Quasiextremal Distance Domains and Integrability of Derivatives of Conformal Mappings

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1. Introduction

Let Ω be a simply connected domain in the complex plane and let f be a conformal mapping from Ω onto the unit disk Δ. Let L be any straight line in the plane. In this note we consider the following question: For which values of p is

$$\int_{L\cap\Omega} |f'(z)|^p\,ds < \infty?$$

That is, for which values of p is it true that $f' \in L^p(L\cap\Omega)$? Without loss of generality, the straight line L may be taken to be the real axis R. For $p=1$, Hayman and Wu [HW] proved that this integral is bounded by a universal constant (see also [GGJ], [FHM], and [Øy]). The Koebe function shows that $f' \in L^2(R\cap\Omega)$ can fail. In [Ha, p. 638], Baernstein conjectured that $f' \in L^p(R\cap\Omega)$ would be true for all $p \in [1,2)$. In the positive direction of this conjecture, Fernandez, Heinonen, and Martio [FHM] proved that there is an absolute constant $\epsilon > 0$ such that $f' \in L^p(R\cap\Omega)$ for all $p \in [1, 1+\epsilon)$. But in [Ba] Baernstein gave a beautiful counterexample to the conjecture. In this paper we give several sufficient conditions which ensure that $f' \in L^p(R\cap\Omega)$ for all $p \in [1,2)$.

In Section 2 we consider quasiextremal distance (or QED) domains and obtain some sharp estimates for the QED constants of certain domains. These estimates have their own interest, but here they are used to prove one of our main results on Baernstein's disproven conjecture. In Sections 3 and 4 we give several geometric conditions, some on $f(L)$ and some on Ω, which ensure that $f' \in L^p(R\cap\Omega)$ for all $p \in [1,2)$. For example, in Section 3 we show that if $f(L)$ is a Jordan curve of bounded rotation, then $f' \in L^p(L\cap\Omega)$ for all $p \in [1,2)$. We also show that if Ω is starlike, then $f' \in L^p(L\cap\Omega)$ for all $p \in [1,2)$.

By modifying Baernstein's example slightly, we construct in Section 5 a chord-arc domain Ω such that $R\cap\partial\Omega$ contains a single point and that $f' \notin L^p(R\cap\Omega)$ for some $p \in (1,2)$. This example suggests that severe conditions

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must be posed in order to have $f' \in L^p(R \cap \Omega)$ for all $p \in [1, 2)$ and it justifies, in some sense, the sufficient conditions we give in Sections 3 and 4.

2. Quasiextremal Distance Domains

A domain $D \subset \bar{\mathbf{C}}$ is called an *M-quasiextremal distance domain* (or *M-QED domain*) for $1 \leq M < \infty$ if, for each pair of disjoint compact sets F_0 and F_1 in $\bar{D}$,

$$\operatorname{mod}(F_0, F_1; \mathbf{C}) \leq M \operatorname{mod}(F_0, F_1; D). \tag{2.1}$$

Here, for any open set $G \subset \bar{\mathbf{C}}$ and for any disjoint compact sets F_0 and F_1, $\operatorname{mod}(F_0, F_1; G)$ denotes the *modulus* of the family $\Gamma(F_0, F_1; G)$ of curves that join F_0 and F_1 in G. The reciprocal of $\operatorname{mod}(F_0, F_1; G)$ is called the *extremal distance* between F_0 and F_1 with respect to G. We refer the reader to [Vä] for definition and basic properties of modulus of a curve family. This class of domains was introduced by Gehring and Martio [GM] (with a slightly different definition) in connection with the theory of quasiconformal mappings. It is known [GM, Thm. 2.22] that if D is a finitely connected domain in $\mathbf{C}$, then D is QED $\Leftrightarrow D$ is uniform $\Leftrightarrow D$ is a quasicircle domain. A domain $D \subset \mathbf{C}$ is said to be *uniform* if there exists a constant c, $1 \leq c < \infty$, such that each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable arc γ in D for which

$$l(\gamma) \leq c|x_1 - x_2| \quad \text{and} \quad \min_{j=1,2} l(\gamma(x_j, x)) \leq c d(x, \partial D)$$

for each $x \in \gamma$. Here $l(\gamma)$ denotes the length of γ, $\gamma(x_j, x)$ the subarc of γ joining x_j and x, and $d(x, \partial D)$ the distance from x to ∂D. A domain D is a *quasicircle* domain if each component of ∂D is either a quasicircle or a point. For more properties of QED domains in more general settings, the reader is referred to [GM], [HK], and [Y1]. In order to study the geometry of QED domains D, the following *QED constant* was introduced [Y1]:

$$M(D) = \sup_{F_0, F_1} \frac{\operatorname{mod}(F_0, F_1; \mathbf{C})}{\operatorname{mod}(F_0, F_1; D)}, \tag{2.2}$$

where the supremum is taken over all pairs of disjoint compact sets F_0 and F_1 in $\bar{D}$ such that $\operatorname{mod}(F_0, F_1; \mathbf{C})$ and $\operatorname{mod}(F_0, F_1; D)$ are not simultaneously zero or infinity. This constant reflects the geometry of D and measures how far D is from being a disk. For example, it was shown in [Y1] that for a domain $D \subset \mathbf{C}$, $M(D) = 2$ if and only if D is Möbius equivalent to the unit disk minus an NED set, a set which is removable for analytic functions with bounded Dirichlet integrals (see [AB]). We refer the reader to [Y1] and [Y2] for more properties and estimates of the QED constant $M(D)$ in the plane and in space.

The following notation will be used. For $z_0 \in \mathbf{C}$ and $r > 0$, we let $\Delta(z_0, r)$ denote the open disk centered at z_0 of radius r. We also write $\Delta(r) = \Delta(0, r)$

and $\Delta = \Delta(0,1)$. For a domain $D \subset \bar{\mathbf{C}}$, its boundary, closure, and exterior $\bar{\mathbf{C}} \setminus \bar{D}$ will be denoted by ∂D, $\bar{D}$ and D^*, respectively.

For $0 < \alpha < 2$, let $A_\alpha = \{z : 0 < \arg(z) < \alpha\pi\}$ be the wedge of angle $\alpha\pi$. By [Y1, Thms. 4.16 & 4.24], for any Jordan domain D with $D \cap \Delta = A_\alpha \cap \Delta$, we have the sharp inequality

$$M(D) \geq 2/\alpha$$

for $0 < \alpha \leq 1$. The main result in this section is the following generalization and extension of the above result. It will also be needed in the next section.

