Unimodular Wavelets for L^2 and the Hardy Space H^2

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1. Introduction

In this paper we construct a family of wavelets ψ for L^2 and the Hardy space H^2 with the property that $|\hat{\psi}(\xi)|=1$ for ξ in the support of $\hat{\psi}$. One of the wavelets constructed is the well-known Journé–Meyer example. We also include a proof of the equivalence of Meyer's equations and wavelet conditions.

Let $\psi \in L^2(\mathbf{R})$ and let, for $j,k\in\mathbf{Z}$,

$$\psi_{j,k}(x)=2^{j/2}\psi(2^jx-k).$$

Let H be a Hilbert subspace of $L^2(\mathbf{R})$. The function $\psi\in H$ is called a *wavelet* for H if $\{\psi_{j,k}\}_{j,k\in\mathbf{Z}}$ forms an orthonormal basis for H; $\{\psi_{j,k}\}$ is called a *wavelet basis.*

There are essentially two methods of constructing wavelets. The first is based on the following equations (W1)–(W4): $\psi\in L^2$ is a wavelet for $L^2(\mathbf{R})$ if and only if ψ satisfies

(W1) $\sum_{k\in\mathbf{Z}}|\hat{\psi}(\xi+2k\pi)|^2=1$,
(W2) $\sum_{k\in\mathbf{Z}}\hat{\psi}(\xi+2k\pi)\hat{\psi}^*(2^j(\xi+2k\pi))=0$ for $j\geq 1$,
(W3) $\sum_{j\in\mathbf{Z}}|\hat{\psi}(2^{-j}\xi)|^2=1$, and
(W4) $\sum_{l\geq 0}\hat{\psi}(2^l(\xi+2p_0\pi))\hat{\psi}^*(2^l\xi)=0$ for $p_0\in 2\mathbf{Z}+1$.

Here, $\hat{\psi}^*$ is the complex conjugate of $\hat{\psi}$. This equivalence appears in [Le] and is attributed to Y. Meyer. However, no proof of it seems available, so we give a complete proof in this paper.

The second method of constructing wavelets is based on the pairing of wavelets and multiresolution analysis (MRA) [Me, D2]. An increasing sequence $\{V_j\}$ of closed subspaces of $L^2(\mathbf{R})$ is called an *MRA* of $L^2(\mathbf{R})$ if the following hold:

(R1) $\bigcap_{j\in\mathbf{Z}}V_j=\{0\}$ and $\overline{\bigcup_{j\in\mathbf{Z}}V_j}=L^2(\mathbf{R})$,
(R2) $f(x)\in V_j$ if and only if $f(2^{-j}x)\in V_0$,

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(R3) $f(x) \in V_0$ if and only if $f(x-k) \in V_0$, and

(R4) there exists a function φ in V_0 such that $\{\varphi(x-k)\}_{j\in\mathbf{Z}}$ is an orthonormal basis of V_0.

The function φ in (R4) is called a *scaling function* for the MRA $\{V_j\}$. Let ψ be a wavelet for $L^2(\mathbf{R})$. For each integer j, we denote by W_j the closure of the span of $\{\psi_{j,k}\}_{k\in\mathbf{Z}}$ in L^2. We say that ψ is *associated with* an MRA if the subspace V_j defined by $V_j = \bigoplus_{l=-\infty}^{j-1} W_l$ forms an MRA for $L^2(\mathbf{R})$. If this is the case and if φ is a scaling function for $\{V_j\}$, then ψ and φ satisfy

$$\hat{\psi}(2\xi) = e^{i\xi} m_0^*(\xi+\pi)\hat{\varphi}(\xi), \tag{1.1}$$

where m_0 is a 2π-periodic function in $L^2(0, 2\pi)$ defined by

$$\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi). \tag{1.2}$$

This pairing of wavelets and multiresolution analysis and an algorithm to construct wavelets from multiresolution analysis were found by Y. Meyer and S. Mallat, and one can find proofs of (1.1) and (1.2) in [Ch, D2, Me].

Even though the second method is a powerful constructive scheme, the first method has its own virtue since there are wavelets that can not be constructed from a multiresolution analysis. One such example of wavelets is the Journé–Meyer example in which $\hat{\psi}$ is even and

$$\hat{\psi}(\xi) = \chi_{[4\pi/7,\pi]}(\xi) + \chi_{[4\pi, 4\pi+4\pi/7]}(\xi) \quad \text{for } \xi > 0. \tag{1.3}$$

However, it is apparent that it is hard to solve equations (W1)–(W4). Therefore, one can naturally put some *a priori* assumptions on ψ so that equations (W1)–(W4) can be reduced to easily solvable ones. Meyer put an assumption that the support of $\hat{\psi}$ lies in a compact set $[-8\pi/3, -2\pi/3] \cup [2\pi/3, 8\pi/3]$, so that the infinite sums in (W1)–(W4) become finite sums—in fact, 2-sums (Meyer's wavelet [Le, Me]). On the other hand, Lemarié assumed that $\hat{\psi}$ can be separated as $\hat{\psi}(\xi) = \omega(\xi)\Omega(\xi)$, where ω is homogeneous and Ω is 4π-periodic (Battle–Lemarié wavelet [Le, Me]). His assumption is based on the observation that all the sums in (W1)–(W4) are made of 2π-translation and dilation. Both Meyer's wavelet and the Battle-Lemarié wavelet can be constructed from MRA [Me].

In this paper, we construct a new class of wavelets based on equations (W1)–(W4). Note that condition (W1) implies that $|\hat{\psi}(\xi)| \le 1$. We thus consider an extreme case when ψ is unimodular—namely, $|\hat{\psi}(\xi)| = 1$ on the support of $\hat{\psi}$. It turns out that, for unimodular functions, (W1) implies (W4) and (W3) implies (W2). Moreover, (W1) and (W3) require that 2π-translates and 2^j-dilates of the support of $\hat{\psi}$ match nicely. This gives a severe restriction on the size and location of the support of $\hat{\psi}$, and we are able to characterize some classes of unimodular wavelets for L^2. We then apply the same method to the Hardy space H^2 to construct a new class of wavelets for H^2.

One class of wavelets constructed in this paper is

$$\hat{\psi}_j(\xi) = \chi_{[2^j\pi/(2^{j+1}-1),\pi]} + \chi_{[2^j\pi, 2^j\pi+2^j\pi/(2^{j+1}-1)]}(\xi) \quad \text{for } j = 1, 2, \ldots,$$

where ψ_j is even. To our surprise, ψ_2 is the Journé–Meyer example. Moreover, we prove that ψ_j cannot be considered from a multiresolution analysis if $j \geq 2$.

We organize this paper as follows. In Section 2 we give a proof of the fact that ψ being a wavelet is equivalent to (W1)–(W4), since no proof of the fact is currently available. In Section 3 we find a necessary and sufficient condition for a unimodular function to be a wavelet. In Section 4 we characterize certain classes of unimodular wavelets for L^2 and deal with the existence and non-existence of corresponding MRAs. In Section 5 we characterize some classes of unimodular wavelets for H^2.

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2. Meyer's Equations

In this section we prove Theorem 2.1, the statement of which appears without proof in [Le]. Note that we prove Theorem 2.1 without any *a priori* assumption on ψ. Bonami, Soria, and Weiss [BSW] proved the theorem when ψ is band-limited (i.e., when $\hat{\psi}$ has a compact support).

THEOREM 2.1. *Let $\psi \in L^2(\mathbf{R})$. Then $\{\psi_{j,k}\}$ is an orthonormal set in $L^2(\mathbf{R})$ if and only if*

(W1) $\sum_{k\in\mathbf{Z}} |\hat{\psi}(\xi+2k\pi)|^2 = 1$ *and*
(W2) $\sum_{k\in\mathbf{Z}} \hat{\psi}(\xi+2k\pi)\hat{\psi}^*(2^j(\xi+2k\pi)) = 0$ *for* $j \geq 1$.

