A Fibered Polynomial Hull without an Analytic Selection

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This note shows that certain polynomial hulls in $\mathbf{C}^3$ have no analytic selection, thus settling a standing question about such hulls.

Recall that the *polynomial hull* $\mathcal{P}(\mathcal{S})$ of a set $\mathcal{S}$ in $\mathbf{C}^M$ is the set

$$\mathcal{P}(\mathcal{S}) = \{w \in \mathbf{C}^M : |p(w)| \le \max_{v \in \mathcal{S}} |p(v)| \text{ for all polynomials } p \text{ on } \mathbf{C}^M\}.$$

We shall be considering sets $\mathcal{S}$ in $\mathbf{C}^{1+N}$ that are fibered over the unit circle $\partial\mathbf{D}$ in $\mathbf{C}$ the complex plane. Thus $\mathcal{S}$ has the form

$$\{(e^{i\theta}, \mathcal{S}_\theta) : 0 \le \theta \le 2\pi\},$$

where each $\mathcal{S}_\theta$ is a subset of $\mathbf{C}^N$. Let $\mathbf{D}$ denote the unit disk in the complex plane and let H_N^∞ denote the $\mathbf{C}^N$-valued functions bounded and analytic on $\mathbf{D}$. Clearly (from the maximum principle) if f is any H_N^∞ function satisfying

$$f(e^{i\theta}) \in \mathcal{S}_\theta \text{ for almost all } \theta,$$

then the graph $\{z, f(z)) : z \in \mathbf{D}\}$ of f lies in $\mathcal{P}(\mathcal{S})$. Such a function f is called an *analytic selection* of $\mathcal{P}(\mathcal{S})$. A significant question about polynomial hulls is which hulls have analytic selections.

An obvious necessary condition is that $\mathcal{P}(\mathcal{S})$ in $\mathbf{C}^{1+N}$ be a set whose projection onto the first coordinate is a set containing $\mathbf{D}$. We shall refer to such $\mathcal{P}(\mathcal{S})$ as having *nontrivial fiber over the unit disk.* Also, if the $\mathcal{S}_\theta$ are not connected then it is easy to make up examples where $\mathcal{P}(\mathcal{S})$ has no analytic selection.

Question (Q). Are these conditions sufficient for $\mathcal{P}(\mathcal{S})$ to have an analytic selection?

By giving a highly pathological example (for $N=1$), Wermer [Wr] showed that in general the answer is No. However, when the $\mathcal{S}_\theta$ are nicely behaved the story is different. For $N=1$ Slodkowski [Sl] and independently Wegert [Wg] and Helton–Marshall [HM] showed that the answer is Yes. This article gives a simple very well-behaved $\mathcal{S}$ in $\mathbf{C}^3$ for which the answer to (Q) is No.

Now we write down the $\mathcal{S}$ that provides our example. Write $\mathbf{C}^2$ as $\{z = (x_1, y_1, x_2, y_2) = (z_1, z_2)\}$. Let $\mathcal{C}$ denote the semicircle

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$$\mathcal{C}=\{(z_1,z_2):|z_1|=1,\ \operatorname{Im} z_1\geq 0,\ z_2=0\}$$

embedded in the first complex coordinate of $\mathbf{C}^2$. Our example is based on rotating $\mathcal{C}$ inside of real 3-dimensional space in a way that varies with θ. Here $\mathbf{R}^3$ is embedded in $\mathbf{C}^2$ in the usual way ($y_2=0$). Let R_θ^1 denote the map which acts on $\mathbf{C}^2$ by the rotation of z in the z_1 plane by θ (z_2 is kept fixed), that is,

$$R_\theta^1(z_1,z_2)=(e^{i\theta}z_1,z_2).$$

Also, let R_θ^2 denote rotation in the (y_1,x_2) plane by θ.

THEOREM 1. *The polynomial hull $\mathcal{P}(\mathcal{S})$ of $\mathcal{S}$ with fibers*

$$\mathcal{S}_\theta=R_{\theta/2}^1(R_{\theta/2}^2(\mathcal{C}))$$

has nontrivial fibers over the disk, since $\mathcal{P}(\mathcal{S})$ contains the graph

$$\{(z,\pm z^{1/2},0):z\in\mathbf{D}\}$$

of $z^{1/2}$. Moreover, each $\mathcal{S}_\theta$ is connected, and the sets $\mathcal{S}_\theta$ vary continuously with θ. However, $\mathcal{P}(\mathcal{S})$ contains no analytic selection.