2.3. THEOREM. *Suppose D is a Jordan domain in* $\mathbf{C}$, *and suppose its boundary* ∂D *has right and left tangents at a point* z_0. *Denote the angle from the right tangent to the left tangent by* $\alpha\pi$. *Then*

$$M(D) \geq \frac{2}{\alpha} \quad \textit{if } 0 < \alpha \leq 1; \tag{2.4}$$

$$M(D) \geq \frac{2}{2-\alpha} \quad \textit{if } 1 \leq \alpha < 2. \tag{2.5}$$

Both inequalities are sharp.

The proof of Theorem 2.3 depends on the following well-known facts about the Grötzsch ring and the Teichmüller ring. For details we refer the reader to [A1] and [Vu]. For $0 < r < 1$ and $t > 0$, let $R_G(r)$ denote the Grötzsch ring bounded by the line segment $[0, r]$ and by the unit circle, and let $R_T(t)$ denote the Teichmüller ring bounded by the line segment $[-1, 0]$ and by the ray $\{z = x : t \leq x < \infty\}$. We define functions $\Phi(r)$, $\mu(r)$, and $\Psi(t)$ by (see [A1, Chap. 3; Vu, Chap. 2])

$$\operatorname{mod}(R_G(r)) = \log \Phi(r) = \mu(r), \qquad \operatorname{mod}(R_T(t)) = \log \Psi(t), \tag{2.6}$$

where $\operatorname{mod}(R_G(r))$ and $\operatorname{mod}(R_T(t))$ denote the conformal moduli of the Grötzsch ring and Teichmüller ring, respectively. It is well known that

$$\operatorname{mod}(F_0, F_1; \mathbf{C}) = 2\pi(\operatorname{mod}(R_T(t)))^{-1} = 2\pi(\log \Psi(t))^{-1},$$

where F_0 and F_1 are the two components of the complement of $R_T(t)$. We also have the following functional identities:

$$\Psi(t) = \left(\Phi\left(\frac{1}{\sqrt{t+1}}\right)\right)^2; \qquad \mu(r)\mu(\sqrt{1-r^2}) = \frac{\pi^2}{4}. \tag{2.7}$$

Furthermore,

$$\lim_{r \to 0}(\mu(r) + \log r) = \log 4; \qquad \lim_{t \to \infty} \frac{\Psi(t)}{t} = 16. \tag{2.8}$$

Proof of Theorem 2.3. Without loss of generality we may assume that $z_0 = 0$, the right tangent of ∂D at 0 is the positive real axis, and the left tangent is the ray

$$L_\alpha = \{re^{i\alpha\pi} : 0 \le r < \infty\}.$$

We may also assume that $\infty \in \partial D$ and that ∂D is parameterized by $\phi(t)$: $[-1,1] \to \partial D$, so that

$$\phi(0)=0; \qquad \phi(-1)=\phi(1)=\infty.$$

Next, for $\theta_1, \theta_2 \in (-2\pi, 2\pi)$ let $A(\theta_1, \theta_2)$ denote the wedge domain

$$A(\theta_1,\theta_2) = \{z : \theta_1 < \arg(z) < \theta_2\}.$$

Since L_0 and L_α are the right and left tangents of ∂D at 0, for $r \in (0,1)$ there exists $\epsilon = \epsilon(r) > 0$ such that $\epsilon(r) \to 0$ and $r \to 0$ and

$$\Delta(r) \cap A(\epsilon, \alpha\pi - \epsilon) \subset \Delta(r) \cap D \subset \Delta(r) \cap A(-\epsilon, \alpha\pi + \epsilon) \tag{2.9}$$

when r is sufficiently small.

For the proof of (2.4), we fix $r \in (0,1)$ and let

$$F_0 = \bar{D} \cap \bar{\Delta}(r^2), \qquad F_1 = \bar{D} \setminus \Delta(r);$$

$$F_0' = \bar{D} \cap \partial\Delta(r^2), \qquad F_1' = \bar{D} \cap \partial\Delta(r).$$

Denote the curve families $\Gamma(F_0, F_1; D)$ and $\Gamma(F_0', F_1'; D \cap (\Delta(r) \setminus \bar{\Delta}(r^2)))$ by Γ and Γ', respectively. Then Γ' is both a subfamily and a minimizing family of Γ. Thus, by [Vä, 6.2 & 6.4],

$$\operatorname{mod}(\Gamma) = \operatorname{mod}(\Gamma'). \tag{2.10}$$

By (2.9), it follows from [Vä, 6.2 & 7.7] that

$$\operatorname{mod}(\Gamma') \le (\alpha\pi + \epsilon)\left(\log \frac{1}{r}\right)^{-1}. \tag{2.11}$$

On the other hand, since F_0 contains a continuum that contains 0 and a point $a \in \partial\Delta(r^2)$ while F_1 contains a continuum that contains ∞ and a point $b \in \partial\Delta(r)$, by the extremal property of the Teichmüller ring (see e.g. [A1, Chap. 3])

$$\operatorname{mod}(F_0, F_1; \mathbf{C}) \ge 2\pi\left(\log \Psi\left(\frac{|b-a|}{|a|}\right)\right)^{-1}. \tag{2.12}$$

Since $\Psi(t)$ is monotone, combining (2.10), (2.11), and (2.12) we obtain

$$\begin{aligned}\frac{\operatorname{mod}(F_0,F_1;\mathbf{C})}{\operatorname{mod}(F_0,F_1;D)} &\ge \frac{2\pi\log(1/r)}{(\alpha\pi+\epsilon)\log\Psi(|b-a|/|a|)} \\ &\ge \frac{2\pi\log(1/r)}{(\alpha\pi+\epsilon)\log\Psi((r+r^2)/r^2)}.\end{aligned}$$

By (2.8), the last expression tends to $2/\alpha$ as $r \to 0$; this proves inequality (2.4). Its sharpness follows from the fact that $M(A_\alpha) = 2/\alpha$ for $0 < \alpha \le 1$ (see [Y1, Cor. 5.3]).

We point out that the above argument for the proof of (2.4) is valid also for $\alpha > 1$. However, in this case, it does not give the sharp estimate (2.5).

