Self-Duality and 4-Manifolds with Nonnegative Curvature on Totally Isotropic 2-Planes

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1. Introduction

In [MM], Micallef and Moore proved a beautiful result which gives a topological classification of simply connected compact manifolds with positive curvature on totally isotropic 2-planes, namely that they are homeomorphic to the sphere. In this paper we want to consider the case of nonnegative curvature on totally isotropic 2-planes (see Definition 3.2) for 4-dimensional compact manifolds. Our first result is the following theorem.

THEOREM 1. *Let M be an irreducible, simply connected compact 4-manifold. If M has nonnegative curvature on totally isotropic 2-planes then M is either homeomorphic to the sphere S^4 or biholomorphic to the complex projective space $\mathbf{CP}^2$.*

Also in [MM, p. 222], the authors investigated some commonly used curvature conditions which imply the nonnegativity of the curvature on totally isotropic 2-planes. (For brevity we denote this by NNC.) For the case of dimension 4, some other conditions will give NNC. For instance, the results of Seaman in [S1] imply that compact, positively curved, real 4-dimensional Kähler manifolds have NNC. Conformally flat 4-manifolds with nonnegative scalar curvature have NNC.

In this paper we will investigate some conditions on a half-conformally flat manifold which will imply nonnegativity of the curvature on totally isotropic 2-planes. For example, although the nonnegativity of the scalar curvature is a necessary condition (Proposition 3.3), Theorem 1 and Theorem B in [Po] combined show that it cannot be sufficient even for positive scalar curvature. We will give in the next theorem a condition in terms of the sectional curvatures which will be a sufficient condition.

THEOREM 2. *Let M^4 be a half-conformally flat manifold with nonnegative scalar curvature. Then M has nonnegative curvature on totally isotropic 2-planes if and only if for any orthonormal basis $\{e_i, e_j, e_m, e_k\}$ of the*

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tangent plane $K_{ij}+K_{mk}\leq S/3$, *where* K_{ij} *denotes the sectional curvature of the plane spanned by* e_i *and* e_j *and* S *is the scalar curvature.*

This theorem will be proved in Section 3. Also in this section, the proof of Proposition (3.6) implies that self-dual Kähler manifolds with nonnegative scalar curvature have NNC. Theorem 1 will be proved in the last section. In Section 2 we review the known result (stated as Proposition 2.4) which guarantees that a manifold is definite. This proposition combined with Proposition (2.5) will enable us to classify topologically compact half-conformally flat manifolds with nonnegative Ricci curvature. For these manifolds we prove the following theorem.

THEOREM 3. *Let* M^4 *be a compact half-conformally flat manifold with nonnegative Ricci curvature. Then one of the following holds.*

(a) *M is an oriented conformally flat* 4*-manifold. In this case M is either conformally equivalent to* S^4 *or is a quotient of* $S^3\times\mathbf{R}$ *or* $\mathbf{R}^4$ *by a group of fixed-point free isometries in the standard metrics.*
(b) *M is not conformally flat; then the universal covering of M is either homeomorphic to* $\mathbf{CP}^2\#\cdots\#\mathbf{CP}^2$ *or diffeomorphic to a* K3 *surface.* (A K3 surface is a complex surface with first Betti number $b_1=0$ and first Chern class $c_1=0$.)

2. Half-Conformally Flat Manifolds with Nonnegative Ricci Curvature

Let M be an oriented Riemannian manifold of dimension 4, and let Λ^2 denote the bundle of exterior 2-forms and $\Lambda^2=\Lambda^2_+\oplus\Lambda^2_-$ the eigenspace splitting for the Hodge *-operator.

The Riemann curvature tensor defines a symmetric operator $\mathfrak{R}\colon\Lambda^2\to\Lambda^2$ given by

$$\mathfrak{R}(e_{ij})=\frac{1}{2}\sum_{k,l}R_{ijlk}e_{kl},$$

where $\{e_i\}$ is a local orthonormal basis of 1-forms, e_{ij} denotes the 2-form $e_i\wedge e_j$, and $R_{ijlk}=\langle R(e_i,e_j)e_l,e_k\rangle$. The operator $\mathfrak{R}$ can be decomposed as

$$\mathfrak{R}=\mathfrak{R}^+_++\mathfrak{R}^+_-+\mathfrak{R}^-_++\mathfrak{R}^-_-$$

with respect to the decomposition $\Lambda^2=\Lambda^2_+\oplus\Lambda^2_-$. This decomposition gives the irreducible components of $\mathfrak{R}$ (see [ST]). They are $\operatorname{tr}\mathfrak{R}^+_+=\operatorname{tr}\mathfrak{R}^-_-=S/4$, where S is the scalar curvature, the traceless Ricci tensor is $\mathfrak{R}^+_-$, and the two components of the Weyl tensor W^+ and W^- are given by $W^+=\mathfrak{R}^+_+-S/12$ and $W^-=\mathfrak{R}^-_--S/12$.

