Geometry of Operator Spaces

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1. Introduction

The results in this paper aim at giving information on the position of $K(X, Y)$, the space of compact linear operators between Banach spaces X and Y, in $L(X, Y)$, the space of bounded linear operators.

It is known that for a number of spaces, for example $X = l^p$ and $Y = l^q$ $(1 < p, q < \infty)$, $K(X, Y)$ is an M-ideal in $L(X, Y)$ (the definition of an M-ideal is given in Section 2); thus the position of $K(X, Y)$ in $L(X, Y)$ vaguely resembles the position of c_0 in l^∞ in this case. We show in Section 2 that, in several instances, a necessary condition for $K(X, Y)$ to be an M-ideal in $L(X, Y)$ is that $K(l^1, Y)$ be an M-ideal in $L(l^1, Y)$; and we go on to investigate those Banach spaces Y for which the latter holds. We prove that such a space is a nonreflexive (unless finite-dimensional) Asplund space; in fact, it is even an M-ideal in its bidual. For the proof of the latter assertion we offer a characterization of M-ideals X in X^{**} which yields in particular that this property is separably determined. Moreover, we prove for a separable space with the metric compact approximation property that $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$ if and only if $K(X, Y)$ is an M-ideal in $L(X, Y)$ for every Banach space X. This class of Banach spaces, called (M_∞)-spaces in [29], was introduced and investigated in [30].

Section 3 deals with the problem of unique Hahn–Banach extensions from $K(X, Y)$ to $L(X, Y)$. The results in this section are motivated by two recent results. For a certain class of Banach spaces X that includes the l^p $(1 < p < \infty)$ spaces, it is proved in [29] that for any Banach space Y, every continuous linear functional on $K(X, Y)$ has a unique norm-preserving extension to a linear functional on $L(X, Y)$. On the other hand, one of us [24] has recently shown that if x is a denting point of the unit ball X_1 of X and y^* is a w^*-denting point of Y_1^* then the functional $x \otimes y^*$ has unique norm-preserving extension from $R(X, Y)$, the space of finite rank operators, to $L(X, Y)$.

We study the properties of a Banach space X for which, for a compact Hausdorff space Ω, extreme points in the unit ball of $K(X, C(\Omega))^*$ have unique

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norm-preserving extensions to $L(X, C(\Omega))$. We extend the result of [24] mentioned above by showing that when Λ is a w^*-denting point of X_1^{**} or y^* is a w^*-denting point of Y_1^*, the functional $\Lambda \otimes y^* \in K(X, Y)^*$ has unique norm-preserving extension. This in turn allows us to enlarge the class of Banach spaces for which the main theorem of [24] is valid. Our approach is based on a characterization of denting points obtained by Lin, Lin, and Troyanski [26]. In particular, it turns out for a separable Banach space X that extreme functionals in the dual unit ball of $K(X, c)$ have unique norm preserving extensions if and only if X is reflexive and every extreme point in the unit ball is a denting point.

In this paper we consider for the most part real Banach spaces, although the transfer to complex spaces should cause no difficulty. For a Banach space X we denote by X_1, $S(X)$, and $\partial_e X_1$ the unit ball, surface of the unit ball, and the set of extreme points of the unit ball, respectively. The symbol $\delta(t)$ stands for the Dirac measure supported at the point t.

2. *M*-Ideals of Operators on l^1

This section deals with M-ideals of compact operators. Recall that a closed subspace F of a Banach space E is called an *M-ideal* if there is a linear projection P from E^* onto $F^\perp := \{x^* \in E^* : x^*(x) = 0 \ \forall x \in F\}$ satisfying

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\| \quad \forall x^* \in E^*.$$

In case P is even weak* continuous, F is the range of a projection Q on E satisfying

$$\|x\| = \max\{\|Qx\|, \|x - Qx\|\} \quad \forall x \in E.$$

In this case F is called an *M-summand*. These definitions are due to Alfsen and Effros [1]; a detailed exposition of M-ideal theory appears in [14].

We shall frequently use the following criterion (known as the *3-ball property*) in order to verify that a subspace is an M-ideal. Its proof can be found in [20, Thm. 6.17] or [14, Thm. I.2.2].

THEOREM 2.1. *A closed subspace F of a Banach space E is an M-ideal if and only if, for all $x \in E_1$, all $y_1, y_2, y_3 \in F_1$, and all $\epsilon > 0$, there is some $y \in F$ such that*

$$\|x + y_i - y\| \le 1 + \epsilon \quad (i = 1, 2, 3).$$

We shall also need the following characterization of M-ideals of compact operators from [41, Thm. 3.1 & Remark].

THEOREM 2.2. *The space of compact operators $K(X, Y)$ is an M-ideal in $L(X, Y)$ if and only if, for all $T \in L(X, Y)$ with $\|T\| \le 1$, there is a net (K_α) in the unit ball of $K(X, Y)$ such that $K_\alpha^* y^* \to T^* y^*$ for all $y^* \in Y^*$ satisfying*

$$\limsup \|S + (T - K_\alpha)\| \le 1 \quad \forall S \in K(X, Y), \ \|S\| \le 1.$$

Our first result explains the importance of the space of operators on l^1 in the present setting.

PROPOSITION 2.3. *Assume that $K(X, Y)$ is an M-ideal in $L(X, Y)$. If X^* contains a subspace isomorphic to c_0, then $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$.*

Proof. If X^* contains a copy of c_0, then, by a theorem due to Bessaga and Pełczyński, X contains a complemented copy of l^1. But more is true: By James's distortion theorem [27, Prop. 2.e.3] X^* contains, for every $\epsilon > 0$, a subspace $(1+\epsilon)$-isomorphic to c_0, and hence (see the proof in [27, Prop. 2.e.8]) X contains, for every $\epsilon > 0$, a subspace $(1+\epsilon)$-isomorphic to l^1 which is the range of a projection of norm $\leq 1+\epsilon$. Now a routine application of Theorem 2.1 yields the claim. □

As an application we will establish the following proposition.

PROPOSITION 2.4. *Assume that $K(X, Y)$ is an M-ideal in $L(X, Y)$ and that Y is an order unit space. Then $K(X, Y) = L(X, Y)$.*

Proof. Recall that an order unit space Y is a subspace of some $C(\Omega)$-space containing the constants. Such a space has a representation as the space of affine continuous functions $A(K)$ on the compact convex set of states on Y.