For the proof of (2.5), we let $r \in (0,1)$ and $\epsilon = \epsilon(r)$ be as above. Let

$$D_1 = A(\epsilon, \alpha\pi - \epsilon) = \{z : \epsilon < \arg(z) < \alpha\pi - \epsilon\},$$
$$D_2 = A(-\epsilon, \alpha\pi + \epsilon) = \{z : -\epsilon < \arg(z) < \alpha\pi + \epsilon\},$$

and let

$$F_0 = [0, r^2 e^{i\epsilon}]; \qquad F_1 = [r^4 e^{i(\alpha\pi - \epsilon)}, r^3 e^{i(\alpha\pi - \epsilon)}].$$

Then, by (2.9), $F_0, F_1 \subset \bar{D} \cap \bar{D}_1$, and it is easy to see that for each curve $\gamma \in \Gamma(F_0, F_1; D)$ either $\gamma \in \Gamma(F_0, F_1; D_2)$ or γ contains a subarc which joins $\partial\Delta(r^2)$ and $\partial\Delta(r)$. Thus, by [Vä, 6.2 & 7.5],

$$\operatorname{mod}(F_0, F_1; D) \le \operatorname{mod}(F_0, F_1; D_2) + 2\pi\left(\log \frac{1}{r}\right)^{-1}.$$

On the other hand, since $\Gamma(F_0, F_1; D_1) \cup \Gamma(F_0, F_1; D_1^*) \subset \Gamma(F_0, F_1; \mathbf{C})$,

$$\operatorname{mod}(F_0, F_1; \mathbf{C}) \ge \operatorname{mod}(F_0, F_1; D_1) + \operatorname{mod}(F_0, F_1; D_1^*),$$

where $D_1^* = \mathbf{C} \setminus \bar{D}_1$. Thus we have

$$\frac{\operatorname{mod}(F_0, F_1; \mathbf{C})}{\operatorname{mod}(F_0, F_1; D)} \ge \frac{\operatorname{mod}(F_0, F_1; D_1) + \operatorname{mod}(F_0, F_1; D_1^*)}{\operatorname{mod}(F_0, F_1; D_2) + 2\pi(\log(1/r))^{-1}}. \tag{2.13}$$

Next, by means of conformal mappings, we see that

$$\operatorname{mod}(F_0, F_1; D_1) = \pi(\operatorname{mod}(R_T(a)))^{-1} = \pi(\log \Psi(a))^{-1},$$

where

$$a = \frac{r^{\alpha_1}(1 + r^{\alpha_1})}{1 - r^{\alpha_1}}, \quad \alpha_1 = \frac{\pi}{\alpha\pi - 2\epsilon}.$$

Similarly,

$$\operatorname{mod}(F_0, F_1; D_1^*) = \pi(\log \Psi(b))^{-1},$$

where

$$b = \frac{r^{\alpha_2}(1 + r^{\alpha_2})}{1 - r^{\alpha_2}}, \quad \alpha_2 = \frac{\pi}{2\pi - (\alpha\pi - 2\epsilon)}.$$

Furthermore, an elementary (but long) argument shows that

$$\lim_{r \to 0} \frac{\operatorname{mod}(F_0, F_1; D_1)}{\operatorname{mod}(F_0, F_1; D_2)} = 1.$$

Therefore, by (2.13),

$$\limsup_{r \to 0} \frac{\operatorname{mod}(F_0, F_1; \mathbf{C})}{\operatorname{mod}(F_0, F_1; D)} \ge \lim_{r \to 0} \frac{(\log \Psi(a))^{-1} + (\log \Psi(b))^{-1}}{(\log \Psi(a))^{-1} + 2(\log(1/r))^{-1}}. \tag{2.14}$$

Since $a \to 0$ as $r \to 0$, it is easy to see that

$$\lim_{r \to 0} \frac{\log \Psi(a)}{\log(1/r)} = 0.$$

It follows from identities (2.6), (2.7), and (2.8) that

$$\lim_{r\to 0}\frac{\log\Psi(a)}{\log\Psi(b)}=\lim_{r\to 0}\frac{\mu(1/(a+1)^{1/2})}{\mu(1/(b+1)^{1/2})}$$
$$=\lim_{r\to 0}\frac{\mu(\sqrt{b/(b+1)})+\frac{1}{2}\log(b/(b+1))-\frac{1}{2}\log(b/(b+1))}{\mu(\sqrt{a/(a+1)})+\frac{1}{2}\log(a/(a+1))-\frac{1}{2}\log(a/(a+1))}$$
$$=\lim_{r\to 0}\frac{\log b}{\log a}=\frac{\alpha}{2-\alpha}.$$

Thus, by (2.14), we have

$$\limsup_{r\to 0}\frac{\operatorname{mod}(F_0,F_1;\mathbf{C})}{\operatorname{mod}(F_0,F_1;D)}\geq\frac{2}{2-\alpha}.$$

This proves (2.5). The sharpness of (2.5) follows from Corollary 2.18. □

Next we consider another class of domains which are closely related to QED domains. For a Jordan domain D in $\bar{\mathbf{C}}$, we say that D is a *K-quasiconformal reflection domain* (or that ∂D admits a K-quasiconformal reflection) if there is a homeomorphism $f:\bar{D}\to\bar{\mathbf{C}}\setminus D$ such that f is K-quasiconformal in D and f fixes the boundary of D pointwise. This class of domains was first considered by Ahlfors [A2], who proved that D is a quasiconformal reflection domain if and only if it is a quasidisk. Thus, by [GM, Thm. 2.22], a Jordan domain D is a quasiconformal reflection domain if and only if it is a QED domain. For such a domain D we define its (quasiconformal) *reflection constant* $K(D)$ as follows:

$$K(D)=\inf\nolimits_f K(f), \tag{2.15}$$

where the infimum is taken over all quasiconformal reflections f in ∂D and $K(f)$ is the maximal dilatation of f in D. More precisely,

$$K(f)=\frac{1+\|k(z)\|_\infty}{1-\|k(z)\|_\infty},$$

where $k(z)=f_z/f_{\bar z}$ is the complex dilatation of the orientation-reversing homeomorphism $f(z)$. The reflection constant $K(D)$, like the QED constant $M(D)$, reflects the geometry of D and measures how far D is from being a disk of half plane (see [Y2]). It is known that for any reflection domain D the infimum in (2.15) can be obtained by some reflection f (see e.g. [Y2, Thm. 5.11]). This reflection is called an *extremal reflection* of ∂D. It is obvious that

$$K(D)=K(D^*), \tag{2.16}$$

and it is also known [Y1, Thm. 5.1] that

$$M(D)\leq K(D)+1. \tag{2.17}$$

From (2.17), Theorem 2.3, and [Y1, 5.3] we obtain the next corollary.