Furthermore, $\{\psi_{j,k}\}$ is complete in $L^2(\mathbf{R})$ if and only if

(W3) $\sum_{j\in\mathbf{Z}} |\hat{\psi}(2^{-j}\xi)|^2 = 1$ *and*
(W4) $\sum_{l\geq 0} \hat{\psi}(2^l(\xi+2p_0\pi))\hat{\psi}^*(2^l\xi) = 0$ *for* $p_0 \in 2\mathbf{Z}+1$.

All the equalities are to hold in the sense of almost everywhere.

Proof. We first note that the infinite series in (W2) converges in $L^1(a, a+2\pi)$ for each a, and that the series in (W4) converges in $L^1(\mathbf{R})$. An easy computation shows that for each pair j and k,

$$\hat{\psi}_{j,k}(\xi) = 2^{-j/2} e^{-i2^{-j}k\xi} \hat{\psi}(2^{-j}\xi). \tag{2.1}$$

We then have

$$\begin{aligned}
&\langle \psi_{j,k}, \psi_{n,l} \rangle \\
&= \frac{1}{2\pi} \langle \hat{\psi}_{j,k}, \hat{\psi}_{n,l} \rangle \\
&= \frac{1}{2\pi} 2^{-j/2} 2^{-n/2} \int_{-\infty}^{\infty} \hat{\psi}(2^{-j}\xi)\hat{\psi}^*(2^{-n}\xi) e^{-i2^{-j}k\xi} e^{i2^{-n}l\xi}\, d\xi
\end{aligned}$$

$$= \frac{1}{2\pi} 2^{(j-n)/2} \int_{-\infty}^{\infty} \hat{\psi}(\xi) \hat{\psi}^*(2^{j-n}\xi) e^{-ik\xi} e^{i2^{j-n}l\xi} \, d\xi$$

$$= \frac{1}{2\pi} 2^{(j-n)/2} \sum_{m\in\mathbf{Z}} \int_0^{2\pi} \hat{\psi}(\xi+2m\pi) \hat{\psi}^*(2^{j-n}(\xi+2m\pi)) e^{-ik\xi} e^{i2^{j-n}l\xi} \, d\xi.$$

Since $\sum_{m\in\mathbf{Z}} \hat{\psi}(\xi+2m\pi)\hat{\psi}^*(2^{j-n}(\xi+2m\pi))$ is in $L^1(0, 2\pi)$, one can interchange the infinite sum with the integral and get

$$\langle \psi_{j,k}, \psi_{n,l} \rangle = \frac{1}{2\pi} 2^{(j-n)/2} \int_0^{2\pi} \nu_{j-n}(\xi) e^{i2^{j-n}l\xi} e^{-ik\xi} \, d\xi, \tag{2.2}$$

where

$$\nu_j(\xi) = \sum_{m\in\mathbf{Z}} \hat{\psi}(\xi+2m\pi) \hat{\psi}^*(2^j(\xi+2m\pi)). \tag{2.3}$$

Suppose that $\{\psi_{j,k}\}$ is an orthonormal system. If $j=n$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \nu_0(\xi) e^{i(l-k)\xi} \, d\xi = \delta_{lk} \quad \text{for all } l, k \in \mathbf{Z}.$$

Since $\nu_0 \in L^1(0, 2\pi)$ and is 2π-periodic, we have $\nu_0(\xi) \equiv 1$ which is (W1). Here, we use the fact that the Cesaró sums of a 2π-periodic function in $f \in L^1(0, 2\pi)$ converge to f in L^1 norm [To]. If $j > n$ then $\nu_{j-n}(\xi) e^{i2^{j-n}l\xi} \in L^1(0, 2\pi)$ and is 2π-periodic; hence $\nu_{j-n}(\xi) \equiv 0$ by (2.2). Thus (W2) is proved. Conversely, that (W1) and (W2) imply the orthonormality of $\{\psi_{j,k}\}$ is trivial.

Suppose now that $\{\psi_{j,k}\}$ is complete in $L^2(\mathbf{R})$. Then $\{\hat{\psi}_{j,k}\}$ is also complete in $L^2(\mathbf{R})$, so we have

$$f = \sum_{j,k} \frac{1}{2\pi} \langle f, \hat{\psi}_{j,k} \rangle \hat{\psi}_{j,k} \quad \text{for all } f \in L^2(\mathbf{R}).$$

Hence, by the Poisson summation formula,

$$f(\xi) = \frac{1}{2\pi} \sum_{j,k} 2^{-j} \left(\int_{-\infty}^{\infty} f(\xi) \hat{\psi}^*(2^{-j}\xi) e^{i2^{-j}k\xi} \, d\xi \right) e^{-i2^{-j}k\xi} \hat{\psi}(2^{-j}\xi)$$

$$= \frac{1}{2\pi} \sum_{j,k} \left(\int_{-\infty}^{\infty} f(2^j\eta) \hat{\psi}^*(\eta) e^{ik\eta} \, d\eta \right) e^{-i2^{-j}k\xi} \hat{\psi}(2^{-j}\xi)$$

$$= \sum_j \sum_k f(2^j(2^{-j}\xi + 2k\pi)) \hat{\psi}^*(2^{-j}\xi + 2k\pi) \hat{\psi}(2^{-j}\xi).$$

In conclusion, we have

$$f(\xi) = f(\xi) \sum_j |\hat{\psi}(2^{-j}\xi)|^2$$
$$+ \sum_j \sum_{k\neq 0} f(\xi + 22^j k\pi) \hat{\psi}(2^{-j}\xi) \hat{\psi}^*(2^{-j}\psi + 2k\pi). \tag{2.4}$$

We note that the infinite series converges in $L^1(\mathbf{R})$ if f has a compact support. We omit the proof of this fact since it is similar to arguments which follow. Let

$$\theta(\xi) = \sum_j |\hat{\psi}(2^{-j}\xi)|^2. \tag{2.5}$$

Let $\xi_0 \neq 0$ be an arbitrary point in $\mathbf{R}$. Choose ϵ so that $0<\epsilon<|\xi_0|/12\pi$ and let $f=\chi_{[\xi_0-\epsilon\pi,\,\xi_0+\epsilon\pi]}(\xi)$, the characteristic function on $[\xi_0-\epsilon\pi, \xi_0+\epsilon\pi]$. It then follows from (2.4) that

$$\int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi} |1-\theta(\xi)|\,d\xi \leq \sum_j \sum_{k\neq 0} \int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi} |f(\xi+2k2^j\pi)||\hat{\psi}(2^{-j}\xi)||\hat{\psi}(2^{-j}\xi+2k\pi)|\,d\xi. \tag{2.6}$$

Note that $f(\xi+2k2^j\pi)\neq 0$ on $[\xi_0-\epsilon\pi, \xi_0+\epsilon\pi]$ only if $|k2^j|<\epsilon$. Thus we have

$$\begin{aligned}\int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi} |1-\theta(\xi)|\,d\xi &\leq \sum_{|k2^j|<\epsilon} \int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi} |\hat{\psi}(2^{-j}\xi)||\hat{\psi}(2^{-j}\xi+2k\pi)|\,d\xi \\ &= \sum_{|k2^j|<\epsilon} 2^j \int_{2^{-j}(\xi_0-\epsilon\pi)}^{2^{-j}(\xi_0+\epsilon\pi)} |\hat{\psi}(\eta)||\hat{\psi}(\eta+2k\pi)|\,d\eta \\ &\leq \sum_{j<\log_2\epsilon} 2^j \left(\int_{2^{-j}(\xi_0-\epsilon\pi)}^{2^{-j}(\xi_0+\epsilon\pi)} |\hat{\psi}(\eta)|^2\,d\eta\right)^{1/2} \\ &\quad\times \sum_{|k|<\epsilon 2^{-j}} \left(\int_{2^{-j}(\xi_0-\epsilon\pi)}^{2^{-j}(\xi_0+\epsilon\pi)} |\hat{\psi}(\eta+2k\pi)|^2\,d\eta\right)^{1/2}.\end{aligned}$$

