Automorphisms of Negatively Curved Polygonal Amalgams

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1. Introduction

In [13] it was shown that the free product with amalgamation of two "nice" (e.g. finite) groups has a very rigidly controlled automorphism group. The argument is based on the action of the amalgamated product on its associated tree. It is natural to ask if similar results hold for groups acting on higher-dimensional objects.

Polygonal amalgams of groups are the 2-dimensional analogues of free products with amalgamation. By studying the actions of negatively curved polygonal amalgams on their corresponding 1-connected 2-complexes, we are able to describe the automorphism group of the amalgam group. In particular, we show that the automorphism group of a negatively curved polygonal amalgam of finite groups is virtually a negatively curved polygonal amalgam of finite groups. It follows from our study of automorphisms that the automorphism group of a Coxeter group acting on the hyperbolic plane with compact fundamental domain has outer automorphism group a finite dihedral group (possibly trivial), and the full automorphism group is itself a Coxeter group.

We extend this analysis of automorphisms of negatively curved polygonal amalgams to the more general case of injective endomorphisms, and show that negatively curved polygonal amalgams of finite groups are co-Hopfian. Because polygonal amalgams where the edge groups generate the group are 1-ended, this result supports Gromov's statement in [11], "Γ *is not isomorphic to any of its proper subgroups.* Probably, the same is true for every word hyperbolic group Γ connected at infinity." (The italics are Gromov's.) We believe Gromov's statement is still an open question for arbitrary 1-ended word hyperbolic groups.

Most of the definitions can be found in Sections 2, 3, and 7. Section 4 describes triangles of groups, and the more general notion of polygons of groups, in more detail. Sections 5 and 6 establish geometric facts about some piecewise hyperbolic 2-complexes, analogous to standard results in the geometry of the hyperbolic plane. These are then applied in the final section to

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study automorphisms of groups acting on these complexes. The main theorems are in Section 8.

We note that similar results about automorphisms have been achieved for polygonal amalgams acting on Euclidean Tits's buildings [16].

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2. Basic Definitions

A *geodesic* in a metric space (X, d) is an isometry $f: I \to X$, where I is a closed subinterval of R. A geodesic will often be identified with its image. A geodesic is a *geodesic segment* if it is the image of a bounded interval. In this case, let $i(f)$ denote the initial point and $\tau(f)$ the terminal point of the geodesic.

A metric space is *geodesic* if, for each pair of points x and y in X, there exists a geodesic segment f with $i(f) = x$ and $\tau(f) = y$. If this geodesic is unique for each pair of points in X then X is a *unique geodesic space.*

A *geodesic polygon* in a metric space (X, d) consists of n distinct points $\{p_1, p_2, \ldots, p_n\}$ (the *corners*) and geodesic segments $\{e_1, e_2, \ldots, e_n\}$ (the *sides*), with $i(e_j) = p_j$ and $\tau(e_j) = p_{j+1}$ (indices taken modulo n). The polygon is *simple* if $K = \bigcup\{e_i\}$ forms a simple closed curve.

Following Gromov [11], a geodesic metric space is said to be *δ-hyperbolic* if there exists a $\delta \geq 0$ such that for all geodesic triangles in X,

$$\sup_{t \in A} d(t, B \cup C) \leq 2\delta,$$

where A, B, and C are the sides of the triangle. For example, the hyperbolic plane is δ-hyperbolic with best possible constant $\log(1+\sqrt{2})$ [18].

A finitely generated group Γ with finite generating set S is *word hyperbolic* if its Cayley graph, viewed as a metric space (each edge being isometric to the unit interval), is δ-hyperbolic. This is independent of the choice of a finite generating set. The reader not familiar with Gromov's notion of hyperbolicity is advised to read Ghys's survey article [9] or the more detailed expositions [10] or [7].

In addition to Gromov's δ-hyperbolicity, there is another notion of negative curvature in an arbitrary geodesic metric space which compares triangles in the metric space and triangles in the hyperbolic plane. We make frequent reference to the hyperbolic plane and denote it by H.