2.18. COROLLARY. *Let A_α be the wedge domain defined above and let P_n denote a regular n-gon in $\mathbf{C}$, $n\geq 3$. Then*

$$M(A_\alpha)=M(A_\alpha^*)=K(A_\alpha)+1=\max\left\{\frac{2}{\alpha},\frac{2}{2-\alpha}\right\};$$

$$M(P_n)=M(P_n^*)=K(P_n)+1=\frac{2n}{n-2}.$$

We end this section by posing a conjecture on the relations between the above constants for general Jordan domains D.

2.19. CONJECTURE. For any Jordan domain D,

$$M(D)=K(D)+1; \tag{2.20}$$

$$M(D)=M(D^*). \tag{2.21}$$

By (2.16), (2.20) obviously implies (2.21). From Corollary 2.18 and results in [Y1] we see that (2.20) and (2.21) hold for some special domains.

3. Conditions on $f(L)$

In this section we give some geometric conditions on $f(R)$ which ensure that $f'\in L^p(R\cap\Omega)$ for all $p\in[1,2)$.

We first recall that a Jordan domain D is said to be *chord-arc* if there is a constant c, $1\le c<\infty$, such that for each pair of points $z,w\in\partial D$

$$\sigma(z,w)\le c|z-w|,$$

where $\sigma(z,w)$ denotes the shorter arc length along ∂D between z and w. A Jordan curve γ is said to be *regular* (in the sense of Ahlfors) if there is a constant c, $1\le c<\infty$, such that $l(\gamma\cap\Delta(z_0,r))\le cr$ for all $z_0\in\mathbf{C}$ and $r>0$. It is well known that chord-arc domains are regular quasidisks.

We employ the following terminologies and results from [WS]. A rectifiable Jordan curve C is said to be of *bounded rotation* if the forward half-tangent exists at every point and the tangent angle $\tau(s)$ that it makes with a fixed direction may be defined as a function of bounded variation of the arc length s. Furthermore, $\tau(s)$ is so determined that its jumps do not exceed π in absolute value. We assume that the arc-length parameterization corresponds to the positive orientation of C. The next result is due to Warschawski and Schober [WS, Thm. 2].

3.1. LEMMA [WS]. *Suppose G is a chord-arc domain and its boundary C is of bounded rotation. Let $v_+(s)$ and $v_-(s)$ be the positive and negative variation functions of $\tau(s)$, respectively, and let*

$$a_+=\sup_s(v_+(s+0)-v_+(s-0));\qquad a_-=\sup_s(v_-(s+0)-v_-(s-0)).$$

Then, for any conformal mapping $f(z)$ from the unit disk onto G,

$$\sup_{0<r<1}\int_0^{2\pi}|f'(re^{i\theta})|^p\,d\theta<\infty \quad \textit{for } 0\le p<\pi/a_+;$$

$$\sup_{0<r<1} \int_0^{2\pi} |f'(re^{i\theta})|^{-p}\, d\theta < \infty \quad \textit{for } 0 \le p < \pi/a_-.$$

Now we present the main result in this section, which states that under certain conditions a stronger version of Baernstein's conjecture holds.

3.2. THEOREM. *Suppose that f is a conformal map of Ω onto Δ and that $\Delta \subset \Omega$. Suppose further that $f(\partial\Delta)$ has both left and right tangents at each point and is of bounded rotation. Then*

$$\int_0^{2\pi} |f'(e^{i\theta})|^p\, d\theta < \infty \tag{3.3}$$

for $0 \le p < 2$.

Proof. Let G denote the domain $f(\Delta)$. We observe that G is 4-QED. In fact, by conformal invariance of modulus and by the fact that Δ is 2-QED, for each pair of disjoint compact sets F_0 and F_1 in $\bar{G}$ we have

$$\begin{aligned} \operatorname{mod}(F_0, F_1; \mathbf{C}) &\le 2 \operatorname{mod}(F_0, F_1; \Delta) = 2 \operatorname{mod}(f^{-1}(F_0), f^{-1}(F_1); \Omega) \\ &\le 4 \operatorname{mod}(f^{-1}(F_0), f^{-1}(F_1); \Delta) = 4 \operatorname{mod}(F_0, F_1; G). \end{aligned}$$

Thus G is 4-QED, and it follows from Theorem 2.3 that at each point of ∂G the angle from the right tangent to the left tangent is at least $\pi/2$. Therefore, in the notation of Lemma 3.1,

$$v_+(s+0) - v_+(s-0) \le \pi - \pi/2 = \pi/2$$

and

$$a_+ \le \pi/2.$$

Next, by the proof of Theorem 2 in [FHM, p. 125], [illegible] is chord-arc. Hence (3.3) follows from Lemma 3.1 for $0 \le p < 2$ as desired. □

3.4. COROLLARY. *Suppose g is a conformal map of a domain G onto Δ and that G contains the upper half plane. Suppose further that $g(R)$ has both left and right tangents at each point and is of bounded rotation. Then*

$$\int_R |g'(x)|^p \left(\frac{x^2+1}{2}\right)^{p-1} dx < \infty \tag{3.5}$$

for $0 \le p < 2$. Hence $g'(x) \in L^p(R)$ for $p \in [1, 2)$.

This follows from Theorem 3.2 by mapping the upper half plane to the unit disk.

4. Conditions on Ω

Let $w = f(z)$ be any conformal map from Ω onto the unit disk Δ, $z = \phi(w) = f^{-1}(w)$, and L any straight line or circle. In this section we consider the following question: What (geometric) conditions on Ω will ensure that $f' \in$

$L^p(L\cap\Omega)$ for $p\in[1,2)$? We also consider L^p-integrability of f' with respect to the area measure on Ω (cf. [Br]).

We recall that a positive measure σ on Δ is a *Carleson measure* if there is a constant $N(\sigma)$ such that $\sigma(S)\le N(\sigma)h$ for every sector

$$S=\{re^{i\theta}: 1-h\le r<1,\ |\theta-\theta_0|\le h\}$$

in Δ. It is well known (see e.g. [Ga]) that σ is a Carleson measure if and only if for some p (hence all p), $0<p<\infty$,

$$\int_\Delta |g(z)|^p\,d\sigma(z)\le C_p\|g\|_p^p$$

for all $g\in H^p$, where H^p is the standard Hardy space on Δ and $\|g\|_p$ is the H^p-norm of g. It is easy to verify that if $g\in H^p$ then $d\sigma=|g|^p\,dx\,dy$ is a Carleson measure on Δ. The argument in this section will be based on this and the fact that the arc length measure on $f(L\cap\Omega)$ is also a Carleson measure (see [GGJ, Thm. 5.1]).