To show that $K(X, Y) = L(X, Y)$, it is enough to prove that $K(X, Y)$ is an M-summand [35, Thm. 2.9]. We distinguish whether or not X^* contains a copy of c_0. If this is not the case, then X^* has the so-called intersection property introduced in [2] and studied further in [13] (see also [14, §II.4]). But then also $K(X, Y)$ $(= A(K, X^*)$ if Y is represented as $A(K))$ has the intersection property, and $K(X, Y)$ is an M-summand [2].

Next suppose that c_0 embeds into X^*. Then, by Proposition 2.3, $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$; and now the above argument shows that $K(l^1, Y) = L(l^1, Y)$ since $l^\infty = (l^1)^*$ satisfies the intersection property. (That $K(l^1, Y) = L(l^1, Y)$ also follows from [21] since both $(l^1)^*$ and Y are order unit spaces.) Hence $\dim Y < \infty$ and $K(X, Y) = L(X, Y)$. □

We are now in a position to provide an answer to a question raised by E. Behrends.

COROLLARY 2.5. *Let X be a Banach space and Ω be a compact Hausdorff space such that $K(X, C(\Omega))$ is an M-ideal in $L(X, C(\Omega))$. Then X or $C(\Omega)$ is finite-dimensional.*

Proof. Proposition 2.4 yields that $K(X, C(\Omega)) = L(X, C(\Omega))$. The assertion now follows since for infinite-dimensional X there is a noncompact bounded linear operator from X into c_0 (the Josefson–Nissenzweig theorem, see [4, Chap. XII]) and for infinite-dimensional $C(\Omega)$ the sequence space c_0 embeds into $C(\Omega)$. □

For a related result involving L^1-predual spaces, see Corollary 2.14.

In the following we try to gain some insight into the nature of those spaces for which $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$. First we show that Y is necessarily an Asplund space. Such spaces are studied in some detail for instance in [32]; we remark that Y is an Asplund space if and only if Y^* has the Radon-Nikodym property (RNP) if and only if every separable subspace of Y has a separable dual [32, pp. 34, 75]. In due course we need the very simple fact that for each bounded sequence (y_n) in Y there exists some $T \in L(l^1, Y)$ such that $Te_n = y_n$. The operator T is compact if and only if $\{y_n : n \in \mathbb{N}\}$ is relatively compact.

LEMMA 2.6. *If $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$, then Y is an Asplund space.*

Proof. Let $Z \subset Y$ be a separable subspace. We shall show that Z^* is separable. Let $\epsilon > 0$ and $0 < c < 1$. We assume that Z^* is not separable. Then, by a result of Stegall [5, Lemma 5, p. 194], there exist sequences (z_n) in the unit ball of Z and (f_n) in the unit ball of Z^* such that

$$f_m(z_n) \geq c \quad \text{for } m \geq n, \tag{1}$$

$$|f_m(z_n)| \leq \epsilon \quad \text{for } m < n. \tag{2}$$

See [22, p. 33] for the argument leading to (1) and (2). We now consider norm-preserving extensions of the (f_m) to all of Y, and we retain the notation (f_m) for these extensions. Let $f \in Y^*$ be a weak* cluster point of the f_m and $z^{**} \in Y^{**}$ be a weak* cluster point of the z_n. Then we have

$$f(z_n) \geq c \quad \forall n, \tag{1'}$$

$$|f_m(z^{**})| \leq \epsilon \quad \forall m. \tag{2'}$$

Define $T: l^1 \to Y$ by $Te_n = z_n$ and $S: l^1 \to Y$ by $Se_n = z_1$ for all n. Then $\|T\| \leq 1$, $\|S\| \leq 1$, and S is compact. By Theorem 2.1, there is some $U \in K(l^1, Y)$ such that $\|T \pm S - U\| \leq 1 + \epsilon$. Consequently

$$\|z_1 \pm (z_n - Ue_n)\| \leq 1 + \epsilon \quad \forall n \in \mathbb{N}.$$

Since U is compact, we may assume that (Ue_n) converges (otherwise we pass to a subsequence), say to $u \in Y$. Then

$$\|z_1 \pm (z_n - u)\| \leq 1 + 2\epsilon \quad \forall n \geq n_0,$$

and again there is no loss of generality in assuming that this holds for all $n \in \mathbb{N}$. Now we deduce from (1′) that

$$\begin{aligned} 1 + 2\epsilon &\geq \max_{\pm} |f(z_1) \pm (f(z_n) - f(u))| \\ &= |f(z_1)| + |f(z_n) - f(u)| \\ &\geq c + |f(z_n)| - |f(u)|, \end{aligned}$$

so that

$$|f(u)| \geq |f(z_n)| + c - 1 - 2\epsilon \quad \forall n \in \mathbb{N}.$$

Thus

$$|f(u)| \geq |f(z^{**})| + c - 1 - 2\epsilon \geq 2c - 1 - 2\epsilon.$$

On the other hand we obtain, from the weak* lower semicontinuity of the norm,

$$\|z_1 \pm (z^{**} - u)\| \leq 1 + 2\epsilon.$$

Hence, by (1) and (2′),

$$\begin{aligned} 1 + 2\epsilon &\geq \max_{\pm} |f_m(z_1) \pm (f_m(z^{**}) - f_m(u))| \\ &= |f_m(z_1)| + |f_m(z^{**}) - f_m(u)| \\ &\geq c + |f_m(u)| - |f_m(z^{**})| \\ &\geq c + |f_m(u)| - \epsilon. \end{aligned}$$

Thus $|f_m(u)| \leq 1 + 3\epsilon - c$ and therefore

$$|f(u)| \leq 1 + 3\epsilon - c.$$

Hence we get $1 + 3\epsilon - c \geq 2c - 1 - 2\epsilon$ and $2 + 5\epsilon \geq 3c$. Choosing ϵ small enough and c close enough to 1 yields a contradiction. □

In the case of separable Y, we can easily supply a stronger result.

LEMMA 2.7. *If $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$ and Y is separable, then Y is an M-ideal in its bidual, and if Y is infinite-dimensional then it is nonreflexive.*

Proof. Let $y_1, y_2, y_3 \in Y_1$, $y^{**} \in Y_1^{**}$, and $\epsilon > 0$. We wish to find some $y \in Y$ such that

$$\|y^{**} + y_i - y\| \leq 1 + \epsilon \quad \text{for } i = 1, 2, 3$$

holds. Then Theorem 2.1 shows that Y is an M-ideal in Y^{**}.

