Noncollarable Ends of 4-Manifolds: Some Realization Theorems

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A fundamental result in manifold theory is Siebenmann's classification of collarable ends of noncompact n-manifolds, $n \geq 6$ (see [Si]). Quinn [Qu] has extended this result to dimension 5 provided the fundamental group at infinity is a Freedman group. Work by Husch and Price [HP] establishes Siebenmann's theorem for 3-manifolds, provided the Poincaré conjecture is true. Remarkably, Siebenmann's theorem fails in dimension 4. Counterexamples are produced by Kwasik and Schultz in [KS]. These examples arise as quotient spaces of certain free G-actions on $S^3 \times \mathbf{R}$ where G is a finite group of even order. In this note we show that in many cases these exotic ends may be realized rather naturally as subsets of closed 4-manifolds. In particular, we show that if E is a 4-dimensional weak collar with $\pi_1(E) \cong \mathbf{Z}_n$ and ∂E is $\mathbf{Z}$-homology equivalent to $L(n,1)$, then there is a closed 4-manifold Y and a compactum $\Sigma \subset Y$ such that Σ has the shape of a 2-sphere and Σ has a neighborhood N with $N - \Sigma$ homeomorphic to E. Moreover, we may choose Y to be $S^2 \times S^2$ if n is even, and $\mathbf{C}P^2 \# (-\mathbf{C}P^2)$ if n is odd. This (the finite cyclic) case is especially interesting to us because it provides negative answers to some questions raised in [LV2]. One such question asks: If Σ is a globally 1-alg shape 2-sphere in a 4-manifold Y, must the end of $Y - \Sigma$ be collarable?

Another class of Kwasik–Schultz counterexamples to a 4-dimensional collaring theorem contains ends with fundamental group isomorphic to the Poincaré dodecahedral group. We show that these may be realized as complements of cell-like subsets of S^4.

1. Background

All results presented here are topological, as opposed to smooth or PL. Manifolds are permitted to have boundary unless stated otherwise. Homology is with $\mathbf{Z}$-coefficients except where noted to the contrary. Throughout the paper the symbols $\approx$ and $\cong$ represent homeomorphisms and (algebraic) isomorphisms, respectively.

Our primary source for terminology and results involving noncompact 4-manifolds will be [FQ, §11.9]. A similar development can be found in [KS].

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A noncompact n-manifold M has a *connected end* if, whenever $C \subset M$ is compact, there exists $D \supset C$, also compact, such that $M-D$ is connected. In this case, a *neighborhood of the end* is any $N \subset M$ for which $\mathrm{cl}(M-N)$ is compact. Following the convention of [FQ] (instead of Siebenmann's original definition), we call the end *tame* if for some neighborhood of U of the end there is a proper map $f\colon U \times [0,1) \to M$ which is the identity on $U \times \{0\}$. We say that π_1 is *stable at infinity* if there is a sequence $N_1 \supset N_2 \supset N_3 \supset \cdots$ of connected neighborhoods of the end such that $\bigcap N_i = \emptyset$ and the sequence

$$\pi_1(N_1, p_1) \xleftarrow{\lambda_1} \pi_1(N_2, p_2) \xleftarrow{\lambda_2} \pi_1(N_3, p_3) \xleftarrow{\lambda_3} \cdots$$

induces isomorphisms $\mathrm{im}(\lambda_i) \xleftarrow{\cong} \mathrm{im}(\lambda_{i+1})$ for all $i \geq 1$, where λ_i is the inclusion homomorphism followed by a change of base point isomorphism. When this occurs we say that $\pi_1(\mathrm{end}\, M) \cong \mathrm{im}(\lambda_i) \cong \varprojlim\{(\pi_1(N_i, p_i), \lambda_i)\}$. The end of M is *collarable* if it has a manifold neighborhood N such that $N \approx \partial N \times [0,1)$. Siebenmann's end theorem may be stated as follows.