Note that if $|k2^j|<\epsilon$ then

$$[2^{-j}(\xi_0-\epsilon\pi), 2^{-j}(\xi_0+\epsilon\pi)]+2k\pi \subset [2^{-j}(\xi_0-3\epsilon\pi), 2^{-j}(\xi_0+3\epsilon\pi)].$$

Hence,

$$\sum_{|k|<\epsilon 2^{-j}} \left(\int_{2^{-j}(\xi_0-\epsilon\pi)}^{2^{-j}(\xi_0+\epsilon\pi)} |\hat{\psi}(\eta+2k\pi)|^2\,d\eta\right)^{1/2} \leq 2\epsilon 2^{-j} \left(\int_{2^{-j}(\xi_0-3\epsilon\pi)}^{2^{-j}(\xi_0+3\epsilon\pi)} |\hat{\psi}(\eta)|^2\,d\eta\right)^{1/2}.$$

It then follows that

$$\int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi} |1-\theta(\xi)|\,d\xi \leq 2\epsilon \sum_{j<\log_2\epsilon} \int_{2^{-j}(\xi_0-3\epsilon\pi)}^{2^{-j}(\xi_0+3\epsilon\pi)} |\hat{\psi}(\eta)|^2\,d\eta.$$

Assume that $\xi_0>0$ without loss of generality. Since $\epsilon<\xi_0/12\pi$, we have $2^{-j+1}(\xi_0-3\epsilon\pi)>2^{-j}(\xi_0+3\epsilon\pi)$ and hence

$$\int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi} |1-\theta(\xi)|\,d\xi \leq 2\epsilon \int_{\epsilon^{-1}(\xi_0-3\epsilon\pi)}^{\infty} |\hat{\psi}(\eta)|^2\,d\eta. \tag{2.7}$$

In particular, $1-\theta\in L^1([\xi_0-\epsilon\pi, \xi_0+\epsilon\pi])$. Let $\xi\in(\xi_0-\epsilon\pi, \xi_0+\epsilon\pi)$ be a Lebesgue point of $1-\theta$ and choose δ so that $[\xi-\delta\pi, \xi+\delta\pi]\subset(\xi_0-\epsilon\pi, \xi_0+\epsilon\pi)$ and that $\delta<\xi/12\pi$. Then, by (2.7),

$$\frac{1}{2\delta\pi}\int_{\xi-\delta\pi}^{\xi+\delta\pi} |1-\theta(\eta)|\,d\eta \leq \frac{1}{\pi}\int_{\delta^{-1}(\xi_0-3\delta\pi)}^{\infty} |\hat{\psi}(\eta)|^2\,d\eta.$$

As $\delta\to 0$, we have $\theta(\xi)=1$ by the Lebesgue differentiation theorem. So, $\theta=1$ a.e. on $[\xi_0-\epsilon\pi, \xi_0+\epsilon\pi]$. Since ξ_0 is arbitrary, we have $\theta=1$ a.e. This proves (W3).

Now, from (2.4) we have

$$\sum_j \sum_{k\neq 0} f(\xi+22^j k\pi)\hat{\psi}(2^{-j}\xi)\hat{\psi}^*(2^{-j}\xi+2k\pi)=0 \quad \text{for all } f\in L^2(\mathbf{R}). \tag{2.8}$$

Let $k=p2^l$, where $l=0,1,2,\dots$ and $p\in 2\mathbf{Z}+1$. Then the identity (2.8) can be written as

$$\begin{aligned} 0&=\sum_{j\in\mathbf{Z}}\sum_{l=0}^{\infty}\sum_{p\in 2\mathbf{Z}+1} f(\xi+2p2^{j+l}\pi)\hat{\psi}(2^{-j}\xi)\hat{\psi}^*(2^{-j}\xi+2p2^l\pi) \\ &=\sum_{p\in 2\mathbf{Z}+1}\sum_{n\in\mathbf{Z}} f(\xi+2p2^n\pi)\theta_p(2^{-n}\xi), \end{aligned} \tag{2.9}$$

where

$$\theta_p(\xi)=\sum_{l=0}^{\infty}\hat{\psi}(2^l\xi)\hat{\psi}^*(2^l(\xi+2p\pi)). \tag{2.10}$$

Fix $p_0\in 2\mathbf{Z}+1$ and let $g(\xi)=f(\xi+2p_0\pi)$. Then, rewrite (2.9) as

$$g(\xi)\theta_{p_0}(\xi)+\sum_{\substack{p\in 2\mathbf{Z}+1 \\ (p,n)\neq(p_0,0)}}\sum_{n\in\mathbf{Z}} g(\xi+2p2^n\pi-2p_0\pi)\theta_p(2^{-n}\xi)=0.$$

As before, let $\xi_0\neq 0$ be an arbitrary point and put $g(\xi)=\chi_{[\xi_0-\epsilon\pi,\,\xi_0+\epsilon\pi]}(\xi)$. It then follows that

$$\int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi}|\theta_{p_0}(\xi)|\,d\xi \le \sum_{\substack{p\in 2\mathbf{Z}+1 \\ n\in\mathbf{Z} \\ (p,n)\neq(p_0,0)}}\int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi}|g(\xi+2p2^n\pi-2p_0\pi)||\theta_p(2^{-n}\xi)|\,d\xi. \tag{2.11}$$

Note that $g(\xi+2p2^n\pi-2p_0\pi)\neq 0$ only if $|2p2^n\pi-2p_0\pi|<2\epsilon\pi$. Thus we have

$$\int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi}|\theta_{p_0}(\xi)|\,d\xi \le \sum_{\substack{|p2^n-p_0|<\epsilon \\ (p,n)\neq(p_0,0)}}\int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi}|\theta_p(2^{-n}\xi)|\,d\xi. \tag{2.12}$$

Then,

$$\begin{aligned} &\sum_{\substack{|p2^n-p_0|<\epsilon \\ (p,n)\neq(p_0,0)}}\int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi}|\theta_p(2^{-n}\xi)|\,d\xi \\ &\quad\le \sum_{\substack{|p2^n-p_0|<\epsilon \\ (p,n)\neq(p_0,0)}} 2^n\int_{2^{-n}(\xi_0-\epsilon\pi)}^{2^{-n}(\xi_0+\epsilon\pi)}|\theta_p(\eta)|\,d\eta \\ &\quad\le \sum_{\substack{|p2^n-p_0|<\epsilon \\ (p,n)\neq(p_0,0)}} 2^n\sum_{l\ge 0}\int_{2^{-n}(\xi_0-\epsilon\pi)}^{2^{-n}(\xi_0+\epsilon\pi)}|\hat{\psi}(2^l\eta)||\hat{\psi}(2^l(\eta+2p\pi))|\,d\eta. \end{aligned}$$