Let (X, d) be a geodesic metric space and let T be a geodesic triangle on the three points $\{p_1, p_2, p_3\}$. A geodesic triangle T^* in H is a *comparison triangle* for T if the side lengths of T and T^* are the same. Let $\cdot^*$ denote the map taking the points of T to their associated points in T^*.

A geodesic triangle T contained in X is said to satisfy the *CAT(−1) inequality* if, for every pair of points p and q of T, $d_X(p, q) \leq d_H(p^*, q^*)$. The metric space X is said to satisfy the CAT(−1) inequality if every geodesic triangle in X satisfies CAT(−1).

A CAT(−1) metric space is δ-hyperbolic, but the converse is false. For details on the CAT inequality, see [18] or W. Ballman's article in [10].

3. Negatively Curved Polyhedra

DEFINITIONS. The *standard n-gon* ($n \geq 3$) S_n is the convex hull of the points $e^{k2\pi i/n}$, $k \in \{1, 2, 3, \ldots\}$, in E. These points are the *vertices*; the lines joining adjacent vertices are *edges*. A *face* is a vertex, an edge, or the entire convex hull. The term *proper face* means either a vertex or an edge, but not the entire convex hull.

An *n-polyhedron* is a topological space P together with an indexed set of ordered pairs consisting of subsets Δ_α of P and maps Φ_α with the following properties.

(1) $P = \bigcup \Delta_\alpha$.
(2) $\Phi_\alpha : \Delta_\alpha \to S_n$ is a homeomorphism.
(3) If $\Delta_\alpha \cap \Delta_\beta \neq \emptyset$ then $\Phi_\alpha(\Delta_\alpha \cap \Delta_\beta)$ is the union of proper faces of S_n. Furthermore, $\Phi_\alpha^{-1} \circ \Phi_\beta |_{\Delta_\alpha \cap \Delta_\beta} = \mathrm{Id}$.
(4) The topology of P is the fine topology; that is, a set is open if and only if its intersection with each cell is open.

The Δ_α are called *chambers*; the pre-images of vertices and edges of the standard n-gon are *vertices* and *edges*, respectively. Arbitrary points of an n-polyhedron will be called *points*, the term "vertices" being reserved for actual vertices in the above sense.

EXAMPLES.

(1) Each labelled simplicial 2-complex is a 3-polyhedron. (For a definition of a *labelling* see [5].)
(2) The Euclidean plane tiled by regular hexagons is a 6-polyhedron.
(3) S^2 can be given the structure of an n-polyhedron for $n \geq 3$ by taking two copies of S_n and identifying corresponding points and edges.

A *piecewise hyperbolic n-polyhedron*, or simply a PH-polyhedron, is an n-polyhedron P such that every connected component has a complete metric d inducing the topology of P, provided that:

(1) every component is geodesic;
(2) for each chamber Δ_α there is an isometry from Δ_α to some nondegenerate n-gon (with its interior) in H mapping edges to edges and vertices to vertices; and
(3) the number of isometry classes of chambers is finite.

DEFINITION. Given any point p in a PH-polyhedron P, there is a natural *space of directions* D_p consisting of the geodesics f with $i(f) = p$, modulo the relation that $f \simeq g$ if $f(t) = g(t)$ on some interval $[0, \epsilon)$ for ϵ greater than zero. The space of directions inherits a natural piecewise spherical metric induced by the piecewise hyperbolic metric on P.

THE LARGE LINK CONDITION. A PH-polyhedron P satisfies the *large link condition* if, for each vertex v in P, the minimal length of a loop in D_v is greater than or equal to 2π.

The following theorem will be used to establish the negative curvature of complexes we will be using in later sections.

THEOREM 3.1 (W. Ballmann, M. Bridson, M. Gromov). *If P is a simply connected PH-polyhedron satisfying the large link condition, then P satisfies* CAT(-1) *and is a unique geodesic space.*

The proof of this theorem can be found in [2].

4. Negatively Curved Polygonal Amalgams

The geometric results of the next section will be applied to negatively curved polygonal amalgams of groups, which are examples of Haefliger's more general notion of negatively curved complexes of groups [12].

DEFINITION. A *polygon of groups* is a contravariant functor from the poset of the faces of a polygon, ordered by inclusion, to the category of groups and monomorphisms. We make two additional assumptions about our polygons of groups.