THEOREM 1.1 ([Si] or [FQ, p. 214]). *Suppose M has a connected tame end with finitely presented fundamental group and* $\dim(M) \geq 6$. *Then there is an obstruction* $\sigma(\mathrm{end}\, M) \in \tilde{K}_0(\mathbf{Z}[\pi_1(\mathrm{end}\, M)])$ *which vanishes if and only if the end of M is collarable.*

Work by Quinn [Qu] extends this theorem to dimension 5 when $\pi_1(\mathrm{end}\, M)$ is a Freedman group. A group G is *Freedman* (called *good* in [FQ]) provided Freedman's disk embedding theorem applies to 4-manifolds with fundamental group isomorphic to G. At this time no examples of non-Freedman groups are known; moreover, all poly-(finite or cyclic) groups are known to be Freedman (see [FQ, p. 99]).

A *weak collar* on M is a neighborhood N of the end of M for which there is a proper map $f\colon N \times [0,1) \to N$ which is the identity on $N \times \{0\}$. It is easy to see that a collar is a weak collar. While the examples produced in [KS] rule out an extension of Theorem 1.1 to dimension 4 (even for Freedman groups), we do have the following theorem.

THEOREM 1.2 [FQ, p. 215]. *Suppose a 4-manifold M has compact boundary and a connected, tame end with finitely presented Freedman fundamental group. Then the obstruction* $\sigma(\mathrm{end}\, M) \in \tilde{K}_0(\mathbf{Z}[\pi_1(\mathrm{end}\, M)])$ *vanishes if and only if the end of M is weakly collarable.*

Note: In particular, the Kwasik–Schultz counterexamples to an extension of Theorem 1.1 to dimension 4 are weakly collarable.

2. Construction of Weak Collars

In this section we review an explicit description of 4-dimensional weak collars found in [FQ]. Let M be a compact 4-manifold, G a finitely presented

Freedman group, and $v: \pi_1(M) \to G$ a surjective homomorphism with perfect kernel. By the "plus construction" (see [FQ, p. 195]), there is a compact cobordism rel boundary (W, M, M^+) with $\pi_1(W) \cong G$, $M \subset W$ a simple $\mathbf{Z}[G]$-homology equivalence, and $M^+ \subset W$ a simple homotopy equivalence. Moreover, W is uniquely determined up to homeomorphism rel M. Given this situation we will denote W by $W(M, v)$ and M^+ by $M^+(M, v)$. By uniqueness, these objects are well-defined. We now employ the plus construction to build a prototypical weak collar. Let L be a closed 3-manifold and let $v: \pi_1(L) \to G$ be a homomorphism onto a Freedman group such that $\ker(v)$ is perfect. For each natural number n, let $M_n^+ = M^+(L \times [n, n+1], v)$. Then $\partial M_n^+ = L \times \{n, n+1\}$. Define $E_\infty(L, v) = M_1^+ \cup M_2^+ \cup M_3^+ \cup \cdots$, with M_k^+ attached to M_{k+1}^+ along $L \times \{k+1\}$ for each k. Then $E_\infty(L, v)$ is a weak collar. (Lemma 4.1 may be used to construct a proper map $f: E_\infty(L, v) \times [0, 1) \to E_\infty(L, v)$.) Moreover, by the following theorem, all 4-dimensional weak collars with Freedman fundamental group are of this type.

THEOREM 2.1 [FQ, p. 222]. *Let E be a 4-dimensional weak collar with compact connected boundary and finitely presented Freedman fundamental group. Then $\partial E \subset E$ is a $\mathbf{Z}[\pi_1(E)]$-homology equivalence (so $i_*: \pi_1(\partial W) \to \pi_1(W)$ is surjective with perfect kernel), and E is homeomorphic to $E_\infty(\partial W, i_*)$.*