To achieve this, consider a quotient mapping Q from l^1 onto Y. Hence there is some $\xi \in (l^1)^{**}$, $\|\xi\| \leq 1$, such that $Q^{**}(\xi) = y^{**}$. Next pick $x^* \in (l^1)^*$ such that $\xi(x^*) = 1$ and $\|x^*\| \leq 1 + \epsilon/2$, and define compact operators $S_i: l^1 \to Y$ by $S_i(x) = x^*(x) y_i$. An appeal to Theorem 2.1 yields a compact operator $S: l^1 \to Y$ such that

$$\|Q + S_i - S\| \leq 1 + \epsilon \quad \text{for } i = 1, 2, 3.$$

Consequently, for $y := S^{**}(\xi) \in Y$ (since S is compact), the estimate

$$\|y^{**} + y_i - y\| = \|Q^{**}(\xi) + S_i^{**}(\xi) - S^{**}(\xi)\| \leq 1 + \epsilon$$

holds.

If Y were reflexive and infinite-dimensional, then $L(l^1, Y) \cong L(Y^*, l^\infty)$ canonically and hence $K(Y^*, l^\infty)$ would be an M-ideal in $L(Y^*, l^\infty)$, contradicting Corollary 2.5. □

Since Banach spaces which are M-ideals in their biduals are Asplund spaces ([22] or [14, Thm. III.3.1]), Lemma 2.7 implies Lemma 2.6 for separable Y.

Actually, we shall eventually prove in Theorem 2.12 that Lemma 2.7 holds without the assumption of separability. However, in order to prove this, we need Lemmas 2.6 and 2.7 and another two auxiliary results. The first one says that the property of being an M-ideal in the bidual is separably determined. Its proof is inspired by [8, Prop. 2.3].

PROPOSITION 2.8. *For a Banach space X, the following assertions are equivalent:*

(i) *X is an M-ideal in its bidual.*

(ii) *For all $x \in X_1$, all sequences (x_n) in X_1, all weak* cluster points x^{**} of (x_n), and all $\epsilon > 0$, there is some $u \in \operatorname{co}\{x_1, x_2, \dots\}$ such that*

$$\|x + x^{**} - u\| \le 1 + \epsilon.$$

(iii) *For all $x \in X_1$, all sequences (x_n) in X_1, and all $\epsilon > 0$, there is some $n \in \mathbb{N}$ and there are $u \in \operatorname{co}\{x_1, \dots, x_n\}$ and $t \in \operatorname{co}\{x_{n+1}, x_{n+2}, \dots\}$ such that*

$$\|x + t - u\| \le 1 + \epsilon.$$

(iv) *For all $x \in X_1$ and all $x^{**} \in X_1^{**}$, there is a net (x_α) in X_1 weak* converging to x^{**} such that*

$$\limsup \|x + x^{**} - x_\alpha\| \le 1.$$

(v) *Every closed separable subspace Y of X is an M-ideal in its bidual.*

Proof. (i) ⇒ (ii): If this were false, then the inequality $\|x + x^{**} - u\| > 1 + \epsilon$ would hold for all $u \in A := \operatorname{co}\{x_1, x_2, \dots\}$. Consequently, A and the ball $B(x + x^{**}, 1 + \epsilon/2)$ could strictly be separated by some $x^{***} \in X^{***}$, with $\|x^{***}\| = 1$ say. By assumption we have a decomposition

$$x^{***} = x^* + x_s^{***} \in X^* \oplus X^\perp, \quad \|x^{***}\| = \|x^*\| + \|x_s^{***}\|.$$

Hence

$$\begin{aligned} 1 + \epsilon/2 &\le x^{***}(u - (x + x^{**})) \\ &\le |x^*(x)| + |x^*(x^{**} - u)| + |x_s^{***}(x^{**})| \\ &\le \max\{\|x\|, \|x^{**}\|\}(\|x^*\| + \|x_s^{***}\|) + |x^*(x^{**} - u)| \\ &\le 1 + |x^*(x^{**} - u)| \end{aligned}$$

for all $u \in A$; however $|x^*(x^{**} - x_n)| < \epsilon/2$ for some n since x^{**} is a weak* cluster point of the sequence (x_n). This leads to a contradiction.

(ii) ⇒ (iii): Let x^{**} be a weak* cluster point of the sequence (x_n). We apply (ii) to obtain

$$\|x + x^{**} - u\| \le 1 + \epsilon/3$$

for some $u \in \operatorname{co}\{x_1, x_2, \dots\}$, say $u \in \operatorname{co}\{x_1, \dots, x_n\}$. Suppose that $\|x + t - u\| > 1 + \epsilon$ holds for each $t \in \operatorname{co}\{x_{n+1}, x_{n+2}, \dots\} =: A$. Again this implies that A

and $B(u-x, 1+\epsilon/2)$ can strictly be separated. Thus, for some $x^* \in X^*$ with $\|x^*\| = 1$,

$$1+\epsilon/2 \le x^*(t-(u-x)) \quad \forall t \in A.$$

But since $x^{**} \in \bar{A}^{w*}$, this yields

$$\begin{aligned} 1+\epsilon/2 &\le x^*(x+x^{**}-u) \\ &\le \|x+x^{**}-u\| \\ &\le 1+\epsilon/3, \end{aligned}$$

a contradiction.

(iii) ⇒ (iv): Again we argue by contradiction. Suppose that, for some $x \in X_1$ and $x^{**} \in X_1^{**}$, there is no such net. Consequently there is, for some $\epsilon > 0$, a convex weak* neighborhood V of x^{**} such that

$$\|x+x^{**}-v\| > 1+\epsilon \quad \forall v \in V. \tag{3}$$

Pick $x_1 \in V \cap X_1$ and put $B_1 = -x + x_1 + (1+\epsilon)X_1^{**}$. This is a weak* compact set not containing x^{**} (by (3)), so there is a convex weak* neighborhood $W_1 \subset V$ of x^{**} such that $W_1 \cap B_1 = \emptyset$. This means

$$\|x+w-x_1\| > 1+\epsilon \quad \forall w \in W_1.$$

Next choose $x_2 \in W_1 \cap X_1$, put $B_2 = -x + \text{co}\{x_1, x_2\} + (1+\epsilon)X_1^{**}$, and find a convex weak* neighborhood $W_2 \subset W_1$ of x^{**} satisfying $W_2 \cap B_2 = \emptyset$, that is,

$$\|x+w-u\| > 1+\epsilon \quad \forall w \in W_2, \; u \in \text{co}\{x_1, x_2\}.$$

Continuing in this manner, we inductively define a sequence of points (x_n) in X_1 and a sequence of convex weak* neighborhoods $V \supset W_1 \supset W_2 \supset \cdots$ such that $x_{n+1} \in W_n$ and

$$\|x+w-u\| > 1+\epsilon \quad \forall w \in W_n, \; u \in \text{co}\{x_1, \ldots, x_n\}$$

for all $n \in \mathbb{N}$. This is a contradiction to (iii), since $\text{co}\{x_{n+1}, x_{n+2}, \ldots\} \subset W_n$.