4.1. THEOREM. *If $\partial\Omega$ is a smooth (i.e. C^1) Jordan curve, then*

$$\int_{L\cap\Omega}|f'(z)|^p|dz|<\infty\quad \textit{for all}\ \ 1<p<\infty; \tag{4.2}$$

$$\int_\Omega|f'(z)|^p\,dx\,dy<\infty\quad \textit{for all}\ \ 2<p<\infty. \tag{4.3}$$

Proof. Since $\partial\Omega$ is smooth, by Zygmund's theorem (see [Ga, Cor. 2.6, p. 114])

$$\sup_{0<r<1}\int_0^{2\pi}\exp(-p\log|\phi'(re^{i\theta})|)\,d\theta=\sup_{0<r<1}\int_0^{2\pi}|\phi'(re^{i\theta})|^{-p}\,d\theta<\infty$$

for all $p<\infty$. Thus $1/\phi'\in H^p$ for all $p<\infty$. Let $d\sigma$ be the arc-length measure on $f(L\cap\Omega)$. By [GGJ, Thm. 5.1], σ is a Carleson measure on Δ. Therefore $1/\phi'\in H^p$ yields that

$$\begin{aligned}\int_{L\cap\Omega}|f'(z)|^p|dz|&=\int_\Delta|\phi'(w)|^{1-p}\,d\sigma(w)\\ &\le N(\sigma)\|1/\phi'(w)\|_{p-1}^{p-1}<\infty\end{aligned} \tag{4.4}$$

for all $1<p<\infty$. This proves (4.2). Similarly, we have

$$\int_\Omega|f'(z)|^p\,dx\,dy=\int_\Delta|\phi'(w)|^{2-p}\,du\,dv. \tag{4.5}$$

Thus, (4.3) follows from (4.5) and the fact that $|1/\phi'(w)|^q\,du\,dv$ is a Carleson measure for all $q>0$. □

4.6. THEOREM. *If Ω is starlike (or close-to-convex), then*

$$\int_{L\cap\Omega}|f'(z)|^p|dz|<\infty\quad \textit{for all}\ \ 1<p<2; \tag{4.7}$$

$$\int_{\Omega} |f'(z)|^p \, dx \, dy < \infty \quad \textit{for all} \quad 2 < p < 4. \tag{4.8}$$

Proof. We recall here that if ϕ is a conformal map of Δ onto Ω and $\phi(0)=0$, then Ω (or ϕ) is called *starlike* with respect to the origin if $\Re[w\phi'(w)/\phi(w)]>0$ for $|w|<1$. Similarly, Ω is *close-to-convex* if $\Re[w\phi'(w)/\psi(w)]>0$ for some starlike function $\psi(w)$. Thus every starlike domain is close-to-convex. Assume that $\phi(0)=0$ and that Ω is close-to-convex with respect to the origin. Then $\Re[\psi(w)/(w\phi'(w))]>0$ for some starlike function $\psi(w)$. By [Ga, Thm. 2.4, p. 114],

$$F(w) = \frac{\psi(w)}{w\phi'(w)} \in H^p(\Delta) \quad \text{for } p<1.$$

Since $\psi(0)=0$, it follows that $1/\phi'(w) \in H^p(\Delta)$ for $p<1$. Thus (4.4) yields (4.7), while (4.8) follows from (4.5) and the fact that $|1/\phi'(w)|^q \, du \, dv$ is a Carleson measure for all $q<1$. □

REMARK. Inequalities (4.3) and (4.8) were known [Br, Thms. 2 & 3], but here we give different proofs using Carleson measure.

4.9. THEOREM. *Suppose* $\arg\phi'(e^{i\theta})$ *exists a.e., and suppose there exist a finite number of intervals* $I_j \subset [0,2\pi]$ *such that* $\bigcup I_j = [0,2\pi]$ *and*

$$|\arg\phi'(e^{i\theta_1}) - \arg\phi'(e^{i\theta_2})| \le \pi$$

for a.e. $\theta_1, \theta_2 \in I_j$, $j=1,\dots,n$. *Then* (4.7) *and* (4.8) *hold.*

Proof. As in the proof of Theorem 4.6, it suffices to show that $1/\phi'(w) \in H^p$ for all $p<1$.

By the hypothesis, there exist constants C_j such that

$$|\arg\phi'(e^{i\theta}) - C_j| \le \pi/2 \text{ a.e.} \quad \text{for } \theta \in I_j.$$

Applying Zygmund's theorem [Ga, Cor. 2.6, p. 114] to the function

$$g(\theta) = (2/\pi)(\arg\phi'(e^{i\theta}) - C_j), \quad \theta \in I_j, \ j \ge 1,$$

and to its conjugate $\tilde{g}(\theta)$, we obtain that $1/\phi'(w) \in H^p$ for all $p<1$ as desired. □

REMARK. The condition in Theorem 4.9 says that $\partial\Omega$ has a tangent almost everywhere and that the angles the tangents make with the positive real axis differ piecewise by at most π.

4.10. THEOREM. *If* $\partial\Omega$ *is of bounded rotation and* $a_- \le \pi/2$ *in the sense of Lemma 3.1, then*

$$\int_{L\cap\Omega} |f'(z)|^p \, |dz| < \infty \quad \textit{for} \quad 1<p<3; \tag{4.11}$$

$$\int_{\Omega} |f'(z)|^p \, dx \, dy < \infty \quad \textit{for} \quad 2<p<6. \tag{4.12}$$

Proof. By Lemma 3.1, $1/\phi'(w) \in H^p$ for $p<2$. Thus (4.11) follows from estimate (4.4), and (4.12) follows from (4.5). □

REMARK. By [FHM, Thm. FH] and by the proof of [GGJ, Thm. 5.1], for any chord-arc curve L the arc-length measure on $f(L\cap\Omega)$ is a Carleson measure on Δ. Therefore Theorems 4.1, 4.6, 4.9, and 4.10 still hold if a straight line L is replaced by a chord-arc curve.

5. A Counterexample

In this section we prove the following result.

5.1. THEOREM. *There exist a chord-arc domain Ω with $R\subset\Omega$ except for one point and a number $p\in(1,2)$ such that*

$$\int_{R\cap\Omega}|f'(x)|^p\,dx=\infty \tag{5.2}$$

for all conformal mappings f of Ω onto Δ.

