear* equations

$$F_{n-1}(t_1,\dots,t_{n-1}) = (\omega_1,\dots,\omega_{n-1})^{\mathrm{T}} \bmod 2\pi, \tag{8}$$

where we define $F_{n-1} = (f_1,\dots,f_{n-1})^{\mathrm{T}}$ with components

$$f_j(t_1,\dots,t_{n-1}) = \sum_{k=1}^{n-1} s_k^{(n-1)} \cot \frac{t_k-\theta_j}{2} \quad (j=1,\dots,n-1).$$

For the present, our aim is to interpolate the nth boundary value by adding a further point mass $s\delta_t$ at a point $e^{it}\in \mathbf{T}$ and a slight movement of the discontinuities $t_k^{(n-1)}$ where the point masses $s_k^{(n-1)}$ remain fixed. Using the implicit function theorem, the following lemma asserts that this is possible in the case where $e^{i\theta_n}\notin \operatorname{supp}\mu_{n-1}$, proceeding on the additional assumption that the functional determinant in (8) does not vanish, that is,

$$\det\left(\frac{\partial F_{n-1}(t_1^{(n-1)},\dots,t_{n-1}^{(n-1)})}{\partial(t_1,\dots,t_{n-1})}\right) \neq 0. \tag{9}$$

A similar method can be found in the work of Belna, Carroll, and Piranian [2, esp. p. 696] for the case of Blaschke products. However, considerable differences arise from the fact that finite Blaschke products extend analytically to a neighborhood of the closed unit disc whereas singular inner functions have a singularity at every point of the support of the generating measure (cf. [9, Thm. II.6.6]). In the following, let $I(\theta,s)=\{t: |t-\theta|<s\}$.

LEMMA 4. *Let the singular inner function $S_{\mu_{n-1}}$ interpolate $n-1$ boundary values*

$$S_{\mu_{n-1}}(e^{i\theta_j}) = e^{i\omega_j}\ (j=1,\dots,n-1) \quad \textit{where} \quad \mu_{n-1} = \sum_{k=1}^{n-1} s_k \delta_{t_k^{(n-1)}} \tag{10}$$

and assume (9). *Then, for any further interpolation point $e^{i\theta_n}\notin \operatorname{supp}\mu_{n-1}$, for any boundary value $e^{i\omega_n}\in\mathbf{T}$, and for each sufficiently small point mass $s_n>0$, there exist discontinuities $t_1^{(n)},\dots,t_n^{(n)}$ such that the singular inner function associated with*

$$\mu_n = \sum_{k=1}^{n} s_k \delta_{t_k^{(n)}} \tag{11}$$

solves the nth *interpolation problem* (3), *also having a nonvanishing functional determinant. Moreover, the location of the new discontinuities can be controlled such that*

$$|t_k^{(n)} - t_k^{(n-1)}| < 2C_{n-1}s_n \ (k=1, \ldots, n-1) \quad \textit{and} \quad t_n^{(n)} \in I(\theta_n, s_n),$$

where the constant C_{n-1} *depends only on* μ_{n-1} *and* $e^{i\theta_n}$.

Except for an additional discontinuity, the new measure μ_n differs arbitrarily little from the original measure μ_{n-1} provided that s_n is chosen small enough.

Proof. (i) We claim that, for every sufficiently small value $s > 0$ and every point $e^{it} \in \mathbf{T}$ that lies sufficiently close to $e^{i\theta_n}$, the interpolation problem (10) for the first $n-1$ boundary values also can be solved by adding a further point mass $s\delta_t$. Consider the function $G = (g_1, \ldots, g_{n-1})^{\mathrm{T}}$, where

$$\begin{aligned} g_j(t_1, \ldots, t_{n-1}, t, s) &= \sum_{k=1}^{n-1} s_k \cot \frac{t_k - \theta_j}{2} + s \cot \frac{t - \theta_j}{2} \\ &= f_j(t_1, \ldots, t_{n-1}) + s \cot \frac{t-\theta_j}{2} \end{aligned} \tag{12}$$

for $j = 1, \ldots, n-1$. Then hypothesis (7) can be written in the form

$$G(t_1^{(n-1)}, \ldots, t_{n-1}^{(n-1)}, \theta_n, 0) = F_{n-1}(t_1^{(n-1)}, \ldots, t_{n-1}^{(n-1)}) = (\omega_1, \ldots, \omega_{n-1})^{\mathrm{T}}.$$