(iv) ⇒ (i): We shall verify that the canonical projection from X^{***} onto X^* is an L-projection. To this end, decompose a given $x^{***} \in X_1^{***}$ into $x^{***} = x^* + x_s^{***} \in X^* \oplus X^\perp$. For an arbitrary $\epsilon > 0$, pick $x \in X_1$ and $x^{**} \in X_1^{**}$ such that $x^*(x) \ge \|x^*\| - \epsilon$ and $x_s^{***}(x^{**}) \ge \|x_s^{***}\| - \epsilon$. By (iv) there is a net (x_α) such that, for sufficiently large α,

$$|x^*(x^{**}-x_\alpha)| \le \epsilon \quad \text{and} \quad \|x+x^{**}-x_\alpha\| \le 1+\epsilon.$$

This yields

$$\begin{aligned} \|x\| + \|x_s^{***}\| - 2\epsilon &\le x^*(x) + x_s^{***}(x^{**}) \\ &\le \langle x^* + x_s^{***}, x + x^{**} - x_\alpha \rangle + \epsilon \\ &\le \|x^{***}\| \cdot (1+\epsilon) + \epsilon, \end{aligned}$$

so that in fact $\|x^*\| + \|x_s^{***}\| = \|x^{***}\|$.

(i) ⇒ (v): This follows from the fact that the class of Banach spaces satisfying (i) is hereditary ([12] or [14, Thm. III.1.6]).

(v) ⇒ (iii): This holds since we already know that (i) implies (iii). □

Next, we have the following stability result. Recall that a Banach space E has the *bounded* compact approximation property if there is a bounded net (u_α) of compact operators on E such that $u_\alpha x \to x$ for all $x \in E$. The *metric* compact approximation property requires in addition that $\sup\|u_\alpha\| \le 1$; if $E = F^*$ is a dual space and the u_α can be chosen to be weak* continuous, we say that F^* has the bounded or metric compact approximation property with adjoint operators.

PROPOSITION 2.9. *Suppose that X^* or Y is an Asplund space and that $K(X, Y)$ is an M-ideal in $L(X, Y)$.*

(a) *If X^* has the bounded compact approximation property with adjoint operators and $V \subset Y$ is a closed subspace, then $K(X, V)$ is an M-ideal in $L(X, V)$.*

(b) *If Y^* has the bounded compact approximation property with adjoint operators and $E \subset X$ is a closed subspace, then $K(X/E, Y)$ is an M-ideal in $L(X/E, Y)$.*

Proof. (a) We apply Theorem 2.2. Let $v \in L(X, V)$, $\|v\| \le 1$, and denote by $J: V \to Y$ the inclusion operator. Pick bounded nets $(u_\alpha) \subset K(X)$ such that $u_\alpha^* x^* \to x^*$ for all $x^* \in X^*$ and $(K_\beta) \subset K(X, Y)$ as in Theorem 2.2, applied to $Jv \in L(X, Y)$. After switching to the product index set we may suppose that the K_β are indexed by the same set as the (u_α), hence we shall write K_α from now on. Then the net $(T_\alpha) = (Jvu_\alpha - K_\alpha)$ converges to 0 in the weak topology $\sigma(K(X, Y), K(X, Y)^*)$; in fact,

$$\langle x^{**}, T_\alpha^* y^* \rangle = \langle T_\alpha, x^{**} \otimes y^* \rangle \to 0,$$

and the linear span of these functionals is dense in $K(X, Y)^*$ since X^{**} or Y^* has the RNP (see [6] and [16]). Consequently, suitable convex combinations of the T_α converge in norm; hence there are $\hat{K}_\alpha \in \operatorname{co}\{K_{\alpha'}: \alpha' > \alpha\}$ and $\hat{u}_\alpha \in \operatorname{co}\{u_{\alpha'}: \alpha' > \alpha\}$ such that $\|Jv\hat{u}_\alpha - \hat{K}_\alpha\| \to 0$. Thus we have, for $w \in K(X, V)$ with $\|w\| \le 1$,

$$\limsup\|w + (v - v\hat{u}_\alpha)\| \le \limsup\|Jw + (Jv - \hat{K}_\alpha)\| \le 1.$$

A second appeal to Theorem 2.2 finishes the proof since $(v\hat{u}_\alpha)^* \to v^*$ pointwise.

(b) The proof is similar, the basic observation being ($q: X \to X/E$ the quotient map) that this time $u_\alpha vq - K_\alpha \to 0$ weakly. □

COROLLARY 2.10. *If $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$ and $V \subset Y$ is a closed subspace, then $K(l^1, V)$ is an M-ideal in $L(l^1, V)$.*

Proof. We know from Lemma 2.6 that Y is an Asplund space. Further, we observe that l^∞ has the metric approximation property and hence, as an easy application of the principle of local reflexivity reveals, the metric approximation property with adjoint operators. It is left to apply Proposition 2.9. □

In connection with the last argument, we mention the recent example of a dual space with the metric compact approximation property which fails the metric compact approximation property with adjoint operators [11].

Concerning quotients of the range space, we have the following easy result.

PROPOSITION 2.11. *If $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$ and V is a closed subspace of Y, then $K(l^1, Y/V)$ is an M-ideal in $L(l^1, Y/V)$.*

Proof. This is an immediate consequence of the lifting property of l^1, which says that for each (compact) $T \in L(l^1, Y/V)$ with $\|T\| < 1$ there is some (compact) $\hat{T} \in L(l^1, Y)$ with $\|\hat{T}\| < 1$ such that $q\hat{T} = T$, and the 3-ball property. (Here $q: Y \to Y/V$ denotes the quotient map.) □

We now come to the main results of this section. A Banach space Y is called an (M_∞)-space if $K(Y \oplus_\infty Y)$ is an M-ideal in $L(Y \oplus_\infty Y)$. This class of Banach spaces is introduced and studied in [30]; see also [29] and [42]. In [30] it is proved that Y is an (M_∞)-space if and only if $K(X, Y)$ is an M-ideal in $L(X, Y)$ for every Banach space X if and only if the same conclusion holds for $X = Y \oplus_\infty Y$. This will be used below. The class of (M_∞)-spaces encompasses $c_0(\Gamma)$ and those of its subspaces or quotient spaces which enjoy the metric compact approximation property [30; 39]. It was recently proved in [18] that a separable (M_∞)-space embeds almost isometrically into c_0.