First we need a lemma. For $k=1,2$, let Proj_k be the map on $\mathbf{C}^3$ given by $\operatorname{Proj}_k(\zeta,z_1,z_2)=z_k$.

LEMMA. $\mathcal{P}(\mathcal{S})\subset\mathbf{D}\times\mathcal{P}(\operatorname{Proj}_1(\mathcal{S}))\times\mathcal{P}(\operatorname{Proj}_2(\mathcal{S}))$.

Proof. $\mathcal{S}\subset\partial\mathbf{D}\times\operatorname{Proj}_1(\mathcal{S})\times\operatorname{Proj}_2(\mathcal{S})$. Thus

$$\mathcal{P}(\mathcal{S})\subset\mathcal{P}(\partial\mathbf{D}\times\operatorname{Proj}_1(\mathcal{S})\times\operatorname{Proj}_2(\mathcal{S}))\subset\mathbf{D}\times\mathcal{P}(\operatorname{Proj}_1(\mathcal{S}))\times\mathcal{P}(\operatorname{Proj}_2(\mathcal{S})).\qquad\square$$

Proof of Theorem 1. Suppose that $f=(f^1,f^2)$ in H_2^∞ has its graph contained in $\mathcal{P}(\mathcal{S})$. By the lemma, the function f^2 is in $\mathcal{P}(\operatorname{Proj}_2(\mathcal{S}))\subset[-1,1]$ in $\mathbf{R}$. In particular $\operatorname{Im} f^2$ is 0; therefore f^2 is constant, but $f^2(1)=0$ since $f(1)\in\mathcal{C}$. We conclude that f^2 is 0.

The value of $f=(f^1,0)$ at θ is in $\mathcal{S}_\theta\cap\{(z_1,0)\}=\{(\pm e^{i\theta/2},0)\}$ except possibly for $\theta=0$ or 2π. That is, $f^1=\pm e^{i\theta/2}$ almost everywhere. No such function exists in H_1^∞, so we have a contradiction.

Now we check that the $\mathcal{S}_\theta$ are smoothly varying. If $0<\theta<2\pi$, then the maps R_θ^1 and R_θ^2 on $\mathbf{C}^2$ are jointly C^∞ in θ and (z_1,z_2). The only potential difficulty is at $\theta=0$ or $\theta=2\pi$. One can easily visualize $\mathcal{S}_\theta$ since it lies in $R^3\subset C^2$. The key is the set of points $\pm z_\theta=(\pm e^{i\theta/2},0)$ which are the image of $\pm i$ under the map $R_{\theta/2}^1R_{\theta/2}^2$ (and which consequently lie in $\mathcal{S}_\theta$). Observe that $\pm i=\pm z_0=\mp z_{2\pi}$, implying that as θ moves from 0 to 2π the points $(+1,0)$ and $(-1,0)$ rotate into each other. Thus $\mathcal{S}_{2\pi}$ is a rotation of $\mathcal{C}$ in the $(z_1,0)$ plane which has the same endpoints as $\mathcal{C}$; we have $\mathcal{S}_{2\pi}=\mathcal{C}$ or $\mathcal{S}_{2\pi}=-\mathcal{C}$. However,

$$R_{2\pi/2}^1(R_{2\pi/2}^2(1,0))=R_\pi^1((-1,0))=(1,0).$$

We have established that the $\mathcal{S}_\theta$ vary continuously in θ. $\square$

Lest one complain that our example hinges on degeneracy of $\mathcal{S}_\theta$, we now give an example where $\mathcal{S}$ is the closure of an open set in $\mathbf{C}^3$ in addition to maintaining other nice properties.

THEOREM 2. *For $n \in \mathbf{N}$, let $\mathcal{S}^n$ be the set in $\mathbf{C}^3$ with fibers $\mathcal{S}_\theta + (1/n)B$, where B is the unit ball of $\mathbf{C}^2$. Then there exists $n_0 \in \mathbf{N}$ such that the polynomial hull of $\mathcal{S}^{n_0}$ has no analytic selection.*

Proof. Suppose not. Then for each n there exists f_n in H_2^∞ whose graph lies in $\mathcal{S}^n$. Let f_0 be a normal families limit of $\{f_n\}$. By polynomial convexity of each $\mathcal{S}_\theta$ and by Corollary 2 in [HM], the function f_0 has graph lying in $\mathcal{P}(\mathcal{S})$. Thus $\mathcal{P}(\mathcal{S})$ has a selection, contrary to Theorem 1. $\square$

References

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