If $|p2^n-p_0|<\epsilon$ and $(p,n)\neq(p_0,0)$, then $n<0$ and hence

$$2^n\le 2^n|p-p_0 2^{-n}|<\epsilon.$$

Hence $n<\log_2\epsilon$. Therefore,

$$\sum_{\substack{|p2^n-p_0|<\epsilon\\(p,n)\neq(p_0,0)}}\int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi}|\theta_p(2^{-n}\xi)|\,d\xi$$
$$\leq \sum_{n<\log_2\epsilon}2^n\sum_{l\geq 0}2^{-l}\left(\int_{2^{-n+l}(\xi_0-\epsilon\pi)}^{2^{-n+l}(\xi_0+\epsilon\pi)}|\hat{\psi}(\eta)|^2\,d\eta\right)^{1/2}$$
$$\times\sum_{|p-p_0 2^{-n}|<\epsilon 2^{-n}}\left(\int_{2^{-n+l}(\xi_0-\epsilon\pi)}^{2^{-n+l}(\xi_0+\epsilon\pi)}|\hat{\psi}(\eta+2p2^l\pi)|^2\,d\eta\right)^{1/2}.$$

Note that if $|p-p_0 2^{-n}|<\epsilon 2^{-n}$, then

$$[2^{-n+l}(\xi_0-\epsilon\pi),2^{-n+l}(\xi_0+\epsilon\pi)]+2p2^l\pi$$
$$\subset[2^{-n+l}(\xi_0+2p_0\pi-3\epsilon\pi),2^{-n+l}(\xi_0+2p_0\pi+3\epsilon\pi)].$$

Hence,

$$\sum_{|p-p_0 2^{-n}|<\epsilon 2^{-n}}\left(\int_{2^{-n+l}(\xi_0-\epsilon\pi)}^{2^{-n+l}(\xi_0+\epsilon\pi)}|\hat{\psi}(\eta+2p2^l\pi)|^2\,d\eta\right)^{1/2}$$
$$\leq 2\epsilon 2^{-n}\left(\int_{2^{-n+l}(\xi_0+2p_0\pi-3\epsilon\pi)}^{2^{-n+l}(\xi_0+2p_0\pi+3\epsilon\pi)}|\hat{\psi}(\eta)|^2\,d\eta\right)^{1/2}.$$

It then follows that

$$\sum_{\substack{|p2^n-p_0|<\epsilon\\(p,n)\neq(p_0,0)}}\int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi}|\theta_p(2^{-n}\xi)|\,d\xi$$
$$\leq 2\epsilon\sum_{l\geq 0}2^{-l}\sum_{n<\log_2\epsilon}\left(\int_{2^{-n+l}(\xi_0-\epsilon\pi)}^{2^{-n+l}(\xi_0+\epsilon\pi)}|\hat{\psi}(\eta)|^2\,d\eta\right)^{1/2}$$
$$\times\left(\int_{2^{-n+l}(\xi_0+2p_0\pi-3\epsilon\pi)}^{2^{-n+l}(\xi_0+2p_0\pi+3\epsilon\pi)}|\hat{\psi}(\eta)|^2\,d\eta\right)^{1/2}$$
$$\leq 2\epsilon\sum_{l\geq 0}2^{-l}\left(\sum_{n<\log_2\epsilon}\int_{2^{-n+l}(\xi_0-\epsilon\pi)}^{2^{-n+l}(\xi_0+\epsilon\pi)}|\hat{\psi}(\eta)|^2\,d\eta\right)^{1/2}$$
$$\times\left(\sum_{n<\log_2\epsilon}\int_{2^{-n+l}(\xi_0+2p_0\pi-3\epsilon\pi)}^{2^{-n+l}(\xi_0+2p_0\pi+3\epsilon\pi)}|\hat{\psi}(\eta)|^2\,d\eta\right)^{1/2}.$$

Assume that $\xi_0>0$ and $\xi_0+2p_0\pi>0$ without loss of generality. We then choose ϵ so that $\epsilon<(1/12\pi)\min(\xi_0,\xi_0+2p_0\pi)$. It then follows that

$$\sum_{\substack{|p2^n-p_0|<\epsilon\\(p,n)\neq(p_0,0)}}\int_{\xi_0-\epsilon\pi}^{\xi_0+\epsilon\pi}|\theta_p(2^{-n}\xi)|\,d\xi$$
$$\leq 2\epsilon\sum_{l\geq 0}2^{-l}\left(\int_{\epsilon^{-1}2^l(\xi_0-\epsilon\pi)}^{\infty}|\hat{\psi}(\eta)|^2\,d\eta\right)^{1/2}\left(\int_{\epsilon^{-1}2^l(\xi_0+2p_0\pi-3\epsilon\pi)}^{\infty}|\hat{\psi}(\eta)|^2\,d\eta\right)^{1/2}$$
$$\leq 2\epsilon\left(\int_{\epsilon^{-1}(\xi_0-\epsilon\pi)}^{\infty}|\hat{\psi}(\eta)|^2\,d\eta\right)^{1/2}\left(\int_{\epsilon^{-1}(\xi_0+2p_0\pi-3\epsilon\pi)}^{\infty}|\hat{\psi}(\eta)|^2\,d\eta\right)^{1/2}.$$

Then, by the same argument as before, we can show that

$$\theta_{p_0} \in L^1([\xi_0 - \epsilon\pi, \xi_0 + \epsilon\pi])$$

and $\theta_{p_0} = 0$ a.e. on $[\xi_0 - \epsilon\pi, \xi_0 + \epsilon\pi]$. Since ξ_0 is arbitrary, we conclude that $\theta_{p_0} = 0$ a.e. This completes the proof of (W4).

If ψ satisfies (W3) and (W4), then $\{\psi_{j,k}\}$ forms a complete system by (2.4) and (2.9). This completes the proof. □

3. Unimodular Wavelets

In this section, we find a necessary and sufficient condition for a unimodular function to be a wavelet for $L^2(\mathbf{R})$.

DEFINITION 3.1. A function $\psi \in L^2(\mathbf{R})$ is called a *unimodular function* if $|\hat{\psi}(\xi)| = 1$ for $\xi \in \operatorname{supp} \hat{\psi}$.

LEMMA 3.2. *Let ψ be a unimodular function in $L^2(\mathbf{R})$. Then* (W1) *implies* (W4).

Proof. Fix ξ. We claim that each term of the summands in (W4) is equal to 0. In fact, if $2^j\xi \in \operatorname{supp} \hat{\psi}$ then $|\hat{\psi}(2^j\xi)| = 1$. Since $\sum_{k\in\mathbf{Z}} |\hat{\psi}(2^j\xi + 2k\pi)|^2 = 1$, $\hat{\psi}(2^j\xi + 2k\pi) = 0$ if $k \neq 0$. In particular, $\hat{\psi}(2^j(\xi + 2p\pi)) = 0$ for every $p \in 2\mathbf{Z}+1$. This completes the proof. □

LEMMA 3.3. *Let ψ be a unimodular function in $L^2(\mathbf{R})$. Then* (W3) *implies* (W2).

Proof. Fix ξ. If $\xi + 2k\pi \in \operatorname{supp} \hat{\psi}$ then $|\hat{\psi}(\xi + 2k\pi)| = 1$. Thus (W3) implies that $\hat{\psi}(2^j(\xi + 2k\pi)) = 0$ if $j \neq 0$. Hence $\hat{\psi}(\xi + 2k\pi)\hat{\psi}(2^j(\xi + 2k\pi)) = 0$ for each j and k. This completes the proof. □

Combining Lemma 3.2 and Lemma 3.3, we obtain the following theorem.

THEOREM 3.4. *Let ψ be a unimodular function in $L^2(\mathbf{R})$. Then ψ is a wavelet for $L^2(\mathbf{R})$ if and only if the following hold:*

(W1) $\sum_{k\in\mathbf{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = 1$; *and*
(W3) $\sum_{j\in\mathbf{Z}} |\hat{\psi}(2^{-j}\xi)|^2 = 1$.