In particular, $K(l^1, Y)$ forms an M-ideal in $L(l^1, Y)$ for an (M_∞)-space Y. Part (c) of the following theorem provides the converse for separable spaces with the metric compact approximation property.

THEOREM 2.12. *Suppose $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$.*

(a) *Then Y is an M-ideal in its bidual, and every infinite-dimensional subspace of a quotient of Y is nonreflexive.*
(b) *If Y has the metric compact approximation property, then $K(X, Y)$ is an M-ideal in $L(X, Y)$ for all separable spaces X.*
(c) *If V is a closed separable subspace of a quotient of Y and if V has the metric compact approximation property, then V is an (M_∞)-space.*

Proof. (a) Let $V \subset Y$ be separable. Then, by Corollary 2.10 and Lemma 2.7, V is an M-ideal in V^{**}. Proposition 2.8 shows that Y is an M-ideal in Y^{**}. Likewise, it follows that no infinite-dimensional subspace of a quotient is reflexive (note Proposition 2.11).

(b) Since Y is an M-ideal in Y^{**}, we deduce from [10] that Y^* has the metric compact approximation property with adjoint operators. Now X is isometric to a quotient of l^1, hence Proposition 2.9(b) shows our claim.

(c) By (b), $K(V \oplus_\infty V, V)$ is an M-ideal in $L(V \oplus_\infty V, V)$, which is enough to prove the assertion of (c). □

COROLLARY 2.13. *If Y is a closed subspace of a quotient of an (M_∞)-space Z, then $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$, and Y contains a copy of c_0 unless it is finite-dimensional.*

Proof. First of all, $K(l^1, Z)$ is an M-ideal in $L(l^1, Z)$ [30]. The first assertion follows from Corollary 2.10 and Proposition 2.11, and the second from Theorem 2.12(a) and [12]. □

COROLLARY 2.14. *Suppose X^* contains a copy of c_0, and let Y denote an L^1-predual space. Then $K(X, Y)$ is an M-ideal in $L(X, Y)$ if and only if Y is isometric with $c_0(\Gamma)$.*

Proof. The 'if' part is clear since $c_0(\Gamma)$ is an (M_∞)-space. On the other hand, Proposition 2.3 and Theorem 2.12(a) imply that Y is an L^1-predual which is an M-ideal in its bidual. Now the converse follows from [12, p. 259]. □

We finish this section with some comments and questions on the above results.

(1) An inspection of the proofs of Proposition 2.9 and Theorem 2.12 reveals that it is enough to assume that V is almost isometric to a subspace of Y, meaning that for every $\epsilon > 0$ there is an operator $T_\epsilon : V \to Y$ satisfying $(1-\epsilon)\|v\| \le \|T_\epsilon v\| \le (1+\epsilon)\|v\|$ for all $v \in V$.

(2) It appears that the relation between the class of spaces for which $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$ and the quotients of subspaces of (M_∞)-spaces is quite intimate; see Theorem 2.12 and Corollary 2.13. Do these classes actually coincide?

(3) It is essentially shown in [17] that a (separable) Banach space Y is an (M_∞)-space if and only if $K(Y)$ is an M-ideal in $L(Y)$ and

$$\limsup \|y + y_n\| = \max\{\|y\|, \limsup \|y_n\|\}$$

whenever $y_n \to 0$ weakly. Let us call this condition (m_∞). What is the exact relation between spaces with the (m_∞)-condition and the M-ideal property of $K(l^1, Y)$ in $L(l^1, Y)$? We would like to mention at this stage the recent result from [18] that an (m_∞)-space with a separable dual embeds almost isometrically into c_0. Therefore, for a Banach space Y with a separable dual, $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$ if Y has property (m_∞). It is tempting to conjecture that the converse holds as well. Let us note that—unlike the (M_∞)-case—property (m_∞) does not imply the approximation property, and neither does the fact that $K(l^1, Y)$ is an M-ideal in $L(l^1, Y)$ (Corollary 2.13). Another result from [18] implies that quotients of subspaces of separable (M_∞)-spaces have (m_∞), so problems (2) and (3) are closely related.

(4) More specifically, we note that the little Bloch space β_0 has the property that $K(l^1, \beta_0)$ is an M-ideal in $L(l^1, \beta_0)$; in fact, it is an (M_∞)-space, as shown in [18]. Recall that the Bloch space β consists of those analytic functions f on the open unit disk $\mathbb{D}$ for which $f(0) = 0$ and

$$\|f\|_\beta := \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) < \infty.$$

The little Bloch space β_0 is the subspace where

$$\lim_{|z|\to 1}|f'(z)|(1-|z|^2)=0.$$

More precisely, Corollary 5.9 in [18] states that β_0 is almost isometric to a subspace of c_0; previously it was shown in [40] that β_0 is an M-ideal in its bidual.

(5) Actually, β_0 is known to be isomorphic to c_0 [37]. We remark that one can deduce this fact using results from M-ideal theory: On the one hand, the space β_0 is a complemented subspace of $C_0(\mathbb{D})$ [36, Thm. 1] and hence a separable $\mathcal{L}^\infty$-space. On the other hand, β_0 is an M-ideal in its bidual (see above). Therefore, β_0 must be isomorphic to c_0, by [9] or [38].

3. Uniqueness of Norm-Preserving Extensions of Functionals on Operator Spaces

Now we turn to the problem of unique Hahn–Banach extensions. It is one aspect of M-ideal theory that, for an M-ideal F in a Banach space E, uniqueness of norm-preserving extension holds. However there is no nontrivial example of an M-ideal $K(X, C(\Omega))$ in $L(X, C(\Omega))$, as shown in Corollary 2.5. On the other hand, uniqueness of norm-preserving extension holds in this setting—for instance, for $X=l^p$ $(1<p<\infty)$ [28]. Hence we shall take a closer look at this situation.