3. The Realization Theorems

Let E be a 4-dimensional weak collar with closed connected boundary and $\pi_1(E) \cong \mathbf{Z}_n$. Then $H_1(\partial E) \cong H_1(E) \cong \mathbf{Z}_n$. It follows that ∂E has the same $\mathbf{Z}$-homology groups as a lens space. We say that ∂E is *Z-homology equivalent* to the lens space $L(n, k)$ if there is a map $f: \partial E \to L(n, k)$ which induces $\mathbf{Z}$-homology isomorphisms in all dimensions. By [LS], ∂E is $\mathbf{Z}$-homology equivalent to a lens space $L(n, k)$ which is unique up to homotopy type. For us, the primary significance of $\mathbf{Z}$-homology type is that a homology lens space J with $H_1(J) \cong \mathbf{Z}_n$ bounds a compact 4-manifold homotopy equivalent to S^2 if and only if J is $\mathbf{Z}$-homology equivalent to $L(n, 1)$. This fact may be deduced from [LS] together with general results on 4-manifolds with boundary found in [Bo], [St], or [Vo]. An elementary exposition of homology lens spaces and 4-manifolds homotopy equivalent to S^2 is presented in [Gu].

As noted in the introduction, some of our main results involve a *shape 2-sphere*—i.e., a compactum with the shape of a 2-sphere. Suppose that $K_1 \supset K_2 \supset K_3 \supset \cdots$ is a nested sequence of compact n-manifolds each homotopy equivalent to S^2, and suppose that $K_{i+1} \subset K_i$ is a homotopy equivalence for each i. It is an elementary observation in shape theory that $\Sigma = \bigcap K_i$ is a shape 2-sphere. For our purposes, those unfamiliar with shape theory may treat this as a definition. Conversely, if a shape 2-sphere Σ is defined by the intersection of a nested sequence of compact n-manifolds $K_1 \supset K_2 \supset K_3 \supset \cdots$, and $K_{i+1} \subset K_i$ is a homotopy equivalence for each $i \geq 1$, then each K_i is homotopy equivalent to S^2. See [MS] for a detailed exposition of shape theory.

We may now state our main results.

THEOREM 3.1. *Let E be a connected 4-dimensional weak collar with compact boundary and $\pi_1(E) \cong \mathbf{Z}_n$. Then there is a compact 4-manifold X (having the homotopy type of S^2) and a shape 2-sphere $\Sigma \subset X$ with $X - \Sigma \approx E$ if and only if ∂E is $\mathbf{Z}$-homology equivalent to $L(n, 1)$.*

If one prefers closed 4-manifolds we have the following theorem.

THEOREM 3.2. *Let E be a connected 4-dimensional weak collar with compact boundary and $\pi_1(E) \cong \mathbf{Z}_n$. Then there is a closed 4-manifold Y, a shape 2-sphere $\Sigma \subset Y$, and a neighborhood N of Σ with $N - \Sigma \approx E$ if and only if ∂E is $\mathbf{Z}$-homology equivalent to $L(n, 1)$. Moreover, we may specify Y to be $S^2 \times S^2$ when n is even and $\mathbf{C}P^2 \# (-\mathbf{C}P^2)$ when n is odd.*

NOTES. (1) Many of the Kwasik–Schultz counterexamples to a 4-dimensional version of Siebenmann's theorem may be realized in the above manner. For example, all homology lens spaces with first homology isomorphic to $\mathbf{Z}_2$ are $\mathbf{Z}$-homology equivalent to $L(2, 1)$. Hence, all weak collars with fundamental group $\mathbf{Z}_2$ occur as shape 2-sphere complements. In fact, given any even integer n, [KS] along with Theorems 3.1 and 3.2 may be used to produce noncollarable tame ends with $\pi_1(\text{end}) \cong \mathbf{Z}_n$ which are realizable as shape 2-sphere complements.

(2) By a classical result on lens spaces (see e.g. [Co, p. 96]), $L(n, k)$ is homotopy equivalent to $L(n, 1)$ if and only if $k = \pm b^2 (\text{mod } n)$. Thus, by the converses of Theorems 3.1 and 3.2, many weak collars with finite cyclic fundamental group can never be realized as shape 2-sphere complements.

Let Δ denote the Poincaré dodecahedral group. Since Δ is finite of even order 120, and since Δ acts on S^3, [KS] guarantees the existence of noncollarable weak collars (connected and having closed boundary) having fundamental group isomorphic to Δ. Hence, the following result is in the same spirit as Theorems 3.1 and 3.2.