Let a be a real number. For each $x \in \mathbf{R}$ there is a unique integer $k(x)$ such that $a \le x + 2k(x)\pi < a + 2\pi$. We define a function $\tau_a : \mathbf{R} \to [a, a+2\pi)$ by

$$\tau_a(x) = x + 2k(x)\pi. \tag{3.1}$$

Let $a > 0$. For each $x > 0$, there is a unique integer $j(x)$ such that $a \le 2^{j(x)}x < 2a$. We define a function $\delta_a : (0, \infty) \to [a, 2a)$ by

$$\delta_a(x) = 2^{j(x)}x. \tag{3.2}$$

If $a<0$, we define $\delta_a(x) = -\delta_{-a}(-x)$ for $x<0$.

LEMMA 3.5. *Let τ_a and δ_a be defined as above. Then:*

(1) *For each pair a and b, $\tau_a\tau_b = \tau_a$ and each τ_a maps $[b, b+2\pi)$ bijectively onto $[a, a+2\pi)$.*
(2) *For positive a and b, $\delta_a\delta_b = \delta_a$ and each δ_a maps $[b, 2b)$ bijectively onto $[a, 2a)$. The same holds for negative a and b.*

Proof. The proof of (1) is trivial. For (2), we let j be the unique integer such that $a \le 2^j b < 2a$. If $2^j b = a$ then there is nothing to be proved. If $2^j b > a$, let $c = 2^{1-j}a$; then $b<c<2b$ and $\delta_a(c) = a$. Hence, δ_a maps $[b, c)$ bijectively onto $[2^j b, 2a)$ and δ_a maps $[c, 2b)$ bijectively onto $[a, 2^j b)$. That $\delta_a\delta_b = \delta_a$ is trivial. This completes the proof. □

THEOREM 3.6. *Let ψ be a unimodular function in $L^2(\mathbf{R})$. Let $K = \operatorname{supp}\hat{\psi}$, $K^+ = K\cap(0,\infty)$, and $K^- = K\cap(-\infty, 0)$. Then ψ is a wavelet for $L^2(\mathbf{R})$ if and only if the following hold:*

(1) *For each $a \in \mathbf{R}$, τ_a is one-to-one on K except on a set of measure zero and $|[a, a+2\pi) - \tau_a(K)| = 0$; and*
(2) *for each $a>0$ and $b<0$, δ_a and δ_b are one-to-one on K^+ and K^- except on a set of measure zero, and*

$$|[a, 2a) - \delta_a(K^+)| = 0 \quad \textit{and} \quad |(2b, b] - \delta_b(K^-)| = 0,$$

respectively.

Here $|\cdot|$ denotes Lebesgue measure.

REMARK 3.7. As a result of Lemma 3.5, "For each a" in (1) and (2) in Theorem 3.6 can be replaced by "For some a".

Proof. Suppose that ψ satisfies (W1). Then $\sum_{k\in\mathbf{Z}}|\hat{\psi}(\xi+2k\pi)|^2 = 1$ a.e. Let H be the set of points where $\sum_{k\in\mathbf{Z}}|\hat{\psi}(\xi+2k\pi)|^2 \ne 1$. Since $\sum_{k\in\mathbf{Z}}|\hat{\psi}(\xi+2k\pi)|^2$ is 2π-periodic, $H+2k\pi = H$ for any integer k and hence $\tau_a(H)\subset H$. Let $\xi\in[a, a+2\pi)-H$. Then $\hat{\psi}(\xi+2k\pi)\ne 0$ for some $k\in\mathbf{Z}$, and hence $\xi+2k\pi\in K$ and $\tau_a(\xi+2k\pi) = \xi$. Thus $[a, a+2\pi)-H\subset\tau_a(K)$. Moreover, such a k is unique since ψ is unimodular, and hence τ_a is one-to-one on $K-H$.

We now suppose that (W3) holds and let a be positive. Let U be the set of all points where $\sum_{j\in\mathbf{Z}}|\hat{\psi}(2^{-j}\xi)|^2 \ne 1$ and let $H = \bigcup_{j\in\mathbf{Z}}(2^j U)$. Then $|H| = 0$ and $\delta_a(H)\subset H$. Let $\xi\in[a, 2a)-H$. Since $\sum_{j\in\mathbf{Z}}|\hat{\psi}(2^{-j}\xi)|^2 = 1$, we have $\hat{\psi}(2^{-j}\xi)\ne 0$ for some j. Therefore, $2^{-j}\xi\in K^+$ and $\delta_a(2^{-j}\xi) = \xi$. It follows that $[a, 2a)-H\subset\delta_a(K^+)$. Moreover, such a j is unique and hence δ_a is one-to-one on K^+-H. The case when $b<0$ can be treated in the same way.

Suppose that (1) holds and that U and V are the sets of measure zero such that $\tau_a(K) = [a, a+2\pi)-V$ and τ_a is one-to-one on $K-U$. Let $H=$

$\bigcup_{k\in\mathbf{Z}}(U\cup V+2k\pi)$. Then $|H|=0$, $\tau_a(H)\subset H\cap[a,a+2\pi)$, and $\tau_a(K-H)=[a,a+2\pi)-H$. We claim that $\sum_{k\in\mathbf{Z}}|\hat{\psi}(\xi+2k\pi)|^2=1$ for all $\xi\in\mathbf{R}-H$. Let $\xi\in\mathbf{R}-H$ and let $\xi_0=\tau_a(\xi)$. Then there exists an integer k_0 such that $\xi=\xi_0+2k_0\pi$. Since $\xi_0\in[a,a+2\pi)-H$ and τ_a is one-to-one on $K-H$, there exists a unique integer k_1 such that $\xi_0-2k_1\pi\in K-H$. Hence, $\xi+2k\pi\in K$ if and only if $k=-(k_0+k_1)$. We thus have

$$\sum_{k\in\mathbf{Z}}|\hat{\psi}(\xi+2k\pi)|^2=|\hat{\psi}(\xi-2(k_0+k_1)\pi)|^2=1.$$

This proves (W1).

Now suppose that (2) holds and that $a>0$. (The case when $a<0$ can be treated in the same way.) Let U and V be sets of measure zero such that $\delta_a(K)=[a,2a)-V$ and δ_a is one-to-one on K^+-U. Let

$$H=\bigcup_{j\in\mathbf{Z}}(2^j(U\cup V)).$$

Then $|H|=0$, $\delta_a(H)\subset H\cap[a,2a)$, and $\delta_a(K^+-H)=[a,2a)-H$. We claim that $\sum_{j\in\mathbf{Z}}|\hat{\psi}(2^{-j}\xi)|^2=1$ for all $\xi\in(0,\infty)-H$. Let $\xi\in(0,\infty)-H$ and let $\xi_0=\delta_a(\xi)$. Then there exists an integer j_0 such that $\xi=2^{j_0}\xi_0$. Since $\xi_0\in[a,2a)-H$, there exists a unique integer j_1 such that $2^{-j_1}\xi_0\in K^+$. Hence $2^{-j}\xi\in K^+$ if and only if $j=j_0+j_1$. We thus have

$$\sum_{j\in\mathbf{Z}}|\hat{\psi}(2^{-j}\xi)|^2=|\hat{\psi}(2^{-j_0-j_1}\xi)|^2=1.$$

This proves (W3). It follows from Theorem 3.4 that ψ is a wavelet for $L^2(\mathbf{R})$. This completes the proof. □

If ψ is a unimodular function and if $K=\operatorname{supp}\hat{\psi}$, then ψ can be written as

$$\hat{\psi}(\xi)=\theta(\xi)\chi_K(\xi), \tag{3.3}$$

where $|\theta(\xi)|=1$. One can see from Theorem 3.6 that ψ is a wavelet, provided that K satisfies conditions (1) and (2) of Theorem 3.6.