Basic to our investigation is a result of Ruess and Stegall [34] (see also [25] and [14, Thm. VI.1.3]) that for any subspace $H\subset K(X,Y)$ containing the finite rank operators, $\partial_e H_1^*=\partial_e X_1^{**}\otimes\partial_e Y_1^*$.

THEOREM 3.1. *Let X be a separable Banach space and Ω a compact Hausdorff space with a convergent sequence $\{t_n\}$ of distinct terms, say $t_n\to t_0$. Let $\Lambda\in\partial_e X_1^{**}$. If the functional $\Lambda\otimes\delta(t_0)$ has unique norm-preserving extension from $K(X, C(\Omega))$ to $L(X, C(\Omega))$, then Λ is w^*-continuous, and if $x_n\in X_1$ and $x_n\to\Lambda$ weakly, then $x_n\to\Lambda$ in the norm.*

Proof. Since X is separable, it is enough to show that Λ is sequentially w^*-continuous. Let $f_n, f\in X_1^*$, $f_n\overset{w^*}{\to}f$. Since, for any $T\in K(X, C(\Omega))$, $T^*(\delta(t_n))\to T^*(\delta(t_0))$ in the norm, we have $\Lambda\otimes\delta(t_n)\overset{w^*}{\to}\Lambda\otimes\delta(t_0)$ in $K(X, C(\Omega))^*$. Since any w^*-accumulation point of the sequence $\{\Lambda\otimes\delta(t_n)\}$ in $L(X, C(\Omega))_1^*$ is a norm-preserving extension of $\Lambda\otimes\delta(t_0)$, we get from the uniqueness assumption that $\Lambda\otimes\delta(t_n)\overset{w^*}{\to}\Lambda\otimes\delta(t_0)$ in $L(X, C(\Omega))^*$.

It is canonical to define an operator $T\in L(X, C(\Omega))$ such that $T^*(\delta(t_n))=f_n$ for all n and $T^*(\delta(t_0))=f$; note that $C(\Omega)$ contains a 1-complemented subspace isometric to c under our assumptions. Now, for this T, $(\Lambda\otimes\delta(t_n))(T)\to(\Lambda\otimes\delta(t_0))(T)$; that is, $\Lambda(f_n)\to\Lambda(f)$.

Ignoring the canonical embedding, let us write $\Lambda=x$. Let $x_n\in X_1$ and $x_n\overset{w}{\to}x$. Choose $f_n\in X_1^*$ such that $\|x_n-x\|=f_n(x_n-x)$. Since X is separable, we may assume that $f_n\overset{w^*}{\to}f$. Now $x_n\otimes\delta(t_n)\overset{w^*}{\to}x\otimes\delta(t_0)$ in $K(X, C(\Omega))^*$ and hence in $L(X, C(\Omega))^*$ by the uniqueness assumption. Let T be defined as before to give $f_n(x_n)\to f(x)$. Since $f(x_n)\to f(x)$, we have that $\|x_n-x\|\to 0$. □

Let us recall that $x \in X_1$ is a *denting point* if for each $\epsilon > 0$ there is a functional $f_\epsilon \in X^*$ such that, for some α, the slice $\{y \in X_1 : f_\epsilon(y) \geq \alpha\}$ contains x and has diameter $\leq \epsilon$. A point $x^* \in X_1^*$ is a weak* denting point if such a functional $f_\epsilon \in X^{**}$ can always be chosen to be weak* continuous. Clearly, a denting point is an extreme point.

COROLLARY 3.2. *Let X and Ω be as above. If all the extreme functionals $\{\Lambda \otimes \delta(t_0) : \Lambda \in \partial_e X_1^{**}\}$ have unique norm-preserving extension from $K(X, C(\Omega))$ to $L(X, C(\Omega))$, then X is reflexive and all extreme points in the unit ball of X are denting points.*

Proof. It follows from the arguments given above that $\partial_e X_1^{**} \subset X$. Since X is separable we can apply a result of Haydon [15] to conclude that X is reflexive. Since every extreme point is a point of continuity for the weak topology on X_1, we get from a result of [26] that all the extreme points in the unit ball are denting points. □

REMARKS. (a) One can relax the assumption of separability on X to "all w^*-sequentially continuous functionals are w^*-continuous". When $\partial_e X_1^{**} \subset X$, one can also apply the celebrated theorem of James [3, pp. 15ff] to conclude that X is reflexive. Therefore, the conclusions of the corollary are valid for a wider class of Banach spaces than separable spaces—namely, weakly compactly generated, weakly k-analytic, or weakly countably determined spaces, etc. (see [19]).

(b) The range space $C(\Omega)$ can be replaced by any Banach space Y that contains an isometric copy of c, the space of convergent sequences, as the range of a norm-1 projection.

These results also go through in the absence of convergent sequences in some special situations. As an illustration, we now prove the following proposition.

PROPOSITION 3.3. *Let X be a separable Banach space. Suppose every extreme point of $K(X, C(\beta\mathbb{N}))_1^*$ has unique norm-preserving extension to $L(X, C(\beta\mathbb{N}))_1^*$. Then X is reflexive and every extreme point in the unit ball is a denting point.*

Proof. Let us note that $\beta\mathbb{N}$ denotes the Stone–Čech compactification of the positive integers.

In this proof we prefer to use the following standard identifications:

$$K(X, C(\beta\mathbb{N})) = C(\beta\mathbb{N}, X^*);$$

$$L(X, C(\beta\mathbb{N})) = C(\beta\mathbb{N}, (X^*, w^*)).$$

(The spaces on the right-hand side consist of continuous functions from $\beta\mathbb{N}$ into X^* when X^* has the norm or w^*-topology respectively, equipped with the sup-norm.)

As before, fix $\Lambda \in \partial_e X_1^{**}$ and $x_n^*, x^* \in X_1^*$ with $x_n^* \to x^*$ in the w^*-topology. Note that this gives an $h \in C(\beta\mathbb{N}, (X^*, w^*))$ with $h(t) = x_n^*$ if $t = n \in \mathbb{N}$ and $h(t) = x^*$ if $t \in \beta\mathbb{N}\setminus\mathbb{N}$.