THEOREM 3.3. *Let E be a connected 4-dimensional weak collar with compact boundary and a finitely generated, perfect, Freedman fundamental group. Then there is a compact contractible 4-manifold C and a cell-like set $A \subset C$ with $E \approx C - A$. Moreover, C may be realized as a neighborhood of $A \subset S^4$.*

4. Proofs

We begin with the following key lemma.

LEMMA 4.1. *Suppose L is a closed connected 3-manifold, G is a finitely presented Freedman group, and $v: \pi_1(L) \to G$ is a surjective homomorphism with perfect kernel. Let $M^+ = M^+(L \times I, v)$, $M_A^+ = M^+(L \times [0, \frac{1}{2}], v)$, $M_B^+ = M^+(L \times [\frac{1}{2}, 1], v)$, and $M_A^+ \cup M_B^+$ be the union M_A^+ and M_B^+ along $L \times \{\frac{1}{2}\}$. Then*

(i) *M^+ is homeomorphic rel $L \times \{0, 1\}$ to $M_A^+ \cup M_B^+$, and*
(ii) *$M_A^+ \cup M_B^+$ deformation retracts onto M_B^+.*

Proof. Recall that M^+ is one end of the unique "plus construction" cobordism rel boundary $(W, L \times I, M^+)$ having the property that $\pi_1(W) \cong G$, $L \times I \subset W$ is a simple $\mathbf{Z}[G]$-homology equivalence, and $M^+ \subset W$ is a simple homotopy equivalence. Similarly, we have cobordisms $(W_A, L \times [0, \frac{1}{2}], M_A^+)$ and $(W_B, L \times [\frac{1}{2}, 1], M_B^+)$ determining M_A^+ and M_B^+. By gluing W_A and W_B together correctly, we may produce a cobordism $(W_A \cup W_B, L \times I, M_A^+ \cup M_B^+)$ which has the properties of a plus construction cobordism. By uniqueness, this cobordism is homeomorphic rel $L \times I$ to $(W, L \times I, M^+)$; hence, $M^+ \approx M_A^+ \cup M_B^+$.

To check (ii), note that $W_B \subset W_A \cup W_B$ is a $\mathbf{Z}[G]$-homology equivalence; because $\pi_1(W_A) \cong \pi_1(W_A \cup W_B) \cong G$, it is also a homotopy equivalence. Thus, $W_A \cup W_B$ deformation retracts onto W_B. Following this with a deformation of W_B onto M_B^+ produces a deformation of $M_A^+ \cup M_B^+$ onto M_B^+. □

Proof of Theorem 3.1. By Theorem 2.1, $E \approx E_\infty(\partial E \times I, i_*) = M_1^+ \cup M_2^+ \cup M_3^+ \cup \cdots$, as described earlier. We begin by embedding this infinite union into M_1^+.

By Lemma 4.1, for each $i \geq 1$ there is a homeomorphism $h_i: M_i^+ \cup M_{i+1}^+ \to M_i^+$ which takes $\partial E \times \{i\}$ and $\partial E \times \{i+2\}$ canonically onto $\partial E \times \{i\}$ and $\partial E \times \{i+1\}$, respectively. Extend each h_i to

$$H_i: M_1^+ \cup M_2^+ \cup \cdots \cup M_{i+1}^+ \to M_1^+ \cup M_2^+ \cup \cdots \cup M_i^+$$

by letting H_i be the identity on $M_1^+ \cup M_2^+ \cup \cdots \cup M_{i-1}^+$, and $H_i|_{M_i \cup M_{i+1}} = h_i$. Next, define $f_1 = H_1$, $f_2 = f_1 \circ H_2: M_1^+ \cup M_2^+ \cup M_3^+ \to M_1^+$, and (inductively) $f_{i+1} = f_i \circ H_{i+1}: M_1^+ \cup M_2^+ \cup \cdots \cup M_{i+1}^+ \to M_1^+$ for each $i \geq 1$. Note that:

(i) each f_i is a homeomorphism;
(ii) for any $k \geq i$, $f_k(x) = f_i(x)$ for all $x \in M_1^+ \cup M_2^+ \cup \cdots \cup M_i^+$; and
(iii) for each i, $\text{cl}(M_1^+ - f_i(M_1^+ \cup M_2^+ \cup \cdots \cup M_i^+)) \approx M_{i+1}^+ \approx M_1^+$.