Because $\theta_j \neq t_k^{(n-1)}$, we can find suitable neighborhoods $U_k \subseteq \mathbf{R}$ of the original discontinuities $t_k^{(n-1)}$ such that G is a continuously differentiable real function of the $n+1$ variables $(t_1, \ldots, t_{n-1}, t, s)$ on

$$U_1 \times \cdots \times U_{n-1} \times I(\theta_n, m_n/3) \times (-m_n/3, m_n/3),$$

where $m_n = \min_{1 \le i < j \le n} |\theta_i - \theta_j|$. Assumption (9) ensures that the functional determinant corresponding to system (12),

$$\begin{aligned} &\det \frac{\partial G}{\partial(t_1, \ldots, t_{n-1})}(t_1^{(n-1)}, \ldots, t_{n-1}^{(n-1)}, \theta_n, 0) \\ &\quad = \det \frac{\partial F_{n-1}}{\partial(t_1, \ldots, t_{n-1})}(t_1^{(n-1)}, \ldots, t_{n-1}^{(n-1)}), \end{aligned}$$

does not vanish. Hence, applying the implicit function theorem, there exist neighborhoods $U \subseteq \mathbf{R}^2$ of $(\theta_n, 0)$ and $V \subseteq U_1 \times \cdots \times U_{n-1}$ of $(t_1^{(n-1)}, \ldots, t_{n-1}^{(n-1)})$ and a continuously differentiable function

$$g = (t_1, \ldots, t_{n-1}) \colon U \to V$$

with

$$\begin{aligned} t_k(\theta_n, 0) &= t_k^{(n-1)} \quad \text{for } k = 1, \ldots, n-1; \\ G(t_1(t,s), \ldots, t_{n-1}(t,s), t, s) &= (\omega_1, \ldots, \omega_{n-1})^{\mathrm{T}} \quad \text{for } (t,s) \in U. \end{aligned} \tag{13}$$

Hence, for each pair $(t, s) \in U$ we can find a solution of (10) whose associated measure is of the form

$$\sum_{k=1}^{n-1} s_k \delta_{t_k(t,s)} + s\delta_t. \tag{14}$$

Moreover, because g is continuously differentiable, there exists a constant C_{n-1}, depending only on $f_1, \ldots, f_{n-1}$ and $e^{i\theta_n}$, with

$$|t_k(t,s) - t_k^{(n-1)}| = |t_k(t,s) - t_k(\theta_n, 0)| \leq C_{n-1}(|t-\theta_n| + |s|) \tag{15}$$

for all $k = 1, \ldots, n-1$ and $(t,s) \in U$, where U possibly has to be replaced by a smaller neighborhood. Thus, as a conclusion of the first step, we can ensure a solution of the original interpolation problem (10) with arbitrarily good control on the location of the new discontinuities.

(ii) The free parameter $(t,s) \in U$ allows us to interpolate the additional nth boundary value. To this end, consider the behavior of the argument function related to the singular inner function with measure (14) at the point $e^{i\theta_n}$,

$$\sum_{k=1}^{n-1} s_k \cot \frac{t_k(t,s) - \theta_n}{2} + s \cot \frac{t-\theta_n}{2}, \tag{16}$$

as $t \to \theta_n$ for fixed s and $(t,s) \in U$. According to (15), for sufficiently small s, the discontinuities $t_k(t,s)$ lie arbitrarily close to $t_k^{(n-1)}$. Hence, because $e^{i\theta_n} \notin \operatorname{supp} \mu_{n-1} = \{e^{it_1^{(n-1)}}, \ldots, e^{it_{n-1}^{(n-1)}}\}$, the discontinuities are bounded away from θ_n as $t \to \theta_n$ with $t \neq \theta_n$. Consequently, the first $(n-1)$ terms in (16) remain bounded, whereas the last summand $s \cot((t-\theta_n)/2)$ tends to $\pm\infty$ as $t \to \theta_n^{\pm}$. Thus, taking $s_n > 0$ small enough, we can find a value $t_n^{(n)} \in I(\theta_n, s_n)$ that satisfies

$$\sum_{k=1}^{n-1} s_k \cot \frac{t_k(t_n^{(n)}, s_n) - \theta_n}{2} + s_n \cot \frac{t_n^{(n)} - \theta_n}{2} = \omega_n,$$

where ω_n is chosen appropriately modulo 2π. Combining this last equation with (13), the points $t_k^{(n)} = t_k(t_n^{(n)}, s_n)$ $(k = 1, \ldots, n-1)$ lead to the required measure (11) which solves the nth interpolation problem (3), and

$$\begin{aligned} |t_k^{(n)} - t_k^{(n-1)}| &= |t_k(t_n^{(n)}, s_n) - t_k(\theta_n, 0)| \leq C_{n-1}(|t_n^{(n)} - \theta_n| + |s_n|) \\ &< 2C_{n-1}s_n \quad (k = 1, \ldots, n-1). \end{aligned}$$