Now fix a $t \in \beta\mathbb{N}\setminus\mathbb{N}$ and consider the functional $\tau: C(\beta\mathbb{N}, (X^*, w^*)) \to \mathbb{R}$ defined by

$$\tau(f) = \lim_t \Lambda(f(n)),$$

where the limit is taken along the ultrafilter t. Note that when $f \in C(\beta\mathbb{N}, X^*)$,

$$\lim_t \Lambda(f(n)) = \Lambda(f(t)).$$

Hence τ is a norm-preserving extension of $\Lambda \otimes \delta(t)$, and therefore by uniqueness we have

$$\Lambda(x^*) = \Lambda(h(t)) = (\Lambda \otimes \delta(t))(h) = \tau(h) = \lim_t \Lambda(x_n^*).$$

Since this is valid for all ultrafilters t, we conclude that $\Lambda(x^*) = \lim \Lambda(x_n^*)$.

Thus Λ is w^*-continuous. It is now very easy to complete the proof of the proposition following the ideas of proofs of Theorem 3.1 and Corollary 3.2. □

Recall that a subspace F of a Banach space E is called *Hahn-Banach smooth* if every $y^* \in F^*$ has a unique norm-preserving extension to an element of E^*. Such subspaces are called U-subspaces in [31].

COROLLARY 3.4. *Suppose X is such that, for all compact Hausdorff spaces Ω, $K(X, C(\Omega))$ is a Hahn–Banach smooth subspace of $L(X, C(\Omega))$. Then X is reflexive and has the Kadec–Klee property, meaning that the weak and norm topologies coincide on the sphere.*

Using the identifications

$$K(l^1, X^*) = K(X, C(\beta\mathbb{N})),$$
$$L(l^1, X^*) = L(X, C(\beta\mathbb{N})),$$

we get that if X is separable and $K(l^1, X^*)$ is a Hahn–Banach smooth subspace of $L(l^1, X^*)$, then X is reflexive and has the Kadec–Klee property. Yost [43] by explicit construction shows that $K(l^1)$ fails the uniqueness of Hahn–Banach extensions in $L(l^1)$ (note that

$$K(c_0, C(\beta\mathbb{N})) = K(l^1) \quad \text{and} \quad L(c_0, C(\beta\mathbb{N})) = L(l^1));$$

this is also done in [28].

Our next proposition gives one more geometric property of this class of Banach spaces.

PROPOSITION 3.5. *Let X be an infinite-dimensional Banach space and Ω an infinite compact Hausdorff space. If $K(X, C(\Omega))$ is a Hahn–Banach smooth subspace of $L(X, C(\Omega))$, then $\overline{\partial_e X_1^{**}} \not\subset S(X^{**})$ (closure taken in the w^*-topology).*

Proof. If $\overline{\partial_e X_1^{**}} \subset S(X^{**})$ then we shall show that, for any $T \in L(X, C(\Omega))$, T^* attains its norm. We shall obtain the required contradiction by exhibiting a $T \in L(X, C(\Omega))$ such that T^* fails to attain its norm.

Since, for any $T \in L(X, C(\Omega))$,

$$\|T^*\| = \sup_{t \in \Omega} \|T^*(\delta(t))\|,$$

it is easy to see that

$$L(X, C(\Omega))_1^* = \overline{\text{co}}\{\Lambda \otimes \delta(t) : \Lambda \in \partial_e X_1^{**}, t \in \Omega\}$$

(w^*-closed convex hull).

Now suppose that $\overline{\partial_e X_1^{**}} \subset S(X^{**})$. For any $\tau \in \partial_e L(X, C(\Omega))_1^*$, write

$$\tau = \lim_\alpha \Lambda_\alpha \otimes \delta(t_\alpha) \quad \text{with } t_\alpha \in \Omega,\ \Lambda_\alpha \in \partial_e X_1^{**}.$$

We may assume that $t_\alpha \to t$ and $\Lambda_\alpha \xrightarrow{w^*} \Lambda$. By our assumption, $\|\Lambda\| = 1$. Now $\Lambda \otimes \delta(t)$ agrees with τ on $K(X, C(\Omega))$ so that by the uniqueness assumption $\tau = \Lambda \otimes \delta(t)$. In other words,

$$\partial_e L(X, C(\Omega))_1^* \subset \{\Lambda \otimes \delta(t) : t \in \Omega, \Lambda \in \partial_e X_1^{**}\}.$$

This clearly implies that, for any $T \in L(X, C(\Omega))$,

$$\|T^*\| = \|T^*(\delta(t))\|$$

for some $t \in \Omega$.

Now fix any $\alpha \in l^\infty$ such that $1 = \|\alpha\| > |\alpha_n|$ for all $n \in \mathbb{N}$. Since X is infinite-dimensional, by the Josefson–Nissenzweig theorem there exist $f_n \in X^*$ such that $\|f_n\| = 1$ and $f_n \xrightarrow{w^*} 0$ (see [4, Chap. XII]). Since Ω is infinite, c_0 is canonically embedded in $C(\Omega)$. Now for $T: X \to c_0 \subset C(\Omega)$ defined by $T(x) = \{\alpha_n f_n(x)\}$, T^* fails to attain its norm. Hence $\overline{\partial_e X_1^{**}} \not\subset S(X^{**})$. □

It may be worth recalling here that some of the Banach spaces considered in [29] are reflexive and locally uniformly rotund.

We next apply some of these methods to a slightly different situation.

PROPOSITION 3.6. *Suppose that $K(X)$ is a Hahn–Banach smooth subspace of* $\operatorname{span} K(X) \cup \{\mathrm{Id}\}$. *Then on $S(X^*)$ the weak and weak* topologies coincide and hence X^* has the Radon–Nikodym property.*

Proof. Let $f_\alpha, f \in S(X^*)$ and $f_\alpha \xrightarrow{w^*} f$. Let $\Lambda \in S(X^{**})$. As before, $\Lambda \otimes f_\alpha \xrightarrow{w^*} \Lambda \otimes f$ in $K(X)^*$ and hence, by the uniqueness assumption, $\Lambda \otimes f_\alpha \xrightarrow{w^*} \Lambda \otimes f$ in $(\operatorname{span} K(X) \cup \{\mathrm{Id}\})^*$. In particular, $(\Lambda \otimes f_\alpha)(\mathrm{Id}) \to (\Lambda \otimes f)(\mathrm{Id})$; that is, $\Lambda(f_\alpha) \to \Lambda(f)$. Hence $f_\alpha \xrightarrow{w} f$. That X^* has the Radon–Nikodym property follows from standard arguments. □

With the help of a characterization of Hahn–Banach smoothness in terms of intersection properties of balls, it is proved in [23] that if $K(X)$ is a Hahn–Banach smooth subspace of $\operatorname{span} K(X) \cup \{\mathrm{Id}\}$ then X is a Hahn–Banach smooth subspace of X^{**}.