The construction of Ω is only a small modification of Baernstein's elegant example [Ba]. For z_1 and z_2 in the plane, let $[z_1,z_2]$ denote the straight line segment connecting z_1 and z_2. Fix $\alpha,\lambda\in(0,1/2)$ (eventually α and λ will be very small). Let

$$S=[0,-1/\sqrt{3}-i]\cup[0,1/\sqrt{3}-i]$$

and define inductively sets T_k, $T_k(\lambda)$, and $B(k)$ as follows:

$$T_0=S+i=\{z+i: z\in S\},\quad B(0)=\{i\},\quad B(1)=T_0\cap\{\Im z=\alpha\};$$

$$T_k=T_{k-1}\cup\Big[\bigcup_{b\in B(k)}(\alpha^kS+b)\Big],\quad k\geq1;$$

$$B(k+1)=T_k\cap\{\Im z=\alpha^{k+1}\},\quad k\geq1;$$

$$T_k(\lambda)=\bigcup_{z\in T_k}[z-\lambda\Im(z),z+\lambda\Im(z)],\quad k\geq0.$$

Next, let

$$T_{-1}=\{z=x+iy: y=-\sqrt{3}x+1,\ y\geq1\};$$

$$T_{-1}(\lambda)=\bigcup_{z\in T_{-1}}[z-\lambda\Im(z),z+\lambda\Im(z)].$$

Then $T_k(\lambda)$ can be regarded as a "λ-neighborhood" of T_k. Define $T=T(\alpha,\lambda)$ to be the closure of $\bigcup\{T_k(\lambda): k\geq-1\}$. It is easy to see that $\Omega=\mathbf{C}\setminus T$ is a Jordan domain containing the lower half plane. One should note that if λ is taken to be 0 then the above domain Ω is the same as in Baernstein's example. We shall show that, for sufficiently small α and λ, Ω is a chord-arc domain and (5.2) holds for some $p\in(1,2)$.

To prove that Ω is a chord-arc domain, we need to show that there is a constant c, $1 \le c < \infty$, such that for each pair of points $z, w \in \partial\Omega$

$$\sigma(z, w) \le c|z-w|,$$

where $\sigma(z, w)$ denotes the arc length along $\partial\Omega$ between z and w.

In what follows we let c and c_i denote absolute constants whose values (possibly depending on α and λ) may vary from line to line. It is not difficult to see that $\partial\Omega$ is locally rectifiable. It is also easy to see that all bounded components of $\partial\Omega\backslash R$ are similar 4-gonal lines with end points on the real axis. If $z, w \in \partial\Omega$ are on the same 4-gonal line or on the same unbounded component of $\partial\Omega\backslash R$, then $\sigma(z, w) \le c|z-w|$ for some constant c, where $\sigma(z, w)$ denotes the shorter arc length along $\partial\Omega$ between z and w. In other cases we project z and w onto the real axis along shortest arcs on $\partial\Omega$ and denote their projections by z^* and w^*, respectively. Then $z^* \ne w^*$, and there exists constant c such that

$$\sigma(z, z^*) \le c|z-z^*|, \quad \sigma(w, w^*) \le c|w-w^*|, \quad \sigma(z^*, w^*) \le c|z^*-w^*|,$$

and

$$\Im(z) \le |z-z^*| \le c\Im(z); \qquad \Im(w) \le |w-w^*| \le c\Im(w).$$

We also see that the inner angle at w^* of the triangle with vertices at z, w, and w^* is no less than a constant $\theta = \theta(\lambda) > 0$. Without loss of generality, we may assume that $\Im(w) \ge \Im(z)$. Then it follows that

$$|w-w^*| \le c|z-w|; \qquad |z-z^*| \le c|w-w^*|.$$

If $|z^*-w^*| \le 3c|w-w^*|$, then

$$\begin{aligned}\sigma(z, w) &\le \sigma(z, z^*) + \sigma(z^*, w^*) + \sigma(w^*, w)\\ &\le c(|z-z^*| + |z^*-w^*| + |w-w^*|) \le c_1|z-w|.\end{aligned}$$

If $|z^*-w^*| > 3c|w-w^*|$, then

$$|z-w| \ge |z^*-w^*| - |z-z^*| - |w-w^*| \ge c_1|z^*-w^*|.$$

Thus it follows that

$$\begin{aligned}\sigma(z, w) &\le c(|z-z^*| + |z^*-w^*| + |w-w^*|)\\ &\le c_1|z^*-w^*| \le c_2|z-w|.\end{aligned}$$

This proves that Ω is a chord-arc domain.

Next let

$$b_0' = \frac{i}{1+\sqrt{3}\lambda}, \quad B'(0) = \{b_0'\};$$

$$B'(n) = \left\{b' = b - \frac{\sqrt{3}\lambda}{1+\sqrt{3}\lambda}\alpha^n i : b \in B(n)\right\} \quad \text{for } n \ge 1.$$

Then $B'(n) \subset \partial\Omega$ for $n \ge 0$. For each $b \in B(n)$ there is a unique simple path in $T(\alpha)$ from b to ∞. This path contains exactly one point of each $B(k)$,

$0 \le k \le n$. We label b by the multiple index $J = (j_0, j_1, \dots, j_n)$ (i.e. let $b_J = b$) in such a way that $j_0 = 0$ and $j_k = 1$ or 0 $(1 \le k \le n)$ depending on whether the path makes a 120-degree turn or continues in a straight line when it passes through the point in $B(k-1)$. For instance, we have

$$b_{(0)} = b_0 = i, \quad b_{(0,0)} = \frac{1-\alpha}{\sqrt{3}} + \alpha i, \quad b_{(0,1)} = -\frac{1-\alpha}{\sqrt{3}} + \alpha i.$$

Then we label the corresponding point $b' \in B'(n)$ by the same index J. For $J = (j_0, j_1, \dots, j_n)$ we write $|J| = n$ and $t = j_0 + j_1 + \dots + j_n$. We see that t is the number of turns the path makes from b_J to ∞.

Let ϕ be a conformal map of Ω onto the right half plane $U = \{z : \Re(z) > 0\}$ such that

$$\phi(b_0') = 0 \quad \text{and} \quad \frac{|\phi(z)|}{|z|^\tau} \to 1 \text{ as } z \to \infty,$$

where

$$\tau = \frac{\pi}{2\pi - (\tau_1 + \tau_2)}, \quad \tau_1 = \tan^{-1}\left(\frac{3\lambda}{4 + \sqrt{3}\lambda}\right), \quad \tau_2 = \tan^{-1}\left(\frac{3\lambda}{4 - \sqrt{3}\lambda}\right).$$

Then

$$f(z) = \frac{1 - \phi(z)}{1 + \phi(z)}$$

is a conformal map of Ω onto the unit disk. We shall show that (5.2) holds for this map and some number $p \in (1, 2)$.