An oriented Riemannian manifold of dimension 4 is called *half-conformally flat* if either $W^+=0$ or $W^-=0$. An oriented Riemannian manifold is *self-dual* if $W^-=0$. It is clear that in a half-conformally flat manifold, self-duality is a property which depends on the orientation.

Let x be an arbitrary point of M and let $\{e_1, e_2, e_3, e_4\}$ be a positively oriented orthonormal basis of the tangent space T_xM. The 2-forms

$$\alpha_1 = \frac{\sqrt{2}}{2}(e_{12}+e_{34}), \quad \alpha_2 = \frac{\sqrt{2}}{2}(e_{13}-e_{24}), \quad \alpha_3 = \frac{\sqrt{2}}{2}(e_{14}+e_{23})$$

are in $\Lambda^2_+(T_xM)$ and are called self-dual; the 2-forms

$$\beta_1 = \frac{\sqrt{2}}{2}(e_{12}-e_{34}), \quad \beta_2 = \frac{\sqrt{2}}{2}(e_{13}+e_{24}), \quad \beta_3 = \frac{\sqrt{2}}{2}(e_{14}-e_{23})$$

are in $\Lambda^2_-(T_xM)$ and are called anti-self-dual. If $W^-=0$ then $\Re^-_- = S/12$; for $\bar{\beta}_i = \sqrt{2}\beta_i$ this implies $\langle \Re(\bar{\beta}_i), \bar{\beta}_i \rangle = S/6$. Therefore

$$\begin{aligned} K_{12}+K_{34}+2R_{1234} &= S/6, \\ K_{13}+K_{24}-2R_{1324} &= S/6, \\ K_{14}+K_{23}+2R_{1423} &= S/6, \end{aligned} \tag{2.1}$$

where K_{ij} denotes the curvature of the plane $\{e_i, e_j\}$.

Let $F\colon \Lambda^2(T_xM) \to \Lambda^2(T_xM)$ be the Weitzenböck operator given by (see [S1])

$$\begin{aligned} \langle F(e_{ij}), e_{kl} \rangle = \operatorname{Ric}(e_i, e_k)\delta_{jl} + \operatorname{Ric}(e_j, e_l)\delta_{ik} \\ - \operatorname{Ric}(e_i, e_l)\delta_{jk} - \operatorname{Ric}(e_j, e_k)\delta_{il} - 2R_{ijlk}, \end{aligned}$$

where Ric denotes the Ricci curvature. This operator satisfies the well-known Weitzenböck formula; that is, $\Delta\omega = -\operatorname{div}\nabla\omega + F(\omega)$. Moreover, F is a symmetric operator and Λ^2_+ and Λ^2_- are F-invariant (see [S2, Prop. 1]).

(2.2) Proposition. *If M is a self-dual manifold then all eigenvalues of the operator $F^- = F\colon \Lambda^2_- \to \Lambda^2_-$ are equal to $S/3$.*

Proof. Let $\{\beta_1, \beta_2, \beta_3\}$ be an orthonormal basis of eigenvectors of F^-. As in [S2, Prop. 2], we consider an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of T_xM such that

$$\beta_1 = \frac{\sqrt{2}}{2}(e_{12}-e_{34}), \quad \beta_2 = \frac{\sqrt{2}}{2}(e_{13}+e_{24}), \quad \beta_3 = \frac{\sqrt{2}}{2}(e_{14}-e_{23}).$$

From the definition of F we have:

$$\begin{aligned} \langle F(\beta_1), \beta_1 \rangle &= \tfrac{1}{2}(\operatorname{Ric}(e_1) + \operatorname{Ric}(e_2) + \operatorname{Ric}(e_3) + \operatorname{Ric}(e_4) - 2k_{12} - 2k_{34} - 4R_{1234}) \\ &= K_{13}+K_{14}+K_{23}+K_{24}-2R_{1234}. \end{aligned} \tag{2.3}$$