(iii) In the third step we shall show that, in addition, $s_n > 0$ can be chosen so small that the functional determinant associated with S_{μ_n} does not vanish. It takes the form

$$\det \frac{\partial F_n(t_1, \ldots, t_n)}{\partial(t_1, \ldots, t_n)} = \det \left(\begin{array}{c|c} \dfrac{\partial F_{n-1}(t_1, \ldots, t_{n-1})}{\partial(t_1, \ldots, t_{n-1})} & \begin{matrix} \alpha_{1n} \\ \vdots \\ \alpha_{n-1,n} \end{matrix} \\ \hline \alpha_{n1} \quad \cdots \quad \alpha_{n,n-1} & \alpha_{nn} \end{array} \right).$$

Here the $(n-1) \times (n-1)$ submatrix satisfies

$$\frac{\partial F_{n-1}(t_1^{(n)}, \ldots, t_{n-1}^{(n)})}{\partial(t_1, \ldots, t_{n-1})} \to \frac{\partial F_{n-1}(t_1^{(n-1)}, \ldots, t_{n-1}^{(n-1)})}{\partial(t_1, \ldots, t_{n-1})} \quad \text{as } s_n \to 0^+,$$

and hence is invertible whenever s_n is chosen sufficiently small. For the elements we have

$$\alpha_{jn}=\frac{\partial f_j}{\partial t_n}=-\frac{s_n}{2}\frac{1}{\sin^2((t_n^{(n)}-\theta_j)/2)}\to 0 \quad \text{as } s_n\to 0 \quad (j=1,\dots,n-1)$$

and

$$\alpha_{nn}=\frac{\partial f_n}{\partial t_n}=-\frac{s_n}{2}\frac{1}{\sin^2((t_n^{(n)}-\theta_n)/2)} \quad \text{where } |\alpha_{nn}|\geq\frac{1}{2s_n}\to\infty \text{ as } s_n\to 0^+,$$

since $t_n^{(n)}\in I(\theta_n,s_n)$. This completes the proof of Lemma 4. □

REMARK 5. The proof of Lemma 4 makes use of the hypothesis $e^{i\theta_n}\notin \operatorname{supp}\mu_{n-1}$. In order to continue our inductive process on the basis of the new solution S_{μ_n}, we need the corresponding condition $e^{i\theta_{n+1}}\notin\operatorname{supp}\mu_n$. To this end, we point out that if μ_{n-1} in Lemma 4 satisfies

$$\det\left(\cot\frac{t_k^{(n-1)}-\theta_j}{2}\right)_{j,k=1}^{n-1}\neq 0, \tag{17}$$

then a similar reasoning as in step (iii) of the proof shows that, for sufficiently small $s_n>0$, the new solution S_{μ_n} constructed according to Lemma 4 satisfies

$$\det\left(\cot\frac{t_k^{(n)}-\theta_j}{2}\right)_{j,k=1}^{n}\neq 0. \tag{18}$$

LEMMA 6. *Corresponding to each countable set $E=\{e^{i\theta_j}: j=1,2,\dots\}$ on the unit circle* **T** *and every sequence $(e^{i\omega_j})$ of complex values of modulus* 1, *there exists a* discrete *singular inner function S_μ having the prescribed radial limits*

$$S_\mu^*(e^{i\theta_j})=e^{i\omega_j} \quad (j=1,2,\dots). \tag{19}$$

In addition, the discrete measure μ can be chosen so that the induced singular inner function S_μ has finite angular derivative at every point $e^{i\theta_j}$ of E and so that S_μ can be extended analytically to $e^{i\theta_j}$ whenever $e^{i\theta_j}$ is an isolated point of E.

REMARK 7. According to a theorem of Calderón, González-Domínguez, and Zygmund (see [17, Thm. VII.7.48]), every singular inner function S_μ ($\mu\neq 0$) assumes each value ζ of modulus 1 infinitely often as a radial limit. It is well known that S_μ can have at most countably many ζ-points where, in addition, the angular derivative is finite. Conversely, for every countable subset E of **T** and $\zeta\in$ **T**, Lemma 6 guarantees the existence of a (nontrivial) discrete singular inner function with $S_\mu^*(e^{i\theta})=\zeta$ and $|S_\mu'(e^{i\theta})|<\infty$ for all $e^{i\theta}\in E$. Thus, in this sense Lemma 6 is best possible.