(i) A polygon of groups is *filled* if the intersection of two edge groups in their associated vertex group is the group of the 2-cell. Our polygons will always be filled.
(ii) Our polygons will also always be *proper*, in the sense that the functor takes proper inclusions to proper monomorphisms.

DEFINITION. The direct limit of such a diagram of groups is a *polygonal amalgam*.

Polygons of groups naturally arise from group actions on n-polyhedra. It is well known that if a group acts on a 1-connected complex with compact fundamental domain mapping homeomorphically to the quotient, then the group is generated by the stabilizers of the faces of the fundamental domain (see [4]). Thus, if a group acts on a 1-connected n-polyhedron with a chamber mapping homeomorphically to the quotient, it can be presented as the polygonal amalgam of the stabilizers of the faces of a chamber. When a polygon of groups arises from such an action on a 1-connected n-polyhedron, the polygon of groups is said to be *developable*.

Triangles of groups can be a rather pathological method of presenting groups. For instance, it is not true in general that the vertex groups inject into the triangular amalgam or that the torsion is contained in the conjugates of the vertex groups. It is even possible for a triangle of finite groups to have a trivial amalgam. For examples of such behavior see [6] or [22].

Such pathological presentations can be avoided by introducing the Gersten–Stallings angles at a vertex. To clarify the discussion we index the edge groups and vertex groups of the polygon of groups by $1, \ldots, n$, with the edge groups E_{i-1} and E_i injecting into the vertex group V_i (indices taken modulo n). The group of the 2-cell we denote C.

DEFINITION. Into each vertex group V_i there is a map, $E_{i-1} \star_C E_i \to V_i$. Let $2n$ be the minimal free product length of an element in the kernel of this map. Then the *Gersten–Stallings angle* at V_i is defined to be $\theta_i = \pi / n$.

DEFINITION. If the sum of the Gersten–Stallings angles of an n-gon of groups is less than $(n-2)\pi$, then the polygon is *negatively curved* and its direct limit is a *negatively curved polygonal amalgam*.

For more details and intuition behind this definition, see Stallings's paper [21], which contains the proof of the following theorem in the case of triangles of groups. It is easy to check that the results also hold for polygons of groups.

THEOREM 4.1 (S. Gersten, J. Stallings). *If $\mathcal{P}$ is a negatively curved polygon of groups then $\mathcal{P}$ is developable. The associated polyhedron P_Γ admits a PH-structure making it a* CAT(-1) *metric space.*

The PH-structure of P_Γ is quite natural. Assign to a chamber of P_Γ the metric structure of a polygon in H whose vertex angles are the same as the Gersten–Stallings angles at each corner of the original polygon of groups, and use the action of the amalgam group on P_Γ to extend this metric equivariantly to the other chambers. It follows from the definition of the Gersten–Stallings angle that the complex satisfies the large link condition, so by Theorem 3.1 the complex P_Γ is CAT(-1).

COROLLARY 4.2. *If Γ is the direct limit of an n-gon of groups, $n > 4$, then the associated complex P_Γ admits a PH-structure. Furthermore, if the vertex groups are finite, Γ is word hyperbolic.*

Proof. Because the polygon of groups is proper and filled, the Gersten–Stallings angles at each vertex are at most $\pi/2$; hence, if there are more than four sides, the polygon of groups is negatively curved. The second sentence follows because Γ acts discretely and co-compactly on a 1-connected CAT(-1) space. □

The corollary below can be proven using the fact that finite groups acting on "nonpositively curved spaces" must have fixed points (see [5]). The corollary follows because the isotropy groups of points in P_Γ are contained in the isotropy groups of the vertices of P_Γ.

COROLLARY 4.3. *If G is a finite subgroup of a negatively curved polygonal amalgam Γ, then G must fix a vertex of the associated PH-polyhedron P_Γ.*

Some examples in Section 7 will exploit the fact (shown below) that, for $n>3$, polygonal amalgams admit nontrivial actions on simplicial trees.