Now define $F: E_\infty(\partial E \times I, i_*) = M_1^+ \cup M_2^+ \cup M_3^+ \cup \cdots \to M_1^+$ by $F(x) = f_i(x)$ whenever $x \in M_i^+$. Using (i) and (ii) above, it is easy to check that F is an embedding.

Now assume that ∂E is $\mathbf{Z}$-homology equivalent to $L(n, 1)$. Then, as noted earlier, ∂E bounds a compact 4-manifold K homotopy equivalent to S^2. Let $X = M_1^+ \cup_{\partial E \times \{2\}} K$, and let $\Sigma = X - F(E_\infty(\partial E, i_*))$.

Claim 1: X is homotopy equivalent to S^2. Examination of the plus construction used to produce M_1^+ reveals that $H_i(M_1^+, \partial E \times \{2\}) = 0$ for all i. Then excision gives $H_i(X, K) = 0$ for all i, so by the Hurewicz theorem,

$\pi_i(X, K) = 0$ for all i. Thus, $K \subset X$ is a homotopy equivalence and our claim is verified.

Claim 2: Σ is a shape 2-sphere. Let

$$K_i = \text{cl}(M_1^+ - F(M_1^+ \cup M_2^+ \cup \cdots \cup M_i^+)) \cup K.$$

By (iii) above and the definition of F,

$$\text{cl}(M_1^+ - F(M_1^+ \cup M_2^+ \cup \cdots \cup M_i^+)) \approx M_1^+$$

for all $i \geq 1$. Thus $K_i \approx M_1^+ \cup K = X$ for all i. In particular, for each i, K_i is homotopy equivalent to S^2 and $K \subset K_i$ is a homotopy equivalence. Then $K_{i+1} \subset K_i$ is a homotopy equivalence for each i. Now $\Sigma = \bigcap K_i$ so, by our earlier discussion, Σ is a shape 2-sphere.

Conversely, suppose there is a compact 4-manifold X and a shape 2-sphere $\Sigma \subset X$ with $X - \Sigma \approx E$. Again by Theorem 2.1, we may view $X - \Sigma$ as $M_1^+ \cup M_2^+ \cup M_3^+ \cup \cdots$. Let $K_i = (M_i \cup M_{i+1} \cup \cdots) \cup \Sigma$. Then $\Sigma = \bigcap K_i$, and an application of Lemma 4.1(ii) shows that K_i deformation retracts to K_{i+1} for any $i \geq 1$. In particular, $K_{i+1} \subset K_i$ is a homotopy equivalence. Our discussion of shape theory now implies that each K_i is homotopy equivalent to S^2. Since ∂E is a homology lens space with $H_1(\partial E) \cong \mathbf{Z}_n$ which bounds a 4-manifold K_i, homotopy equivalent to S^2, we know that ∂E is $\mathbf{Z}$-homology equivalent to $L(n, 1)$. □

Proof of Theorem 3.2. If ∂E is $\mathbf{Z}$-homology equivalent to $L(n, 1)$, then let X be the compact 4-manifold and $\Sigma \subset X$ the shape 2-sphere promised by Theorem 3.1. Let $Y = X \cup X^-$ be the double of X along its boundary (X^- denotes a copy of X with reversed orientation) and let $N = X \subset Y$. Conversely, if there is a closed 4-manifold Y, a shape 2-sphere $\Sigma \subset Y$, and a neighborhood N of Σ with $N - \Sigma \approx E$, then N satisfies the conditions on X in Theorem 3.1. Hence, $\partial N\ (= \partial E)$ is $\mathbf{Z}$-homology equivalent to $L(n, 1)$.