Proof of Lemma 6. Let $m_1=1$, $m_n=\min_{1\leq i<j\leq n}|\theta_i-\theta_j|$ for $n\geq 2$, and $\epsilon_n=(m_n/2^n)^2$. By an inductive process we shall construct discrete measures

$$\mu_n = \sum_{k=1}^{n} s_k^{(n)} \delta_{t_k^{(n)}} \quad \text{where } 0 < s_k^{(n)} < \epsilon_k \text{ and } t_k^{(n)} \in I(\theta_k, \epsilon_k) \quad (k = 1, \ldots, n), \tag{20}$$

so that the associated singular inner function S_{μ_n} interpolates the first n boundary values according to (3) under the restriction that μ_n differs only little from μ_{n-1}; this means that

$$|t_k^{(n-1)} - t_k^{(n)}| < \frac{1}{2^n} \quad \text{and} \quad |s_k^{(n-1)} - s_k^{(n)}| < \frac{1}{2^n} \quad \text{for } k = 1, \ldots, n-1 \quad (n \geq 2). \tag{21}$$

Taking $d_j = \operatorname{dist}(\operatorname{supp} \mu_j, e^{i\theta_j})$, the distance of the discontinuities $t_k^{(n)}$ to the interpolation points $e^{i\theta_j}$ will be controlled by

$$|t_k^{(n)} - \theta_j| \geq d_j \left(1 - \sum_{\nu=2}^{n} \frac{1}{2^\nu}\right) \quad \text{for all } j = 1, \ldots, n \text{ and } k = 1, \ldots, j, \tag{22}$$

where in the case $n = 1$ we have the empty sum $\sum_{\nu=2}^{n} 1/2^\nu = 0$.

This inductive process will now be carried out. In the case $n = 1$, let $0 < s_1^{(1)} < \epsilon_1$ and choose $t_1^{(1)} \in I(\theta_1, s_1^{(1)})$ satisfying

$$s_1^{(1)} \cot \frac{t_1^{(1)} - \theta_1}{2} \equiv \omega_1 \bmod 2\pi.$$

Then the induced singular inner function possesses the radial limit $S_{\mu_1}(e^{i\theta_1}) = e^{i\omega_1}$. Here, relation (22) is trivial since $|t_1^{(1)} - \theta_1| = d_1$. If, in addition, $s_1^{(1)} < m_2/3 = |\theta_1 - \theta_2|/3$, then we also have $e^{i\theta_2} \notin \operatorname{supp} \mu_1 = \{e^{it_1^{(1)}}\}$.

Now suppose that for an integer $n \geq 2$ we have constructed a solution $S_{\mu_{n-1}}$ of the $(n-1)$th interpolation problem (10) satisyfing (20) and (22) with n replaced by $n-1$. Moreover, the first step of our induction allows us to assume that $e^{i\theta_n} \notin \operatorname{supp} \mu_{n-1}$ and that $S_{\mu_{n-1}}$ fulfills the hypothesis (9) of Lemma 4 and (17) of Remark 5. For every sufficiently small value $s_n \in (0, \epsilon_n)$, Lemma 4 ensures a solution of the nth interpolation problem (3) with associated measure

$$\sum_{k=1}^{n-1} s_k^{(n-1)} \delta_{t_k^{(n)}} + s_n \delta_{t_n^{(n)}},$$

where $t_n^{(n)}$ lies close to the new interpolation point with $t_n^{(n)} \in I(\theta_n, s_n)$, and whose discontinuities $t_k^{(n)}$ for $k = 1, \ldots, n-1$ lie close to the respective discontinuities of μ_{n-1} with

$$|t_k^{(n-1)} - t_k^{(n)}| < 2C_{n-1} s_n \quad (k = 1, \ldots, n-1). \tag{23}$$

Hence we can again assume $t_k^{(n)} \in I(\theta_k, \epsilon_k)$ whenever s_n is sufficiently small. Recall that the constant C_{n-1} depends only on μ_{n-1} and $e^{i\theta_n}$. If, furthermore, s_n is so small that

$$2C_{n-1} s_n < \frac{d_j}{2^n} \quad (j = 1, \ldots, n-1), \tag{24}$$

then by (23), (24), and the inductive hypotheses corresponding to (22), for $j=1,\dots,n-1$ and $k=1,\dots,j$ we obtain

$$\begin{aligned} |t_k^{(n)}-\theta_j| &\geq |t_k^{(n-1)}-\theta_j|-|t_k^{(n)}-t_k^{(n-1)}| \\ &> d_j\left(1-\sum_{\nu=2}^{n-1}\frac{1}{2^\nu}\right)-\frac{d_j}{2^n}=d_j\left(1-\sum_{\nu=2}^{n}\frac{1}{2^\nu}\right). \end{aligned} \tag{25}$$

Since for $j=n$ and $k=1,\dots,n$ this last estimation is trivial, μ_n satisfies (22). Because $d_j \leq |t_j^{(j)}-\theta_j|<\epsilon_j<1$, it follows by (23) and (24) that

$$|t_k^{(n-1)}-t_k^{(n)}|<\frac{1}{2^n} \quad \text{for } k=1,\dots,n-1. \tag{26}$$

We choose s_n so small that μ_n also has nonvanishing functional determinant (in the sense of Lemma 4) and that μ_n satisfies (18) of Remark 5.