For each $b_J' \in B'(n)$, $n \ge 0$, define

$$m_J = \Re(b_J') = \Re(b_J), \qquad z_J = f(m_J),$$

and

$$I(b_J') = I(b_J) = [m_J - 10^{-2}\alpha^n, m_J + 10^{-2}\alpha^n].$$

Fix $r > 0$ so that $E_0 = \partial\Omega \cap \bar{\Delta}(b_0', r)$ is connected, and define

$$E_J = \partial\Omega \cap \bar{\Delta}(b_J', \alpha^n r)$$

for $b_J' \in B'(n)$, $n \ge 0$. Using the comparison principle and the conformal invariance of harmonic measures, one can see that there exists a constant $c > 0$ such that

$$\omega(\phi(m_J), \phi(E_J), U) = \omega(m_J, E_J, \Omega) \ge c$$

for all $J = (j_0, j_1, \dots, j_n)$ and all $n \ge 0$, where $\omega(\cdot)$ denotes harmonic measure. Since ϕ is bounded on bounded sets, it follows from standard distortion estimates that

$$1 - |z_J| \approx |f(E_J)| \approx |\phi(E_J)| \approx \Re(\phi(m_J));$$
$$|f'(x)| \approx \frac{1 - |z_J|}{|I(b_J')|} \quad \text{for } x \in I(b_J'). \tag{5.3}$$

Here we use $|E|$ to denote 1-dimensional measure of set E; the notation $A \approx B$ means that $c^{-1}A \le B \le cA$ for some constant c. Thus, by (5.3),

$$\sum_{b'_J \in B'(n)} \int_{I(b'_J)} |f'(x)|\,dx \ge c \sum_{|J|=n} |\phi(E_J)|. \tag{5.4}$$

To prove (5.2) we need the following important estimates. Let Ω_0^* be the λ-shape domain

$$\Omega_0^* = \mathbf{C} \setminus \sum_{z \in T_0 \cup T_{-1}} [z - \lambda\Im(z), z + \lambda\Im(z)],$$

where T_0 and T_{-1} are defined as above. Let ϕ_0^* be a conformal map of Ω_0^* onto the right half plane U such that

$$\phi_0^*(b'_0) = 0 \quad \text{and} \quad \frac{|\phi_0^*(z)|}{|z|^\tau} \to 1 \text{ as } z \to \infty.$$

Define $\beta_0, \beta_1 \in (0, \infty)$ by

$$\beta_0^\tau = \lim_{z \to 1/\sqrt{3}} \frac{|\phi_0^*(z) - \phi_0^*(1/\sqrt{3})|}{|z - 1/\sqrt{3}|^\tau}; \qquad \beta_1^\tau = \lim_{z \to -1/\sqrt{3}} \frac{|\phi_0^*(z) - \phi_0^*(-1/\sqrt{3})|}{|z + 1/\sqrt{3}|^\tau}.$$

We see that when $\lambda = 0$, Ω_0^* is the same as the "fork domain" in [Ba, §2]. Hence, by [Ba, Thm. 2], we have the following lemma.

5.5. LEMMA. *If $\lambda > 0$ and $\alpha > 0$ are sufficiently small, then*

$$\beta_0^\tau + \beta_1^\tau > 2^\tau.$$

The main idea in the proof of [Ba, (2.5)] is the following estimate about harmonic measures. In our case, it follows from [Ba] via normal families, but we also give a direct proof which is a little different from the one in [Ba, §3].

5.6. LEMMA. *Given $\epsilon > 0$, there exist $\alpha, \lambda > 0$ such that for $T = T(\alpha, \lambda)$, all $b'_J \in B'(n)$, and $n = |J| \ge 0$,*

$$|\phi(E_J)| \ge c(e^{-2n\epsilon}\alpha^n \beta_0^{n-t}\beta_1^t)^\tau, \tag{5.7}$$

where $J = (j_0, j_1, \ldots, j_n)$ and $t = j_0 + j_1 + \cdots + j_n$.

Proof. For $J = (j_0, j_1, \ldots, j_n)$, define

$$\Gamma_J = \{z : |z - b'_J| = \alpha^{n-1/2}\} \quad \text{and}$$

$$\Omega_J = \Omega \cap \Delta(b'_J, \alpha^{n-1/2}) \setminus [\bar{\Delta}(b'_{(J,0)}, \alpha^{n+1/2}) \cup \bar{\Delta}(b'_{(J,1)}, \alpha^{n+1/2})],$$

and let T_J be a linear or conjugate linear conformal map of Ω_J onto Ω_0 with

$$T_J(b'_J) = b'_0, \quad T_J(m_J) = m_0, \quad T_J(\Gamma_J) = \Gamma_0, \quad T_J(E_J) = E_0.$$

Recall that ϕ is a conformal map of Ω onto U, the right half plane, with $\phi(b'_0) = 0$. Let

$$\delta_J = \frac{\inf\{|\phi(z)-\phi(b'_J)| : z \in \Gamma_J\}}{\inf\{|\phi(z)| : z \in \Gamma_0\}}.$$

Then the function

$$\phi_J(z) = \begin{cases} \dfrac{\phi(T_J^{-1}(z)) - \phi(b'_J)}{\delta_J}, & T_J \text{ linear}, \\ \dfrac{-\overline{\phi(T_J^{-1}(z))} + \overline{\phi(b'_J)}}{\delta_J}, & T_J \text{ conjugate linear}, \end{cases}$$

maps Ω_0 into U with

$$\phi_J(b'_0) = 0, \quad |\phi(E_J)| = \delta_J |\phi_J(E_0)|,$$

and

$$\inf\{|\phi_J(z)| : z \in \Gamma_0\} = \inf\{|\phi(z)| : z \in \Gamma_0\} \sim \alpha^{-\tau/2}.$$

Here $A \sim B$ denotes that $A/B \to 1$ as $\alpha \to 0$.

Fix $\lambda > 0$ so small that Lemma 5.5 holds for all $\alpha < \alpha_0 = \alpha_0(\lambda, \epsilon)$. We claim that, uniformly in J and uniformly in $z \in \bar{\Omega}_0$,

$$\lim_{\alpha \to 0} \phi_J(z) = \phi_0^*(z) \tag{5.8}$$

in spherical metric. Here $\phi_0^*(z)$ is a conformal map of Ω_0^* onto U defined above.