4. Unimodular Wavelets for L^2

According to Theorem 3.6 and Remark 3.7, a unimodular wavelet is essentially a characteristic function on a set K. In this section, we shall consider those unimodular wavelets whose supports are disjoint unions of finitely many intervals. In fact, we characterize three special classes of unimodular wavelets. The same method may be applied to more general classes of unimodular wavelets.

Let ψ be a unimodular wavelet and let $K=\operatorname{supp}\hat{\psi}$. As before, let $K^+=K\cap(0,\infty)$ and $K^-=K\cap(-\infty,0)$. Then K^+ and K^- are non-empty, by Theorem 3.6. From now on, we deal with those K which are finite unions of intervals. If K is a finite union of intervals, then $|[a,a+2\pi)-\tau_a(K)|=0$ if and only if $\tau_a(K)=[a,a+2\pi)$, and τ_a is one-to-one on K except on a set of measure zero if and only if τ_a is one-to-one on K except at finitely many points. So, if this is the case, we simply say $\tau_a(K)=[a,a+2\pi)$ and τ_a is one-to-one on K. The same thing applies to δ_a.

I. The case when both K^+ and K^- are intervals

Suppose that $K^+=[a,b]$ for some $b>a>0$. Since $\tau_a(K)=[a,a+2\pi)$ and τ_a is one-to-one on K, $\tau_a(K^-)=[b,a+2\pi)$. Hence, $b<a+2\pi$ and there exists an integer p such that $a-2(p-1)\pi<0$ and $K^-=[b-2p\pi,a-2(p-1)\pi]$.

Since $\delta_a(K^+)=[a,2a)$ and δ_a is one-to-one on K^+, we have $b=2a$. Likewise, one can see that $b-2p\pi=2(a-2(p-1)\pi)$. We thus have the following:

$$b=2a<a+2\pi \quad \text{and} \quad b-2p\pi=2(a-2(p-1)\pi)<0.$$

THEOREM 4.1. *Let $\hat{\psi}(\xi)=\theta(\xi)\chi_K(\xi)$, where*

$$K=[2a-4\pi,a-2\pi]\cup[a,2a] \tag{4.1}$$

for some $0<a<2\pi$ and $|\theta(\xi)|=1$. Then ψ is a wavelet for L^2. Moreover, each unimodular wavelet ψ for which the support of $\hat{\psi}$ consists of two disjoint intervals is of this form.

REMARK 4.2. The wavelets in Theorem 4.1 are associated with MRAs. In fact, if

$$\hat{\psi}_\theta(\xi)=\theta(\xi)(\chi_{[2a-4\pi,2a-2\pi]}+\chi_{[a,2a]})$$

with θ defined on $[2a-4\pi,a-2\pi]\cup[a,2a]$, then we can extend θ to be a 2π-periodic function. Hence the closure of the span of $\{\psi_\theta(x-k): k\in\mathbf{Z}\}$ is the same no matter what θ is. Hence, it is enough to show that the wavelet $\hat{\psi}(\xi)=e^{i\xi/2}(\chi_{[2a-4\pi,2a-2\pi]}+\chi_{[a,2a]})$ is associated with an MRA. But one can easily see that the wavelet $\hat{\psi}(\xi)=e^{i\xi/2}(\chi_{[2a-4\pi,2a-2\pi]}+\chi_{[a,2a]})$ is associated with the MRA whose scaling function is given by $\hat{\varphi}(\xi)=\chi_{[a-2\pi,a]}(\xi)$.

II. The case when $K^-=-K^+$ and K^+ consists of two disjoint intervals

Suppose that $K^+=[a,b]\cup[c,d]$ for some $0<a<b<c<d$. Since $|\tau_a(K)|=2|\tau_a(K^+)|$ and $|\tau(K)|=2\pi$, $|\tau(K^+)|=\pi$. Moreover, $a=|\delta_a(K^+)|\le|K^+|\le\pi$. We claim that $b=\pi$. In fact, if $b>\pi$ then $\tau_0([a,\pi])\cap\tau_0([-b,-\pi])$ contains an interval and hence τ_0 is not one-to-one on K. Now, suppose that $b<\pi$. Then, since $\pi\in\tau_0(K)$, there exists an integer k such that $\pi+2k\pi\in[c,d]$ and $-\pi-2k\pi\in[-d,-c]$. Hence

$$\tau_0([-d,-\pi-2k\pi])\cap\tau_0([c,\pi+2k\pi])$$

contains an interval and so τ_0 is not one-to-one on K. Note that $[c,d]\subset(2^ja,2^{j+1}a]$ since, if not, then $[a,a+\epsilon)\subset\delta_a([c,d])\cap[a,b]$ for some $\epsilon>0$;

hence δ_a is not one-to-one on K. Since $\delta_a(K^+)=[a,2a)$, we have $c=2^j b$ and $d=2^{j+1}a$ for some positive integer j. Therefore

$$K=[-2^{j+1}a,-2^j\pi]\cup[-\pi,-a]\cup[a,\pi]\cup[2^j\pi,2^{j+1}a].$$

Note that $\tau_{-\pi}(K)=[-\pi,\pi)$, that $\tau_{-\pi}(2^j\pi)=0$ if $j\geq 1$, and that $\tau_{-\pi}([a,\pi])=[a,\pi)$. Thus we must have $\tau_{-\pi}([2^j\pi,2^{j+1}a])=[0,a]$. It then follows that $2^{j+1}a-2^j\pi=a$ and hence $a=2^j\pi/(2^{j+1}-1)$.

THEOREM 4.3. *Let* $\hat{\psi}(\xi)=\theta(\xi)\chi_K(\xi)$, *where*

$$K^+=\left[\frac{2^j}{2^{j+1}-1}\pi,\pi\right]\cup\left[2^j\pi,2^j\pi+\frac{2^j}{2^{j+1}-1}\pi\right],\quad K^-=-K^+,\qquad(4.2)$$

and where j is a positive integer and $|\theta(\xi)|=1$. Then ψ is a wavelet for L^2. Moreover, each unimodular wavelet for which $K^-=-K^+$ and K^+ consists of two disjoint intervals is of this form.

If $j=2$ and $\theta\equiv 1$, then

$$\hat{\psi}(\xi)=\chi_{[-32\pi/7,-4\pi]}+\chi_{[-\pi,-4\pi/7]}+\chi_{[4\pi/7,\pi]}+\chi_{[4\pi,32\pi/7]}.$$

This ψ is the Journé–Meyer example of a wavelet which is not associated with an MRA [D2, Ma]. We will show that each ψ is not associated with an MRA if $j\geq 2$.

THEOREM 4.4. *Let* $\hat{\psi}(\xi)=\theta(\xi)\chi_K(\xi)$, *where K is given as in* (4.2) *and* $|\theta|=1$.

(1) *If $j=1$, then the corresponding ψ is associated with an MRA.*
(2) *If $j\geq 2$, then the corresponding ψ is not associated with an MRA.*

Proof. (1) If $j=1$ and $\hat{\psi}(\xi)=e^{i\xi/2}\chi_K(\xi)$, then one can easily check by (1.2) that ψ is associated with an MRA whose scaling function is given by

$$\hat{\varphi}(\xi)=\chi_{[-4\pi/3,-\pi]}(\xi)+\chi_{[-2\pi/3,2\pi/3]}(\xi)+\chi_{[\pi,4\pi/3]}(\xi).\qquad(4.3)$$

It can also be checked by Proposition 5.3.1 and 5.3.2 of [D1] that this φ really defines a scaling function. For arbitrary θ, the theorem follows from the same argument as in Remark 4.2.