In the case $e^{i\theta_{n+1}} \notin \operatorname{supp}\mu_n$ we can continue our induction with the measure (20), which essentially possesses the old point masses $(s_1^{(n)},\dots,s_n^{(n)})=(s_1^{(n-1)},\dots,s_{n-1}^{(n-1)},s_n)$. If, in contrast, $e^{i\theta_{n+1}}=e^{it_{k_0}^{(n)}}$ for an index $k_0\in\{1,\dots,n\}$, then a further modification is necessary: Consider the point masses $(s_1^{(n-1)},\dots, s_{n-1}^{(n-1)},s_n)$ as a solution of the system of linear equations for the argument function whose matrix

$$\left(\cot\frac{t_k^{(n)}-\theta_j}{2}\right)_{j,k=1}^{n}$$

is invertible according to (18) in our construction. For sufficiently small movement of the critical point $t_{k_0}^{(n)}$, the modified system of linear equations remains uniquely solvable and the solution $(s_1^{(n)},\dots,s_n^{(n)})$ depends continuously on $t_{k_0}^{(n)}$. Furthermore, if (25), (26), and the conditions on the two determinants are preserved, we then obtain a solution of the interpolation problem (3) whose point masses $s_k^{(n)}$ differ only little from the original ones. Hence, we can assume $0\leq s_k^{(n)}<\epsilon_k$ $(k=1,\dots,n)$ and

$$|s_k^{(n-1)}-s_k^{(n)}|<\frac{1}{2^n} \quad \text{for } k=1,\dots,n-1.$$

Altogether the induction can be continued for all $n\in\mathbf{N}$.

According to (21), the sequences $(t_k^{(n)})_{n\geq k}$ and $(s_k^{(n)})_{n\geq k}$ are (for fixed k) Cauchy sequences. Thus there exist numbers t_k, s_k satisfying

$$t_k^{(n)}\to t_k \text{ and } s_k^{(n)}\to s_k \quad \text{as } n\to\infty \quad (k=1,2,\dots).$$

Using $0\leq s_k\leq\epsilon_k$, it is esily verified that the sequence (μ_n) converges weak* to the discrete measure $\mu=\sum_{k=1}^{\infty}s_k\delta_{t_k}$, which has finite total variation $\mu(\mathbf{T})\leq \sum_k\epsilon_k<\infty$. We claim that the induced singular inner function S_μ solves the interpolation problem (19): Let $j\in\mathbf{N}$ be fixed. On the one hand, by (22) we have, for $n\geq j$,

$$|t_k^{(n)}-\theta_j|\geq\frac{d_j}{2}>0 \quad \text{for } k=1,\dots,j.$$

On the other hand, as $t_k^{(n)} \in I(\theta_k, \epsilon_k)$ and $\epsilon_k \leq m_k/2$ we get

$$|t_k^{(n)} - \theta_j| \geq |\theta_j - \theta_k| - |\theta_k - t_k^{(n)}| \geq m_k - \frac{m_k}{2} = \frac{m_k}{2} \quad \text{for } k = j+1, \ldots, n.$$

Thus, using $\epsilon_k \leq \pi^2/4^k$, for $n \geq j$ we have

$$\int_{\mathbf{T}} \frac{1}{|t-\theta_j|^2}\, d\mu_n(t) = \sum_{k=1}^{n} \frac{s_k^{(n)}}{|t_k^{(n)} - \theta_j|^2} \leq \sum_{k=1}^{j} \frac{\epsilon_k}{(d_j/2)^2} + \sum_{k=j+1}^{n} \frac{\epsilon_k}{(m_k/2)^2}$$
$$\leq \frac{\pi^2}{d_j^2} \sum_{k=1}^{j} \frac{1}{4^{k-1}} + \sum_{k=j+1}^{\infty} \frac{1}{4^{k-1}} \leq C_j,$$

where $C_j = 2(\pi^2/d_j^2 + 1)$. Hence, by the theorem of M. Riesz mentioned in Section 1, the sequence (S_{μ_n}) has bounded angular derivatives at each of the points $e^{i\theta_j}$. Consequently, by Lemma 2, the locally uniform limit S_μ possesses a finite angular derivative for all $e^{i\theta_j} \in E$, and by (5) we obtain the existence of the radial limits

$$S_\mu^*(e^{i\theta_j}) = \lim_{n\to\infty} S_{\mu_n}^*(e^{i\theta_j}) = e^{i\omega_j}$$

for all $j \in \mathbf{N}$. □

REMARK 8. The restriction on N steps in the inductive process leads to an alternative proof of Theorem 1 in [8]. However, the implicit function theorem involves nonconstructive elements compared with the direct approach of [8].

Now, based on Lemma 6, the statement of Theorem 1 can easily be derived. Instead of completing its proof here, we shall immediately demonstrate the method in the more general situation of prescribing radial cluster sets.