Accepting (5.8) for now, we have

$$\begin{aligned} \frac{|\phi(E_{(J,j_{n+1})})|}{|\phi(E_J)|} &= \frac{|\phi_{(J,j_{n+1})}(E_0)|}{|\phi_J(E_0)|} \frac{\inf\{|\phi(z)-\phi(b'_{(J,j_{n+1})})| : z \in \Gamma_{(J,j_{n+1})}\}}{\inf\{|\phi(z)-\phi(b'_J)| : z \in \Gamma_J\}} \\ &= \frac{|\phi_{(J,j_{n+1})}(E_0)|}{|\phi_J(E_0)|} \frac{\inf\{|\phi_J(z)-\phi_J(T_J(b'_{(J,j_{n+1})}))| : z \in T_J(\Gamma_{(J,j_{n+1})})\}}{\inf\{|\phi_J(z)| : z \in \Gamma_0\}} \\ &\geq \begin{cases} e^{-\epsilon}\alpha^\tau \beta_0^\tau, & j_{n+1} = 0, \\ e^{-\epsilon}\alpha^\tau \beta_1^\tau, & j_{n+1} = 1. \end{cases} \end{aligned}$$

By induction on J, this yields (5.7) and completes the proof of Lemma 5.6 except for the proof of (5.8).

To prove (5.8), choose $r' > r$ so that the components of $\partial\Omega \cap \bar{\Delta}(b'_0, r')$ are three nontrivial arcs E'_0, E'_1, and E'_2. Then simple comparisons yield the harmonic measure estimates

$$0 < c_1 \leq \omega(m_0, E'_j, \Omega) \leq c_2, \tag{5.9}$$

with c_1 and c_2 independent of α, and

$$\omega(m_0, \Gamma_0 \cup \Gamma_{(0,0)} \cup \Gamma_{(0,1)}, \Omega_0) \leq c_3 \alpha^{1/4}. \tag{5.10}$$

Together (5.9) and (5.10) show there is a compact set $K \subset U$ such that $\varphi_J(m_0) \in K$ for all J and all small α. Also, (5.9) and (5.10) show that in the spherical Hausdorff metric, $\varphi_J(\partial\Omega_0) \to \partial U$ uniformly in J as $\alpha \to 0$. Now (5.8) follows easily from Courant's theorem [Ts, p. 383] on the continuity of conformal mappings to varying domains.

It follows from (5.4) and Lemma 5.6 that

$$\sum_{b'_J\in B'(n)}\int_{I(b'_J)}|f'(x)|\,dx\ge c[e^{-\epsilon}\alpha^{\tau}(\beta_0^{\tau}+\beta_1^{\tau})]^n = c(q\alpha)^{\tau n}, \tag{5.11}$$

where, by Lemma 5.5,

$$q=[(\beta_0^{\tau}+\beta_1^{\tau})e^{-\epsilon}]^{1/\tau}>2$$

when $\epsilon>0$ is sufficiently small. By (5.11) and Hölder's inequality, one sees that there is a number $p_0\in(1,2)$ such that

$$\int_{R\cap\Omega}|f'(x)|^p\,dx=\infty,\quad p_0<p<2. \tag{5.12}$$

Finally, to obtain a domain described in Theorem 5.1, we still need to modify the domain Ω constructed above. For this we define

$$I'(b_J)=I'(b'_J)=[m_j-\alpha^n3^{-1/2},\,m_j+\alpha^n3^{-1/2}]$$

for $J=(j_0,j_1,\dots,j_n)$, $n\ge1$. Then $I'(b_J)$ contains all intervals $I(b_{J'})$ for index J' such that the path from $b_{J'}$ to ∞ along $T(\alpha)$ passes through b_J. Using (5.4) and (5.7), an argument similar to the above shows that

$$\int_{I'(b_J)\cap\Omega}|f'(x)|^p\,dx=\infty,\quad p_0<p<2 \tag{5.13}$$

for all J, where $p_0\in(1,2)$ is the same number as in (5.12).

Next we define $b_n\in B(n)$, $n\ge1$, inductively as follows. Let b_1 be the leftmost point in $B(1)$. Define b_{j+1} to be the leftmost point in $B(j+1)$ such that the path from b_{j+1} to ∞ does not pass through any of the points $b_1,b_2,\dots,b_j$. Then $I'(b_j)\cap I'(b_k)=\emptyset$ for $j\ne k$, and $\partial\Omega\cap R$ is contained in the closure of $\bigcup\{I'(b_n):n\ge1\}$. Fix $p'_0\in(p_0,2)$. By (5.13) we can choose compact sets

$$G_n\subset I'(b_n)\cap\Omega,\quad n\ge1,$$

such that

$$\int_{G_n}|f'(x)|^p\,dx>1,\quad n\ge1,\ p'_0\le p<2. \tag{5.14}$$

For $n\ge1$ define $d_n=d(G_n,\,\partial\Omega)$ and

$$\Omega_n=\Omega\cup\{z=x+iy\colon x\in I'(b_n),\,y<d_n\},$$

where $d(\cdot,\cdot)$ denotes the Euclidean distance. Let

$$\Omega_*=\bigcup_{n\ge1}\Omega_n.$$

It is easy to see that Ω_* is a Jordan domain and that $\partial\Omega_*\cap R=\{1/\sqrt{3}\}$. Furthermore, since Ω is chord-arc, it is also not difficult to see that Ω_* is chord-arc.

Let

$$f_n : \Omega_n \to \Delta \quad \text{and} \quad f_* : \Omega_* \to \Delta$$

be conformal maps with $f_n(f^{-1}(0)) = 0$ and $f_*(f^{-1}(0)) = 0$. Since $\Omega \subset \Omega_n \subset \Omega_*$, for $x \in G_n$ we have

$$d(x, \partial\Omega_n) \leq 3d(x, \partial\Omega) \quad \text{and} \quad d(x, \partial\Omega_*) = d(x, \partial\Omega_n);$$

by Schwarz's Lemma and Koebe's $\frac{1}{4}$-theorem,

$$|f_*'(x)| \geq \tfrac{1}{4}|f_n'(x)| \geq \tfrac{1}{48}|f'(x)|.$$

Thus, by (5.14),

$$\int_{R \cap \Omega_*} |f_*'(x)|^p \, dx \geq (48)^{-p} \sum_{n=1}^{\infty} \int_{G_n} |f'(x)|^p \, dx = \infty$$

and

$$f_*' \notin L^p(R \cap \Omega_*) \quad \text{for } p \in [p_0', 2).$$

Finally, it is easy to see that $g' \notin L^p(R \cap \Omega_*)$ for any conformal map g of Ω_* onto Δ and each $p \in [p_0', 2)$. This completes the proof of Theorem 5.1. □

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