(2) Suppose that $j\geq 2$ and that ψ is associated with an MRA with a scaling function φ. Then, $\hat{\psi}(2\xi)=l(\xi)\hat{\varphi}(\xi)$ and $\hat{\psi}(4\xi)=m(\xi)\hat{\varphi}(\xi)$ for some 2π-periodic functions l and m in $L^2(0,2\pi)$. Since $\hat{\psi}(2\xi)=1$ on

$$\left[\frac{2^{j-1}}{2^{j+1}-1}\pi,\frac{\pi}{2}\right]\cup\left[2^{j-1}\pi,2^{j-1}\pi+\frac{2^{j-1}}{2^{j+1}-1}\pi\right],$$

$l(\xi)\neq 0$ on the same set. Since l is 2π-periodic and $j>1$, $l(\xi)\neq 0$ on $[0,\pi/2]$. Hence

$$\hat{\psi}(4\xi)=\frac{m(\xi)}{l(\xi)}\hat{\psi}(2\xi)\quad\text{on }[0,\pi/2].\qquad(4.4)$$

But equality (4.4) cannot hold, since $\hat{\psi}(4\xi)\neq 0$ on

$$\left[\frac{2^{j-2}}{2^{j+1}-1}\pi, \frac{2^{j-1}}{2^{j+1}-1}\pi\right]$$

while $\hat{\psi}(2\xi)=0$ on the same set. This completes the proof. □

REMARK 4.5. For the scaling function φ given in (4.3), m_0 is given by

$$m_0(\xi)=\chi_{[-2\pi/3,\,-\pi/2]}(\xi)+\chi_{[-\pi/3,\,\pi/3]}(\xi)+\chi_{[\pi/2,\,2\pi/3]}(\xi). \tag{4.5}$$

Note that m_0 in (4.5) does not satisfy

$$\inf_{|\xi|\le\pi/2}|m_0(\xi)|>0. \tag{4.6}$$

In [Ma], Mallat introduced the condition (4.6) to construct a scaling function φ for an MRA from a 2π-periodic function m_0 by the formula

$$\hat{\varphi}(\xi)=\prod_{j\ge1} m_0(2^{-j}\xi). \tag{4.7}$$

III. The case when K^- is an interval and K^+ consists of two disjoint intervals

As before, it follows from Theorem 3.6 that

$$K^-=[-2c,-c] \quad\text{and}\quad K^+=[a,b]\cup[2^jb,2^{j+1}a]$$

for some a, b, c, and j satisfying

$$c>0,\quad 2a>b>a>0 \quad\text{and}\quad j\ge1. \tag{4.8}$$

Since $\tau_a(K)=[a,a+2\pi]$, there are positive integers p and q such that either

$$2^jb-2p\pi=b,\quad -c+2q\pi=a+2\pi,\quad\text{and}\quad 2^{j+1}a-2p\pi=-2c+2q\pi \tag{4.9}$$

or

$$-2c+2q\pi=b,\quad 2^{j+1}a-2p\pi=a+2\pi,\quad\text{and}\quad -c+2q\pi=2^jb-2p\pi. \tag{4.10}$$

From (4.9), we have

$$a=\frac{p-q+2}{2^j-1}\pi,\quad b=\frac{2p}{2^j-1}\pi,\quad\text{and}\quad c=2(q-1)\pi-2^{j+1}a.$$

It then follows from (4.8) that $1+2a/\pi<q<2$, so (4.9) cannot occur. From (4.10), we have

$$a=\frac{2(p+1)}{2^{j+1}-1}\pi,\quad b=\frac{2(2p+q)}{2^{j+1}-1}\pi,\quad\text{and}\quad c=\left(q-\frac{2p+q}{2^{j+1}-1}\right)\pi.$$

Since $b<2a$, $q<2$ and hence $q=1$. Using the other inequalities in (4.8), we conclude that

$$a=\frac{2(p+1)}{2^{j+1}-1}\pi,\quad b=\frac{2(2p+1)}{2^{j+1}-1}\pi,\quad\text{and}\quad c=\left(1-\frac{2p+1}{2^{j+1}-1}\right)\pi \tag{4.11}$$

for some $j\ge2$ and $1\le p\le 2^j-2$.

THEOREM 4.6. *Let* $\hat{\psi}(\xi)=\theta(\xi)\chi_K(\xi)$, *where*

$$K^-=\left[-2\left(1-\frac{2p+1}{2^{j+1}-1}\right)\pi, -\left(1-\frac{2p+1}{2^{j+1}-1}\right)\pi\right];$$
$$K^+=\left[\frac{2(p+1)}{2^{j+1}-1}\pi, \frac{2(2p+1)}{2^{j+1}-1}\pi\right]\cup\left[\frac{2^{j+1}(2p+1)}{2^{j+1}-1}\pi, \frac{2^{j+2}(p+1)}{2^{j+1}-1}\pi\right] \tag{4.12}$$

for $j\geq 2$ *and* $1\leq p\leq 2^j-2$ *and* $|\theta(\xi)|=1$. *Then* ψ *is a wavelet for* L^2. *Moreover, each unimodular wavelet for which* K^- *is an interval and* K^+ *consists of two disjoint intervals is of this form.*

If $j=2$ and $p=1$, then

$$K=[-\tfrac{8}{7}\pi, -\tfrac{4}{7}\pi]\cup[\tfrac{4}{7}\pi, \tfrac{6}{7}\pi]\cup[\tfrac{24}{7}\pi, \tfrac{32}{7}\pi].$$

The inverse Fourier transform of the characteristic function on this particular K is known to be a wavelet which is not associated with an MRA [Le].

THEOREM 4.7. *Let* $\hat{\psi}=\theta\chi_K$, *where* K *is given in* (4.12) *and* $|\theta|=1$. *If* p *is odd, then* ψ *is not associated with an MRA.*

Proof. Fix j and p in (4.12) and let a, b, and c be the numbers given in (4.11). Suppose that ψ is associated with an MRA with a scaling function φ. Then, as before,

$$|\hat{\varphi}(\xi)|^2=\sum_{j\geq 1}|\hat{\psi}(2^j\xi)|^2=\chi_{[-c,a]}(\xi)+\sum_{k=0}^{j-1}\chi_{[2^kb,2^{k+1}a]}(\xi),$$

so

$$|\hat{\varphi}(2\xi)|^2=\chi_{[-c/2,a/2]}(\xi)+\sum_{j=0}^{j-1}\chi_{[2^{k-1}b,2^ka]}(\xi).$$

Hence,

$$|m_0(\xi)|=\begin{cases}1 & \text{on } [-c/2,a/2]\cup(\bigcup_{k=0}^{j-1}[2^{k-1}b,2^ka]),\\ 0 & \text{on } [-c,-c/2]\cup[a/2,b/2]\cup[2^{j-1}b,2^ja].\end{cases} \tag{4.13}$$

We claim that this m_0 cannot be extended to be a 2π-periodic function. In order to prove this, we observe that

$$2^{j-1}b-(p+1)\pi=-c/2 \quad\text{and}\quad 2^ja-(p+1)\pi=a/2. \tag{4.14}$$

(4.14) implies that $[2^{j-1}b,2^ja]-(p+1)\pi=[-c/2,a/2]$, and hence m_0 cannot be extended as a 2π-periodic function, provided that p is odd. This completes the proof. □

REMARK 4.8. If p is even then the situation is totally different. For example, if $j=p=2$ then $a=\frac{6}{7}\pi$, $b=\frac{10}{7}\pi$, $c=\frac{2}{7}\pi$ in (4.11). Hence, by (4.13),

$$|m_0(\xi)|=\begin{cases}1 & \text{on } [-\tfrac{1}{7}\pi,\tfrac{3}{7}\pi]\cup[\tfrac{5}{7}\pi,\tfrac{6}{7}\pi]\cup[\tfrac{10}{7}\pi,\tfrac{12}{7}\pi],\\ 0 & \text{on } [-\tfrac{2}{7}\pi,-\tfrac{1}{7}\pi]\cup[\tfrac{3}{7}\pi,\tfrac{5}{7}\pi]\cup[\tfrac{20}{7}\pi,\tfrac{24}{7}\pi].\end{cases}$$