3. Radial Cluster Sets of Discrete Singular Inner Functions

The radial cluster set $C_\rho(f, e^{i\theta})$ of f at $e^{i\theta} \in \mathbf{T}$ is the set of all points $a \in \mathbf{C}$ for which there exists a sequence (r_n) with $0 < r_n < 1$ and $r_n \to 1$ with $f(r_n e^{i\theta}) \to a$ as $n \to \infty$. If f is inner then necessarily $C_\rho(f, e^{i\theta})$ is a non-empty, closed, and connected subset K of the closed unit disc $\bar{\mathbf{D}}$; hence it is a continuum or a singleton in $\bar{\mathbf{D}}$ (see [6, pp. 1–3]).

THEOREM 9. *Let $E = \{e^{i\theta_j}: j = 1, 2, \ldots\}$ be an arbitrary countable subset of the unit circle $\mathbf{T}$ and let (K_j) be a sequence of non-empty, closed, and connected subsets of the closed unit disc $\bar{\mathbf{D}}$. Then there exists a discrete singular inner function S_μ that has prescribed radial cluster sets at every point of E; that is,*

$$C_\rho(S_\mu, e^{i\theta_j}) = K_j \quad (j = 1, 2, \ldots).$$

Since the function f possesses a radial limit $f^*(e^{i\theta})=a$ at $e^{i\theta}$ if and only if $C_\rho(f,e^{i\theta})=\{a\}$, Theorem 9 contains Theorem 1. On the other hand, a proper continuum $C_\rho(f,e^{i\theta})=K$ describes an oscillating behavior of f on the radius from 0 to $e^{i\theta}$. Inner functions possess radial limits almost everywhere on $\mathbf{T}$, with the possible exception of a set $E\subseteq\mathbf{T}$ of measure zero. Conversely, for the countable sets E, Theorem 9 allows one to prescribe precisely divergence phenomena for singular inner functions.

Even though both K_j and E can have rather complicated structures, the proof of Theorem 9 will be a straightforward combination of the special case in Lemma 6 and the *local* cluster set result of [8, Thm. 10]. The constructive approach used there allows a strengthened version of the latter result involving a precise control of the participating measures.

LEMMA 10. *Let K be a continuum in $\bar{\mathbf{D}}$ and let $e^{i\theta}\in\mathbf{T}$. Then, for each countable subset E of $\mathbf{T}$ and each $\epsilon>0$, there exists a discrete singular inner function S_μ with $C_\rho(S_\mu,e^{i\theta})=K$ such that $\mu(\mathbf{T})\le 2\epsilon$,*

$$\operatorname{supp}\mu\subseteq I(\theta,\epsilon),\quad \textit{and}\quad \operatorname{supp}\mu\cap E\subseteq\{e^{i\theta}\}.$$

Proof. Without loss of generality, assume $e^{i\theta}=1$. Since E is countable we can find $\alpha\in(0,\epsilon)$ such that $E\cap\{e^{\pm i\alpha/k}: k=1,2,\dots\}=\emptyset$. Consider the measure

$$\eta^{(\alpha)}=\sum_{k=1}^{\infty}s_k^{(\alpha)}(\delta_{t_k^{(\alpha)}}+\delta_{-t_k^{(\alpha)}}),$$

where $t_k^{(\alpha)}=\alpha/k$ and $s_k^{(\alpha)}=t_k^{(\alpha)}-t_{k+1}^{(\alpha)}$. Then $\eta^{(\alpha)}$ possesses the structure of the measure $\eta=\eta^{(1)}$ used in the construction of [8]. In particular, the basic asymptotic behavior of the discontinuities remains invariant, which means that

$$\frac{t_k^{(\alpha)}}{t_{k+1}^{(\alpha)}}=\frac{k+1}{k}\to 1\quad (k\to\infty)$$

and $\eta^{(\alpha)}(\mathbf{T})=\alpha$. Thus, using $\eta^{(\alpha)}$ instead of η, the construction of [8, Thm. 10] gives μ with $C_\rho(S_\mu,e^{i\theta})=K$, $\mu(\mathbf{T})\le 2\eta^{(\alpha)}(\mathbf{T})\le 2\epsilon$, and $\operatorname{supp}\mu\subseteq\operatorname{supp}\eta^{(\alpha)}\subseteq(-\epsilon,\epsilon)$, completing the proof of Lemma 10. □

We are now in a position to prove Theorem 9.