PROPOSITION 4.4. *For $n>3$, polygonal amalgams can be represented as amalgamated free products.*

Proof. We can think of "chopping off" two vertex groups, V_1 and V_2, and forming their amalgamated product $G_1 = V_1 \star_{E_1} V_2$ and the amalgamated product of the remaining vertex groups, $G_2 = V_3 \star_{E_3} \cdots \star_{E_{n-1}} V_n$. Because we have assumed that our polygons of groups are filled, there are no "hidden" intersections of the face groups which are not explicitly presented in the original polygon of groups. Hence the intersection of G_1 and G_2 is the group generated by E_2 and E_n. Once again, because the polygon is filled and proper, this group is simply $E_2 \star_C E_n$. This gives a presentation of the polygonal amalgam as the free product of G_1 and G_2 along $E_2 \star_C E_n$. □

A triangle of groups cannot be presented in this manner, because the group generated by two edge groups is not always the free product of the edge groups along the group of the 2-cell. For instance, if the vertex groups of a triangle of groups are all finite, then the group generated by two edge groups must also be finite.

The idea of polygonal amalgamation is not new. There are many Coxeter groups which act on H with compact polygonal fundamental domain. Residual properties of some polygonal amalgams have been studied in [1] and [14], and some Bianchi groups have been given decompositions as direct limits of triangles and rectangles of groups (see [8] or [19]).

5. Mapping Disks

P will always denote a 1-connected PH-polyhedron for the remainder of this paper.

Intuitively, one thinks of simple polygons in PH-polyhedra as bounding disks. However, it is easy to construct examples where this is *not* the case (see [15]). Even with this sort of difficulty there is an intuitive notion of the "interior" of a simple geodesic polygon, which we describe below.

DEFINITION. Given a simple geodesic polygon K in $P^{(1)}$, the 1-skeleton of P, a *solid n-gon* $\bar{K}$ is the union of closed chambers occurring in a 2-chain with K as (homological) boundary cycle. If K were an arbitrary simple polygon in P, not necessarily in the 1-skeleton, then $\bar{K}$ could also be defined by subdividing the cells of P by finitely many geodesics and embedding K into the 1-skeleton of the subdivided complex. Two such geodesic subdivisions have a common refinement, and it is easily checked that a set $\bar{K}$ which is a solid n-gon is still a solid n-gon after a finite refinement.

PROPOSITION 5.1. *Let K be a simple polygon. Then $\bar{K}$ exists and is unique.*

Proof. We assume that P has been subdivided so that K is a simple closed curve in the 1-skeleton. Thus, the set of edges composing K forms a (homological) 1-cycle. P is contractible; hence it has the homology of a point, so there is a 2-chain with K as boundary. Because P is a 2-complex there are no 3-chains, so $\bar{K}$ must be unique. □

When the original polygon K is a triangle, $\bar{K}$ is a topological disk with a fairly simple description.

PROPOSITION 5.2. *Let p be a point contained in a simple geodesic triangle K, and let e be the union of corners and sides of K whose closure does not contain p. If T is the union of geodesics from p to e, then T is $\bar{K}$.*

Proof. To prove the proposition we will first construct a map from a simplex to P. To make the argument more clear, assume that p is a corner of K so that e is the closure of its opposite side. By Lemma 2 of [3], T is the continuous image of a 2-simplex in the Euclidean plane, Δ^2, obtained by mapping geodesics on the simplex to the corresponding geodesics in P. That is, let p' be a vertex of a fixed Euclidean 2-simplex Δ^2 and let e' be the side of Δ^2 opposite p'. Define f to be a constant speed map from e' to e. The map F from Δ^2 to T is then defined by sending the geodesic between p' and a point $x \in e'$ in Δ^2 to the geodesic between p and $f(x)$ in P.

This map might not be one-to-one, since it is possible for two geodesics in P to agree for some initial segment and then diverge. Thus the pre-image of a point in T may not be a single point in Δ^2. However, once two geodesic segments based at a common point p begin to diverge, they cannot intersect later by the CAT inequality. Further, since K is a simple geodesic triangle, no two points on e are on a common geodesic from p. Thus the pre-image of a point in T will be a connected, piecewise smooth arc in Δ^2. Since K is a simple geodesic triangle, the geodesic segments forming the sides adjacent to p intersect only at p. The pre-image of each point in T cannot intersect both sides of Δ^2, so the pre-images of points in T give a nice cell-like decomposition of Δ^2. By a famous theorem of Moore [17], it follows that T is a disk. □

Because arguments are easier when $\bar{K}$ is a topological disk, we define the associated notion of a *mapping disk*.