We next prove a result that extends Lemma 11 of [24], and as a consequence obtain the characterization result mentioned in the introduction.

THEOREM 3.7. *Let $\Lambda \in \partial_e X_1^{**}$ and $y^* \in \partial_e Y_1^*$. Suppose that either Λ or y^* is a w^*-denting point. Then the functional $\Lambda \otimes y^*$ has unique norm-preserving extension from $R(X, Y)$ to $L(X, Y)$.*

Proof. We shall make use of the fact that a denting point is a point of continuity (weak or weak* appropriately). Also, if Λ is a w^*-denting point of X_1^{**} then $\Lambda = x$ is a denting point of X_1; see [26] and [33].

By the result of Ruess and Stegall mentioned in the introduction to this section, $\Lambda \otimes y^* \in \partial_e R(X, Y)_1^*$ and hence

$$F = \{\phi \in L(X, Y)_1^* : \phi = \Lambda \otimes y^* \text{ on } R(X, Y)\}$$

is a w^*-closed extreme subset of the unit ball. Since $\Lambda \otimes y^* \in F$, it is enough to show that $\partial_e F$ is a single point.

It is easy to see that $L(X, Y)_1^* = \overline{\mathrm{co}}(X_1^{**} \otimes Y_1^*)$ (w^*-closed convex hull), so that

$$\partial_e L(X, Y)_1^* \subset \overline{X_1^{**} \otimes Y_1^*}.$$

Now let $\phi \in \partial_e F$. Since F is an extreme subset, $\phi \in \partial_e L(X, Y)_1^*$; hence there exist nets $\{\Lambda_\alpha\}$ in X_1^{**} and $\{y_\alpha^*\}$ in Y_1^* and $\phi = \lim_\alpha \Lambda_\alpha \otimes y_\alpha^*$ in the w^*-topology of $L(X, Y)^*$.

Since $\phi = \Lambda \otimes y^*$ on $R(X, Y)$ it is easy to see that $\Lambda_\alpha \xrightarrow{w^*} \Lambda$ and $y_\alpha^* \xrightarrow{w^*} y^*$.

Now suppose that Λ is a w^*-denting point and that $\Lambda = x$. Then $\Lambda_\alpha \to x$ in the norm topology so that, for any $T \in L(X, Y)$,

$$\phi(T) = \lim_\alpha \Lambda_\alpha(T^*(y_\alpha^*)) = T^*(y^*)(x) = (x \otimes y^*)(T).$$

A similar argument works when y^* is a w^*-denting point. Hence $\Lambda \otimes y^*$ has unique norm-preserving extension to $L(X, Y)$. □

COROLLARY 3.8. *For a separable Banach space X, all the functionals in $\partial_e K(X, c)_1^*$ have unique norm-preserving extensions if and only if X is reflexive and every point of $\partial_e X_1$ is a denting point.*

COROLLARY 3.9. *Let X be a Banach space which is strictly convex and has the Kadec–Klee property, with X_1^* the norm closed convex hull of its extreme points. Then X has the metric approximation property if and only if $R(X)^\perp$ is the kernel of a norm-1 projection in $L(X)^*$.*

Proof. This is one of the 10 statements in Theorem 13 of [24]. The results of [26] and [33] imply that every point of $S(X) = \partial_e X_1$ is a denting point. Now if $x \in S(X)$ and $x^* \in \partial_e X_1^*$, since $x \otimes x^*$ has unique norm-preserving extension to $L(X)$, in the notation of [24] (proof of (10) ⇒ (1)), we get $x^*(T_\alpha x) \to x^*(x)$. Since X_1^* is the norm closed convex hull of its extreme points, we deduce that $T_\alpha x \to x$ weakly and, since X has the Kadec–Klee property, $\|T_\alpha x - x\| \to 0$. □

A Banach space is said to have property (**) if the norm topology and the weak* topology agree on its dual unit sphere. If Y has property (**) then, by the arguments in [33, Prop. 3.3], every $y^* \in \partial_e Y_1^*$ is a w^*-denting point. Therefore Theorem 3.7 implies the following corollary.

COROLLARY 3.10. *If X^* or Y has the property (**), then every extreme point of $K(X, Y)_1^*$ has unique norm-preserving extension to $L(X, Y)^*$.*

We finish this section with an application of Theorem 3.7 to point-set topology.

Look at

$$K(l^1, C(\beta\mathbb{N})) \subset L(l^1, C(\beta\mathbb{N})).$$

Using the identifications $K(l^1, C(\beta\mathbb{N})) = C(\beta\mathbb{N}, C(\beta\mathbb{N})) = C(\beta\mathbb{N} \times \beta\mathbb{N})$ and $L(l^1, C(\beta\mathbb{N})) = \bigoplus_\infty C(\beta\mathbb{N}) = C(\beta\mathbb{N})$, we see that the inclusion $K(l^1, C(\beta\mathbb{N})) \subset L(l^1, C(\beta\mathbb{N}))$ corresponds to the following.

Let $\phi: \mathbb{N} \to \mathbb{N} \times \mathbb{N} \subset \beta\mathbb{N} \times \beta\mathbb{N}$ be any bijection. Denote still by ϕ its extension from $\beta\mathbb{N}$ onto $\beta\mathbb{N} \times \beta\mathbb{N}$. Since $\beta\mathbb{N}$ is not homeomorphic to $\beta\mathbb{N} \times \beta\mathbb{N}$ [7, p. 97] this map ϕ is not one-to-one. Consider the inclusion $C(\beta\mathbb{N} \times \beta\mathbb{N}) \subset C(\beta\mathbb{N})$ induced by this map ϕ.

For $(t, s) \in \beta\mathbb{N} \times \beta\mathbb{N}$, look at $E = \phi^{-1}\{(t, s)\}$.

For any $x \in E$, note that $\delta(x) \in \partial_e C(\beta\mathbb{N})_1^*$ and is a norm-preserving extension of $\delta((t, s))$. The set $F = \overline{\mathrm{co}}\, E$ is precisely the set of norm-preserving extensions. Hence $\delta((t, s))$ has unique norm-preserving extension if and only if E is a singleton. Hence, these are precisely the points where ϕ is one-to-one. From Theorem 3.7 it follows that this set contains $\{(t, s): t \text{ or } s \text{ is in } \mathbb{N}\}$.

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