Proof of Theorem 9. Let $m_1=1$ and $m_n=\min_{1\le i<j\le n}|\theta_i-\theta_j|$ for $n\ge 2$. According to Lemma 10, for every $j\in\mathbf{N}$ and $\epsilon_j=m_j/2^j$ there exists a discrete singular inner function S_{μ_j} with radial cluster set $C_\rho(S_{\mu_j},e^{i\theta_j})=K_j$, $\mu_j(\mathbf{T})\le 2\epsilon_j$ (ensuring that the measure $\mu_1+\mu_2+\cdots$ is finite), and

$$\operatorname{supp}\mu_j\subseteq I(\theta_j,\epsilon_j)\quad\text{and}\quad \operatorname{supp}\mu_j\cap E\subseteq\{e^{i\theta_j}\}. \tag{27}$$

Hence, at every point $e^{i\theta_k}\ne e^{i\theta_j}$ the function S_{μ_j} extends analytically and therefore (trivially) possesses a radial limit of modulus 1:

$$|S_{\mu_j}^*(e^{i\theta_k})|=1\quad\text{for } k\ne j. \tag{28}$$

Even the infinite product $\prod_{j=1, j\neq k}^{\infty} S_{\mu_j}(z)$ is well-behaved when z radially approaches $e^{i\theta_k}$; that is, it assumes a radial limit of modulus 1. For $k = 1, \ldots, j-1$ we obtain $|\theta_k - \theta_j| \geq m_j$ and therefore, by definition of ϵ_j and (27), we have the estimate $\operatorname{dist}(\operatorname{supp} \mu_j, e^{i\theta_k}) > m_j/2$; consequently,

$$\int_{\mathbf{T}} \frac{1}{|t-\theta_k|} \, d\mu_j(t) \leq \frac{\mu_j(\mathbf{T})}{m_j/2} \leq 4\frac{\epsilon_j}{m_j} \leq \frac{1}{2^{j-2}} \quad \text{for } k<j. \tag{29}$$

Hence, for all $k \in \mathbf{N}$,

$$\sum_{j=k+1}^{\infty} \int_{\mathbf{T}} \frac{1}{|t-\theta_k|} \, d\mu_j(t) \leq 2 < \infty$$

and therefore, by [8, Thm. 5], the discrete singular inner function associated with $\mu_{k+1} + \mu_{k+2} + \cdots$ possesses a radial limit of modulus 1 at $e^{i\theta_k}$. Together with (28), for all $k \in \mathbf{N}$ there exists $\omega_k \in \mathbf{R}$ such that, even with the additional finitely many factors $S_{\mu_1}, \ldots, S_{\mu_{k-1}}$,

$$\prod_{j=1, j\neq k}^{\infty} S_{\mu_j}(re^{i\theta_k}) \to e^{i\omega_k} \quad \text{as } r \to 1 \quad (k=1,2,\ldots).$$

If we compensate these values by Lemma 6 with a discrete measure σ satisfying $S_\sigma^*(e^{i\theta_k}) = e^{-i\omega_k}$ $(k=1,2,\ldots)$, then $\mu = \sigma + \mu_1 + \mu_2 + \cdots$ gives the desired result $C_\rho(S_\mu, e^{i\theta_k}) = K_k$ for all k. □

REMARK 11. The geometric properties of S_μ in Theorem 9 can be improved by considering the angular derivatives. If we choose $\epsilon_j = (m_j/2^j)^2$ in the above proof, then instead of (29) we have

$$\int_{\mathbf{T}} \frac{1}{|t-\theta_k|^2} \, d\mu_j(t) \leq \frac{2\epsilon_j}{(m_j/2)^2} \leq \frac{2}{4^{j-1}} \quad \text{for } k<j.$$

In the case of a radial limit of modulus 1 (i.e., $K_k = \{e^{i\alpha_k}\}$), we fix $\mu_k = 0$. Then the analyticity of each S_{μ_j} at $e^{i\theta_k}$ $(j=1,\ldots,k-1)$ gives

$$\sum_{j=1}^{\infty} \int_{\mathbf{T}} \frac{1}{|t-\theta_k|^2} \, d\mu_j(t) < \infty.$$

Hence the solving singular inner function S_μ possesses finite angular derivative at each point $e^{i\theta_k}$ where a radial limit of modulus 1 is pre-assigned, provided that we ensure the same for S_σ according to Lemma 6.

REMARK 12. Notice that, in the special case of Theorem 1, the corresponding measures μ_j can be computed explicitly (cf. [8, Ex. 7]):

$$\mu_j = p_j \eta_j + \tau_j,$$

where

$$p_j = \frac{1}{2\pi} \ln \frac{1}{|a_j|};$$

$$\eta_j = \eta^{(\alpha)} \quad \text{with discontinuities } \theta_j \pm t_k^{(\alpha)} \ (k=1,2,\ldots), \quad \text{and}$$

$$\tau_j = s_j \delta_{t_j}, \quad \text{where } t_j \notin E \text{ and } s_j \cot \frac{t_j - \theta_j}{2} \equiv \arg a_j \bmod 2\pi.$$

4. The Case of Continuous Singular Measures

The following section deals with analogous questions for the class of continuous singular inner functions.