DEFINITIONS. A *combinatorial map* is a cellular map between cell complexes that is a homeomorphism on the closed cells. (Note that this is stronger than the usual definition of combinatorial map.) Given a combinatorial map from some n-polyhedron S to a PH-polyhedron P, the chambers of S can be given the metric structure of their images in P. Such a combinatorial map is *locally large* if S with its induced metric structure satisfies the large link condition.

A *mapping disk* for a simple closed polygon K is a combinatorial map ϕ of D^2 into P such that:

(1) the image of D^2 contains $\bar{K}$;
(2) ϕ maps S^1 homeomorphically to K; and
(3) ϕ is locally large.

Since the mapping disk is locally large, the disk with its PH-structure is a PH-polyhedron.

PROPOSITION 5.3. *Let K be a simple polygon in P. Then K has a mapping disk.*

Proof. Proposition 5.2 shows that geodesic triangles have mapping disks, so we will piece together mapping disks of triangles to form a mapping disk for a simple polygon. As a first step, subdivide P so that K is contained in $P^{(1)}$.

Let S^1 map to the polygon K. Add vertices to S^1 so that the map takes vertices of S^1 to the corners of K. Add additional edges to S^1 from a chosen vertex v to the other vertices of S^1 not adjacent to v. Call this graph $\hat{S}^1$. Extend the map from S^1 to K to a map from $\hat{S}^1$ into P by mapping the new edges of $\hat{S}^1$ to the geodesics joining the corresponding corners of K. Let $\hat{K}$ denote this union of geodesics. After a finite subdivision, we may assume that $\hat{K}$ is contained in $P^{(1)}$.

The graph $\hat{S}^1$ is composed of loops of combinatorial length 3 all mapping to geodesic triangles in $P^{(1)}$. By filling in each one of these loops in $\hat{S}^1$ by a disk, we create a triangulated disk $\hat{D}^2$. Since geodesic triangles in P have mapping disks (by Proposition 5.2), there are mapping disks onto the solid triangles for each of the geodesic triangles composing $\hat{K}$. It follows that each simplex of $\hat{D}^2$ can be given a mapping disk structure.

Because the geodesic triangles of $\hat{K}$ have mapping disks, each of the simplices in $\hat{D}^2$ can be given a cellular structure along with a map onto the solid triangles in P. In a slight abuse of notation, we once again use $\hat{D}^2$ to denote $\hat{D}^2$ with the cellular structure it inherits from the mapping disk structure on each of the original simplices of $\hat{D}^2$. Because mapping disks are homeomorphisms on their boundaries, these maps are consistent along their borders and give a combinatorial map from $\hat{D}^2$ to P which is a homeomorphism on its boundary. The complex $\hat{D}^2$ maps onto $\bar{K}$ because the homological boundary of the set of 2-cells in $\hat{D}^2$ is the 1-cycle of edges forming S^1.

Mapping disks are locally large, so the map from $\hat{D}^2$ to P is locally large away from $\hat{S}^1$. To finish the proof, it remains only to establish that the map from $\hat{D}^2$ to P is locally large at the vertices mapping to the geodesics in $\hat{K}\setminus K$. The angle between two halves of a geodesic containing a vertex in a PH-polyhedron is greater than or equal to π [2]. By the description of the solid triangles given in Proposition 5.2, it follows that the angle sum about any vertex mapping to $\hat{K}\setminus K$ must be greater than or equal to 2π. Thus this map is locally large at all the vertices of $\hat{D}^2$. □

6. Angles

In this section we state two combinatorial Gauss–Bonnet theorems similar to the Gersten weight tests in [21]. They combinatorially describe the geometric property that as polygons in hyperbolic spaces get bigger, their interior angles become smaller.

For both propositions the proof proceeds by first finding a mapping disk for the polygon K contained in the 1-skeleton of a PH-polyhedron. This gives a PH-disk, and the results follow using standard small cancellation arguments based on the fact that the Euler number of a sphere is 2.