To complete the proof, we show that $X \cup X^- \approx S^2 \times S^2$ if n is even and $X \cup X^- \approx \mathbf{C}P^2 \# (-\mathbf{C}P^2)$ if n is odd. This will be accomplished by applying Freedman's classification of simply connected 4-manifolds (see [Fr]). The following facts, which may be found in Lemma 5.1 of [Gu], will help us calculate the intersection pairing of $X \cup X^-$:

(i) there is a framed proper immersion of an oriented disk D in X such that $[\partial D]$ generates $H_1(\partial X)$ and $[D]$ generates $H_2(X, \partial X)$;
(ii) if $S \subset X$ is the (oriented) image of a homotopy equivalence $g: S^2 \to X$, then $[S]$ generates $H_2(X)$ and the orientation may be chosen so that the intersection number $[S] \cdot [D] = 1$;
(iii) given a collection $\{D_i\}_{i=1}^n$ of n distinct parallel copies of D, there is an oriented surface $A \subset \partial X$ with $\partial A = \bigcup \partial D_i$ and $[D_1 \cup D_2 \cup \cdots \cup D_n \cup A] = [S]$ in $H_2(X)$.

Since $X \cup X^-$ is a closed simply connected 4-manifold, $H_2(X \cup X^-)$ is free. Then, by the Mayer–Vietoris sequence

$$\begin{array}{ccccccccc}\cdots\longrightarrow H_2(\partial X) & \longrightarrow & H_2(X)\oplus H_2(X^-) & \longrightarrow & H_2(X\cup X^-) & \xrightarrow{\partial_*} & H_1(\partial X) & \longrightarrow 0,\\ \| & & \| & & & & \| & \\ 0 & & \mathbf{Z}\oplus\mathbf{Z} & & & & \mathbf{Z}_n & \end{array}$$

$H_2(X\cup X^-)$ has rank 2 and is generated by $\{[S],[S^-],[D\cup D^-]\}$. Furthermore,

$$\begin{aligned} n[D\cup D^-] &= n[D\cup D^-\cup A\cup(-A)]\\ &= [(D_1\cup D_2\cup\cdots\cup D_n)\cup(D_1^-\cup D_2^-\cup\cdots\cup D_n^-)\cup A\cup A^-]\\ &= [D_1\cup D_2\cup\cdots\cup D_n\cup A]+[D_1^-\cup D_2^-\cup\cdots\cup D_n^-\cup A^-]\\ &= [S]+[S^-]. \end{aligned}$$

Thus, $[S^-]=n[D\cup D^-]-[S]$, so $\{[S],[D\cup D^-]\}$ is a basis for $H_2(X\cup X^-)$.

Next we calculate the intersection pairing for $X\cup X^-$. By (ii) above, it is clear that $[S]\cdot[D\cup D^-]=1$. Since self-intersection points in D all have corresponding self-intersections in D^- with opposite sign, $[D\cup D^-]\cdot[D\cup D^-]=0$. By (iii) above, $[S]\cdot[S]=[S]\cdot[D_1\cup D_2\cup\cdots\cup D_n\cup A]$, and since $A\subset\partial X$, we may assume $S\cap A=\emptyset$. Hence, $[S]\cdot[S]=n([S]\cdot[D])=n$. The intersection pairing for $X\cup X^-$ can thus be represented by the matrix $\omega_X=\begin{bmatrix}0&1\\1&n\end{bmatrix}$. If n is even and $A=\begin{bmatrix}1&0\\-n/2&1\end{bmatrix}$, then $A\cdot(\omega_X)\cdot A^T=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ and so, by Freedman's classification theorem, $X\cup X^-\approx S^2\times S^2$. Similarly, when n is odd, ω_X is equivalent to $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$. By the classification theorem, a closed, simply connected 4-manifold with this intersection pairing is either $\mathbf{C}P^2\#(-\mathbf{C}P^2)$, or a manifold homotopy equivalent to $\mathbf{C}P^2\#(-\mathbf{C}P^2)$ but with nontrivial Kirby–Siebenmann invariant. The Kirby–Siebenmann invariant (hereafter denoted "ks") lies in $\mathbf{Z}_2$ and is additive for manifolds joined along a component of their boundaries (see [FQ, p. 164]), so $\text{ks}(X\cup X^-)=\text{ks}(X)+\text{ks}(X^-)=2\cdot\text{ks}(X)=0$ (in $\mathbf{Z}_2$). Thus $X\cup X^-\approx\mathbf{C}P^2\#(-\mathbf{C}P^2)$ when n is odd. □