COROLLARY 13. *The boundary interpolation problem* (2) *for the radial limit function can also be solved by* S_μ *where the associated singular measure* μ *is continuous.*

Proof. Let ν be an arbitrary (nontrivial) continuous singular Borel measure on $\mathbf{T}$. S_ν assumes 0 and each value of modulus 1 as a radial limit (cf. the theorem of Caldéron et al. in [17, Chap. VII]). Thus, if $a_j = 0$ or $a_j \in \mathbf{T}$ we can find $z_j \in \mathbf{T}$ where $S_\nu^*(z_j) = a_j$. If $0 < |a_j| < 1$ then there may be a value $z_j \in \mathbf{D}$ where $S_\nu(z_j) = a_j$. Otherwise, in the case where S_ν omits the value a_j on $\mathbf{D}$, a theorem of Seidel (see [13, p. 37, Thm. 6(ii)]) asserts that the inner function S_ν possesses a_j as radial limit, hence once more $S_\nu^*(z_j) = a_j$ for a suitable boundary point $z_j \in \mathbf{T}$. According to Theorem 1, there exists a singular inner function S_σ with radial limits $S_\sigma^*(e^{i\theta_j}) = z_j e^{-i\theta_j}$ $(j = 1, 2, \ldots)$ having a finite angular derivative at those points $e^{i\theta_j} \in E$ where $z_j \in \mathbf{T}$. Then the inner function $zS_\sigma(z)$ maps the respective radius from 0 ending at $e^{i\theta_j}$ onto an arc that completely lies in a Stolz angle at $z_j \in \mathbf{T}$. This ensures that the composite function

$$f = S_\nu \circ (zS_\sigma)$$

approaches the required radial limits $\lim_{r\to 1} f(re^{i\theta_j}) = a_j$ $(j = 1, 2, \ldots)$. As a composition of inner functions, f again is inner [see e.g. [15, p. 323]), f does not vanish on $\mathbf{D}$ and is positive at the origin, and $f(0) = S_\nu(0) > 0$. Hence we have $f = S_\nu \circ (zS_\sigma) = S_\mu$ with a suitable singular measure μ on T. Since ν is continuous, by a result in [15] (case (3) in the proof of Theorem 5.7 and Remark 6.1d), μ has no discrete part, thus proving that the interpolation problem (2) is always solvable in the class of purely continuous singular inner functions. In order to guarantee well-behaved solutions of (2) with respect to the angular derivative, let ν satisfy $\operatorname{supp} \nu \neq \mathbf{T}$ and choose the points $z_j \in \mathbf{T}$ with $S_\nu^*(z_j) = a_j$ such that $z_j \notin \operatorname{supp} \nu$ if $|a_j| = 1$ in the construction above. Then S_ν can be extended analytically to z_j, and S_μ has finite angular derivative at $e^{i\theta_j}$ provided that S_σ has a finite angular derivative at $e^{i\theta_j}$. □

REMARK 14. (i) We must leave as an open question the problem of whether there is a similar easy composition argument that yields the more general *cluster-set* result for continuous singular inner functions from the corresponding discrete case (Theorem 9). Nevertheless, it should be pointed out that the existence of continuous singular inner functions having prescribed radial cluster sets at finitely many points of **T** can be established by adapting the constructive methods given in [8]. We will only sketch the main ideas of the necessary modifications. Let $(t_k) \subseteq (0,1)$ be strictly decreasing with $t_k \to 0$ and $t_{k+1}/t_k \to 1$ as $k \to \infty$, and choose $\epsilon_k > 0$ such that the arcs $I_k = \{e^{it} : t \in [t_k - \epsilon_k, t_k]\}$ are mutually disjoint. Consider $\nu = \sum_k \nu_k + \tilde{\nu}_k$, where ν_k is a continuous singular measure that assigns the mass $s_k = t_k - t_{k+1}$ to I_k and $\tilde{\nu}_k$ has the corresponding mass distribution symmetric to the real axis compared with ν_k. Then it is easily seen that $S_\nu^*(1) = \exp(-2\pi)$ (compare Lemma 3 and Example 7 in [8]). Hence the construction in [8, §3] (with ν playing the role of η) gives a continuous singular measure μ having a prescribed radial cluster set at 1. Furthermore, the closed support of the suitable measure can be controlled by $\operatorname{supp} \mu \subseteq \{1\} \cup \bigcup_k (I_k \cup \tilde{I}_k)$ and can be chosen arbitrarily close to 1. The final statement now follows as in the proof of [8, Cor. 12]—keeping in mind that our Corollary 13 allows one to interpolate radial limits by continuous singular inner functions.

(ii) It would be interesting to know whether Nicolau's results in [12] on the boundary values of Blaschke products hold also for the class of singular inner functions.

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Mathematisches Institut I
Universität Karlsruhe
Englerstraße 2
D 76128 Karlsruhe
Germany