PROPOSITION 6.1. *Let P be a PH-polyhedron where the chambers are all isometric to the same n-gon in H. Assume that the corner angles of each chamber sum to A. Let K be a simple geodesic n-gon in the 1-skeleton of P whose corner angles sum to greater than or equal to A. Then K bounds a single chamber.*

PROPOSITION 6.2. *Let P be a PH-polyhedron where the chambers are all isometric to the same n-gon in H and each vertex is in at least four chambers. Let K be a simple geodesic m-gon in $P^{(1)}$. Then m is greater than or equal to n.*

If Γ is a negatively curved polygonal amalgam and P_Γ its associated PH-polyhedron, then the condition in Proposition 6.2 that every vertex be in at least four chambers is always satisfied because the polygon of groups is filled and proper.

7. Characteristic Actions

Because we study automorphisms by their actions on stabilizers, we need some condition relating stabilizers to automorphisms. This is the notion of a "characteristically bounded" action. Proposition 7.1 will be used in studying the automorphisms of polygonal amalgams, since an automorphism is determined by the image of the generators, and the face groups of a polygon of groups generate the polygonal amalgam.

DEFINITIONS. A set of subgroups of a given group Γ is *characteristic* if each automorphism induces a permutation of this set of subgroups. The action of a group Γ on a negatively curved polyhedron P is *characteristically bounded* if the set of subgroups stabilizing n-cells is characteristic, for every n. That is, an automorphism takes the isotropy groups of n-cells to the isotropy groups of n-cells.

PROPOSITION 7.1. *Let Γ be a negatively curved polygon of groups with characteristically bounded action on P_Γ. Then any automorphism ϕ takes the stabilizers of the faces of the chamber to the stabilizers of the faces of a chamber.*

Proof. The map ϕ sends vertex stabilizers to vertex stabilizers, so it induces a bijective map $\tilde{\phi}$ on the vertices, where $\tilde{\phi}: x \mapsto \mathrm{Fix}(\phi(\Gamma_x))$. Let $\{v_i\}$ be the vertices of a chamber Δ. The images of these vertices, $\tilde{\phi}(v_i)$, can then be connected by geodesics from $\tilde{\phi}(v_i)$ to $\tilde{\phi}(v_{i+1})$ for all i, giving some new polygon $\tilde{\Delta}$. Conceivably this might not be simple, and quite possibly it could be larger than the original chamber.

Since edge stabilizers are mapped to edge stabilizers, the geodesic sides of $\tilde{\Delta}$ run in the 1-skeleton of P. No two geodesics in $\tilde{\Delta}$ can intersect in more than one point. If they did, they would have to share some common edge. But then either ϕ was not an automorphism, or an isotropy group of an edge is mapped to a proper subgroup of an isotropy group of an edge, which is impossible by hypothesis. Thus $\tilde{\Delta}$ cannot be a tree, and it must contain some geodesic m-gon for $m \leq n$. This m-gon is contained in $P^{(1)}$ which, by Proposition 6.2, is impossible unless $m = n$.

Let v be the vertex where two edges of $\tilde{\Delta}$ meet. The geometric angle between these two edges cannot be smaller than the Gersten–Stallings angle associated to the original pair of edge groups. The Gersten–Stallings angle being equal to π/n implies there is an alternating sequence of $2n$ elements coming from the edge groups which is trivial in the group generated by the edge groups. If the geometric angle is smaller than π/n, then in the space of directions there would be a circuit of length less than 2π, contradicting the large link condition.

It follows that $\tilde{\Delta}$ contains an n-gon with geometric angles at the corners greater than or equal to the geometric angles at the corners of a chamber. By Proposition 6.1, this n-gon must be the boundary of a chamber. By the argument at the start of the proof, it is impossible for any of the geodesic edges to overlap in more than a single point, so $\tilde{\Delta}$ is simply the boundary of a chamber. □

Having a characteristically bounded action is not uncommon, as the next two propositions demonstrate. However, it is not always the case that a negatively curved polygonal amalgam has a characteristically bounded action on its associated polyhedron. For instance, the free group of rank 3 can be presented as the limit of a negatively curved triangle of groups. Let a, b, and c be generators for F_3 and let the vertex groups be $F(a, b)$, $F(b, c)$, and $F(c, a)$, with edge groups the infinite cyclic groups $\langle a \rangle$, $\langle b \rangle$, and $\langle c \rangle$.