This m_0 can be extended as a 2π-periodic function. In this case, the wavelet given by $\hat{\psi}(\xi)=e^{i\xi/2}(\chi_{[-4\pi/7,\,-2\pi/7]}(\xi)+\chi_{[6\pi/7,\,10\pi/7]}(\xi)+\chi_{[40\pi/7,\,48\pi/7]}(\xi))$ is associated with the MRA whose scaling function is given by

$$\hat{\varphi}(\xi)=\chi_{[-2\pi/7,6\pi/7]}(\xi)+\chi_{[10\pi/7,12\pi/7]}(\xi)+\chi_{[20\pi/7,24\pi/7]}(\xi).$$

On the other hand, if $j=3$ and $p=2$, then $a=\frac{6}{15}\pi$, $b=\frac{10}{15}\pi$, and $c=\frac{10}{15}\pi$. Hence

$$|m_0(\xi)|=\begin{cases} 1 & \text{on } [-\frac{5}{15}\pi,\frac{3}{15}\pi]\cup[\frac{5}{15}\pi,\frac{6}{15}\pi]\cup[\frac{10}{15}\pi,\frac{12}{15}\pi]\cup[\frac{20}{15}\pi,\frac{24}{15}\pi], \\ 0 & \text{on } [-\frac{10}{15}\pi,-\frac{5}{15}\pi]\cup[\frac{3}{15}\pi,\frac{5}{15}\pi]\cup[\frac{40}{15}\pi,\frac{48}{15}\pi]. \end{cases}$$

Note that $[\frac{10}{15}\pi,\frac{12}{15}\pi]\subset[\frac{40}{15}\pi,\frac{48}{15}\pi]-2\pi$ and hence m_0 cannot be extended to be a 2π-periodic function.

5. Unimodular Wavelets for the Hardy Space H^2

We first recall that the classical Hardy space $H^2=H^2(\mathbf{R})$ is the collection of all functions f in L^2 whose Fourier transforms are supported in $[0,\infty)$. The only known example of a wavelet for H^2 is the inverse Fourier transform of the characteristic function of $[2\pi,4\pi]$ [BSW]. Moreover, P. Auscher recently proved that there is no continuously differentiable wavelet ψ for H^2 with the property $|\hat{\psi}(\xi)|+|\hat{\psi}'(\xi)|\le C|\xi|^{-\alpha}$ for $\xi\ge 1$ and $\alpha>\frac{1}{2}$. In this section, we construct a family of wavelets for H^2 by using the patchwork developed in previous sections.

Analogously to Theorem 3.6, we first have the following theorem regarding unimodular wavelets for H^2.

THEOREM 5.1. *Let ψ be a unimodular function in $L^2(\mathbf{R})$. Let $K=\operatorname{supp}\hat{\psi}$. Then ψ is a wavelet for H^2 if and only if the following hold:*

(1) $K\subset[0,\infty)$;
(2) *for each $a\in\mathbf{R}$, τ_a is one-to-one on K except on a set of measure zero and $|[a,a+2\pi)-\tau_a(K)|=0$; and*
(3) *for each $a>0$, δ_a is one-to-one on K except on a set of measure zero and $|[a,2a)-\delta_a(K)|=0$.*

Theorem 5.1 can be proved in the same way as Theorem 3.6.

Let us now construct unimodular wavelets for H^2 based on Theorem 5.1. As before, we first consider the case when $K=[a,b]$ with $a\ge 0$. Since $\tau_0(K)=[0,2\pi)$, we must have $a=2k\pi$ and $b=2(k+1)\pi$ for some integer k. Since $\delta_a(K)=[a,2a)$ and $[a,b)\subset[a,2a)$, we have $2(k+1)\pi=4k\pi$. Hence $k=1$, and so each unimodular wavelet with connected support is of the form

$$\hat{\psi}(\xi)=\theta(\xi)\chi_{[2\pi,4\pi]}(\xi), \tag{5.1}$$

where $|\theta(\xi)|=1$.

Let us now consider the case when $K=\operatorname{supp}\hat{\psi}$ consists of two disjoint intervals, say $[a,b]$ and $[c,d]$ with $b<c$. We first observe that $a\neq 0$ since

δ_1 cannot be one-to-one on $(0, b]$, whatever b is. We also observe that $|K| = |\tau_0(K)| = 2\pi$ by (2). Then, since $K \subset [a, \infty)$, $|\delta_a(K)| \leq |K|$; therefore, $a = |\delta_a(K)| \leq |K| = 2\pi$. Since $b < c$, $d > a + 2\pi \geq 2a$ and hence $|\delta_a([c, d])| < d - c$. So $|\delta_a(K)| < |K|$ and $a < 2\pi$. We claim that $c = b + 2k\pi$ and $d = a + 2(k+1)\pi$ for some integer k. To prove this claim, we first observe that the interval $[c, d)$ cannot contain a point of the form $a + 2l\pi$ with $l \in \mathbf{Z}$. In fact, if $[c, d)$ contains $a + 2l\pi$ for some l, then $[a, \epsilon) \subset \tau_a([c, d)) \cap [a, b)$ for some $\epsilon > 0$ and hence τ_a cannot be one-to-one on K. If $d \neq a + 2l\pi$ for any integer l, then $\tau_a(d) < a + 2\pi$ and hence $\tau_a(K)$ is properly contained in $[a, a + 2\pi)$. Therefore, $\tau_a(d) = a$ and $\tau_a(c) = b$. On the other hand, by a similar argument, $[c, d)$ cannot contain a point of the form $2^j a$ with $j \in \mathbf{Z}$. So we need to have $c = 2^j b$ and $d = 2^{j+1} a$ for some j. It follows that $b + 2k\pi = 2^j b$, $a + 2(k+1)\pi = 2^{j+1} a$, and $0 < a < 2\pi$. Hence we have

$$a = \frac{2(k+1)}{2^{j+1} - 1}\pi \quad \text{and} \quad b = \frac{2k}{2^j - 1}\pi \quad \text{for } k < 2(2^j - 1). \tag{5.2}$$

THEOREM 5.2. *Let* $\hat{\psi}(\xi) = \theta(\xi)\chi_K(\xi)$, *where*

$$K = \left[\frac{2(k+1)}{2^{j+1}-1}\pi, \frac{2k}{2^j - 1}\pi\right] \cup \left[\frac{2^{j+1}k}{2^j - 1}\pi, \frac{2^{j+2}(k+1)}{2^{j+1}-1}\pi\right] \tag{5.3}$$

for some integers $j > 0$ *and* $0 < k < 2(2^j - 1)$ *and* $|\theta(\xi)| = 1$. *Then* ψ *is a wavelet for* H^2. *Moreover, these are the only unimodular wavelets for* H^2 *such that the support of their Fourier transform is the union of two disjoint intervals.*

Proof. It is enough to prove the first statement. Fix $j \in \mathbf{N}$ and $0 < k < 2(2^j - 1)$. Let $a = 2(k+1)\pi/(2^{j+1} - 1)$ and $b = 2k\pi/(2^j - 1)$. We show that τ_a and δ_a satisfy (2) and (3) of Theorem 5.1.

Since $K = [a, b] \cup [2^j b, 2^j 2a]$, $\delta_a(K) = [a, 2a)$ and δ_a is one-to-one on K except at the endpoints. Since $K = [a, b] \cup [b + 2k\pi, a + 2(k+1)\pi]$, $\tau_a(K) = [a, a + 2\pi)$ and τ_a is one-to-one on K except at the endpoints. This completes the proof. ❑

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