Proof of Theorem 3.3. Recall that $\pi_1(E)$ perfect means that $H_1(E)=0$. Since $H_1(\partial E)\cong H_1(E)$, duality and universal coefficients imply that ∂E is a homology 3-sphere. By [Fr], ∂E bounds a compact contractible 4-manifold C'. The proof is now a simpler version of the proof of Theorem 3.1, with C' playing the role of K and C the role of X. In the end, $A=C-F(E)$, where $F\colon E\ (\approx E_\infty(\partial E\times I,i_*))\ \to M^+(\partial E\times I,i_*)$ is an embedding which takes ∂E onto $\partial E\times\{0\}$, and $C=M^+(\partial E\times I,i_*)\cup_{\partial E\times\{1\}}C'$. Now A may be viewed as the intersection of a nested sequence of compact contractible manifolds, and is therefore cell-like. To embed C in S^4, simply note that the double of C is a homotopy 4-sphere, and hence homeomorphic to S^4 by [Fr]. □

5. An Application to Embedding Theory

Let X^n be an n-manifold, and let $A\subset X^n$. A is *globally 1-alg* in X if for any neighborhood U of A there is a neighborhood V of A, $V\subset U$, such that loops that are null-homologous in $V-A$ are null-homotopic in $U-A$. This condition has proven to be valuable for studying complements of certain embeddings. For example, if $\Sigma^k \subset S^n$ is an embedded k-sphere (or shape k-sphere) with $k \leq n-3$, then $S^n - \Sigma^k \approx S^n - S^k$ if and only if Σ^k is globally 1-alg (see [Du] and [Ve]). Analogous results, but with knotting taken into consideration, are known when $k = n-2$. One example, due to Liem and Venema, is the following.

THEOREM 5.1 [LV1]. *Let $\Sigma^2 \subset S^4$ be an embedded shape 2-sphere. Then $S^4 - \Sigma^2 \approx S^4 - K^2$ for some locally flat 2-sphere K^2, or (equivalently) Σ^2 has a neighborhood $N \approx S^2 \times D^2$ with $N - \Sigma^2 \approx (S^2 \times S^1) \times [0,1)$ if and only if Σ^2 is globally 1-alg in S^4.*

In [LV2], the following question is raised: If $\Sigma^2 \subset X^4$ is a 1-alg shape 2-sphere in a 4-manifold, does there exist a locally flat 2-sphere $K^2 \subset X^4$ with $X^4 - \Sigma^2 \approx X^4 - K^2$? Equivalently, one may ask whether every globally 1-alg shape 2-sphere Σ^2 in a 4-manifold X^4 has a neighborhood N homeomorphic to a disk bundle D over S^2 with $N - \Sigma^2$ homeomorphic to $D - S_0^2$, where S_0^2 is the 0-section of D. A weaker version simply asks whether the end of $X^4 - \Sigma^2$ must be collarable. It is easy to check that, when Theorem 3.1 or 3.2 is applied to a weak collar E with boundary $\mathbf{Z}$-homology equivalent to $L(n,1)$, the resulting shape 2-sphere Σ is globally 1-alg. Furthermore, when E does not contain an actual collar (as in the Kwasik–Schultz examples), we have produced an example that answers the above questions negatively.

6. Questions

It is natural to ask whether the results in this paper are true with actual 2-spheres taking the place of shape 2-spheres. In particular:

(1) Can all 4-dimensional weak collars with fundamental group isomorphic to $\mathbf{Z}_n$ and boundary $\mathbf{Z}$-homology equivalent to $L(n,1)$ be realized as complements of (wildly) embedded 2-spheres?
(2) Does there exist a globally 1-alg 2-sphere Σ in a 4-manifold X^4 for which the end of $X^4 - \Sigma$ is not collarable, or for which there are not locally flat 2-spheres $K \subset X^4$ with $X^4 - \Sigma \approx X^4 - K$?

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