PROPOSITION 7.2. *The action of a negatively curved polygonal amalgam with finite vertex groups on its associated PH-polyhedron P_Γ is characteristically bounded.*

Proof. It is immediate that the set of maximal finite subgroups of Γ is characteristic, and it follows from the fact that finite groups fix vertices (Corollary 4.3) that the set of stabilizers of vertices is the set of maximal finite subgroups. Thus the set of vertex stabilizers is characteristic.

Let ϕ be an automorphism, and let E be the isotropy group of the edge between two vertices fixed by the groups V_1 and V_2. Then $\phi(E)$ must fix the geodesic between the vertices fixed by $\phi(V_1)$ and $\phi(V_2)$. Because the edge groups are of larger order than the isotropy groups of the chambers, this geodesic must be contained in $P_\Gamma^{(1)}$. As before, call the n-gon given by these geodesics $\tilde{\Delta}$.

The polygon $\tilde{\Delta}$ cannot be a tree. To see this we examine two cases. First, if the original polygon were a triangle, this would imply that the edges of $\tilde{\Delta}$

all shared a common point. But then the group generated by the edge groups, which is an infinite group, would fix a point, contradicting our assumption of finite stabilizers. Second, if there are more than three sides, two edge groups of edges which were not adjacent in the original n-gon of groups would be mapped into the same edge stabilizer. But because we have assumed that the polygon of groups is filled and proper, the group generated by nonadjacent edge groups is infinite, so the stabilizer of the edge fixed by their images would be infinite. However, stabilizers of edges are all finite.

Since $\tilde{\Delta}$ cannot be a tree, it must contain some m-gon for $m \leq n$. The remainder of the argument is the same as in Proposition 7.1. □

The arguments above really only used the fact that automorphisms are injective. The following result can be proven by using essentially the same arguments.

COROLLARY 7.3. *Let ϕ be an injective endomorphism of a negatively curved polygonal amalgam of finite groups. Then ϕ takes the stabilizers of the faces of a chamber to the stabilizers of the faces of a chamber.*

It is not true that the only groups admitting characteristically bounded actions are the groups with finite stabilizers. For instance, we have the following set of examples.

PROPOSITION 7.4. *Let Γ be the direct limit of a polygon of groups with more than three sides whose vertex groups are copies of $SL_3(Z)$ and with finite edge groups. The action of Γ on its space P_Γ is characteristically bounded.*

Proof. By Proposition 4.4, Γ can be presented as a free product with amalgamation; hence it will admit a nontrivial action on a simplicial tree. We will be studying Γ both as a polygonal amalgam and as a free product with amalgamation. To avoid confusion with the terminology (i.e., is it a stabilizer of a vertex in the tree associated to the decomposition of Γ as an amalgamated free product, or the stabilizer of a vertex in the PH-polyhedron), the terms "vertex", "edge", etc. will always refer to the original polyhedron. When the tree decomposition is being considered there will be "tree-vertices" and "tree-edges".

Let G be any copy of $SL_3(Z)$ in Γ, not necessarily a vertex group. In [20] it is shown that G cannot act on a tree without a fixed point, and hence G must be contained in a tree-vertex stabilizer. The tree-vertex stabilizers are the fundamental groups of a tree of groups, so once again, since G cannot act nontrivially on a tree, it must be contained in one of the stabilizers of this tree of groups. But these are just the original vertex groups of the polygon.

Just as in the previous propositions, this allows us to define a new polygon $\tilde{\Delta}$. Borrowing the argument from Proposition 7.1, and using the fact that the edge stabilizers are finite, it can be shown that $\tilde{\Delta}$ is the boundary of a chamber. From here it is easy to show that the isotropy groups are characteristic. □

FURTHER EXAMPLES. Essentially the same proof works for polygons of groups whose vertex stabilizers cannot act on a simplicial tree without a fixed point and whose edge groups are finite. For instance, if the vertex groups of an n-gon of groups ($n \geq 4$) were copies of $G(A)$, where G is a simple Chevalley group of rank greater than or equal to 2 and A is the ring of integers of an algebraic number field, then the action of the limit group Γ on P_Γ would be characteristically bounded. See [20], and the references cited there, for proofs that these groups cannot act on trees without fixing a point.

Another example is given by groups studied in [1], where the vertex groups are finitely generated free abelian groups and the polygon of groups has trivial 2-cell group. By a more careful Kurosh subgroup argument than was used in our Proposition 7.4, it can be shown that these polygonal amalgams (with five or more sides) have characteristically bounded actions on their associated polyhedron. For details, see [15].

8. Automorphisms and Injective Endomorphisms

Using Theorem 7.1, it is now easy to establish the main theorems. We denote the full automorphism group of a group Γ by $\mathrm{Aut}(\Gamma)$ and its outer automorphism group by $\mathrm{Out}(\Gamma)$.

THEOREM 8.1. *If Γ is a negatively curved polygonal amalgam whose action is characteristically bounded, then $\mathrm{Aut}(\Gamma)$ is virtually a negatively curved polygonal amalgam.*

Proof. Let $\mathrm{Con}(\Gamma)$ be the subgroup of elements of the automorphism group which preserve conjugacy classes of vertex groups. Notice that $\mathrm{Con}(\Gamma)$ contains a copy of Γ as the inner automorphisms. $\mathrm{Con}(\Gamma)$ then acts on P_Γ with the same fundamental domain and a polygon as quotient; hence it is a negatively curved polygon of groups. $\mathrm{Con}(\Gamma)$ is normal and gives the exact sequence

$$1 \to \mathrm{Con}(\Gamma) \to \mathrm{Aut}(\Gamma) \to D \to 1.$$

Since Γ acts transitively on chambers, $\mathrm{Con}(\Gamma)$ does too. It follows that D acts faithfully on a copy of the fundamental chamber and hence is a subgroup of a finite dihedral group. If the vertex stabilizers are finite then it is easy to see that this short exact sequence splits, hence $\mathrm{Aut}(\Gamma)$ is $D \ltimes \mathrm{Con}(\Gamma)$. □

COROLLARY 8.2. *If Γ is a negatively curved polygonal amalgam with finite vertex groups, then the outer automorphism group $\mathrm{Out}(\Gamma)$ is finite.*

Proof. The group Γ is generated by the isotropy groups of a chamber's vertices. By Theorem 8.1, modulo an inner automorphism, any automorphism will map the vertex stabilizers of a given chamber back to the vertex stabilizers of that chamber. Thus the vertex groups can only be mapped amongst themselves, and since they are finite this gives only finitely many possibilities. □

The following corollary is an easy example of the previous statements. It is also true for Coxeter groups acting on Euclidean space with compact fundamental domain [15].

COROLLARY 8.3. *Let* Γ *be a Coxeter group acting on* H *with a compact polygon* K *for fundamental domain. That is,* Γ *has a presentation* $\langle r_1, r_2, \ldots, r_n \mid r_i^2 = 1, (r_i r_{i+1})^{m_i} = 1 \rangle$, *where the indices are taken modulo* n, $2 \le m_i < \infty$, *and* $\sum_{i \in \{1, \ldots, n\}} (1/m_i) < (n-2)$. *Then* $\mathrm{Aut}(\Gamma)$ *is isomorphic to a Coxeter group acting on* H *with a compact polygon for fundamental domain.*

PROPOSITION 8.4. *Negatively curved polygonal amalgams of finite groups are co-Hopfian.*

Proof. Recall that a group is co-Hopfian if any injective endomorphism is an automorphism. Let ϕ be an injective endomorphism. By Proposition 7.3, ϕ will send the stabilizers of the faces of a chamber to the stabilizers of the faces of a chamber. Since these are finite groups and ϕ is injective, it follows that ϕ restricted to the stabilizers is an isomorphism between the stabilizers. Because the stabilizers of the faces of any chamber generate the amalgam group, an inverse to ϕ can be constructed. □

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