Using the first Bianchi identity and (2.1), we conclude:

$$\langle F(\beta_1), \beta_1 \rangle = K_{13}+K_{24}-2R_{1324}+K_{14}+K_{23}+2R_{1423} = S/3.$$

Similarly, we obtain

$$\begin{aligned} \langle F(\beta_2), \beta_2 \rangle &= K_{12}+K_{34}+K_{14}+K_{23}+2R_{1324} = S/3, \\ \langle F(\beta_3), \beta_3 \rangle &= K_{12}+K_{34}+K_{13}+K_{24}-2R_{1423} = S/3. \end{aligned}$$

□

(2.4) PROPOSITION. *Let M be a half-conformally flat manifold with nonnegative scalar curvature. If there is a point in M such that the scalar curvature is positive, then M is definite.*

Proof. Integrating by parts, the Weitzenböck formula over M yields

$$(\Delta\omega, \omega) = (\nabla\omega, \nabla\omega) + \int_M \langle F(\omega), \omega\rangle \, dV,$$

where (,) is the inner product on $\Lambda^2(M)$ given by:

$$(\phi, \psi) = \int_M \langle \phi, \psi \rangle \, dV;$$

dV is the volume form of M, and $\langle\, ,\rangle$ is the naturally induced inner product on the space of 2-forms $\Lambda^2(T_x M)$. Let us suppose that the orientation was chosen so that M is self-dual. The hypothesis about the sign of the scalar curvature together with Proposition (2.2) implies that if ω is anti-self-dual then $(\Delta\omega, \omega)$ is positive. Therefore, if there are nonzero harmonic 2-forms then they must be self-dual, proving the proposition. □

In order to prove Theorem 3 (stated in the introduction), we will study the universal covering of compact half-conformally flat manifolds with nonnegative Ricci curvature and prove the next proposition.

(2.5) PROPOSITION. *Let M^4 be a compact half-conformally flat manifold with nonnegative Ricci curvature. Then either the fundamental group $\pi_1(M)$ is finite or M is covered by $\mathbf{R}^4$ or $S^3 \times \mathbf{R}$ with their standard metrics.*

Proof. It follows by a theorem of Cheeger and Gromoll [CG] that the universal covering $\tilde{M}$ of M splits isometrically as $\bar{M} \times \mathbf{R}^k$, where $\bar{M}$ is compact.

Let us suppose that M is self-dual and that $\{e_1, e_2, e_3, e_4\}$ is an orthonormal positively oriented basis of the tangent space of an arbitrary point of $\tilde{M}$. Consider the anti-self-dual forms defined by this basis. If $k=1$ let us suppose that e_1, e_2, e_3 are tangent to $\bar{M}$. Then we have $K_{14} = K_{24} = K_{34} = 0$ and $R_{1234} = R_{1324} = 0$ which implies $R_{1423} = 0$. It follows from (2.1) that $K_{12} = K_{13} = K_{23} = S/6$ and so $\bar{M} = S^3$.

If $k=2$ we consider $\{e_1, e_2, e_3, e_4\}$ such that e_1 and e_2 are tangent to $\bar{M}$. We have $K_{13} = K_{14} = K_{23} = K_{24} = 0$ and $R_{1234} = 0$. It follows from (2.3) that $S = 0$. But $S = K_{12}$, contradicting that $\bar{M}$ is compact and simply connected.

The cases $k=0$ and $k=4$ obviously imply $\pi_1(M)$ finite and $\tilde{M} = \mathbf{R}^4$, respectively, and the case $k=3$ cannot occur since it would contradict the simple connectivity of $\bar{M}$. □

(2.6) PROOF OF THEOREM 3. Notice that by definition a half-conformally flat manifold is oriented. Therefore, if M is conformally flat then the result will follow from the proof of Proposition (2.5).

If M is half-conformally flat but not a conformally flat manifold, we conclude from Proposition (2.5) that $\pi_1(M)$ is finite. Therefore the universal covering is still compact and we suppose, without losing generality, that M is simply connected. Moreover, a well-known formula for the signature of M^4 (see [AHS, p. 428]) implies that half-conformally flat manifolds which are not conformally flat have the second Betti number $b_2 > 0$. Therefore, if M is definite, it follows from [Do] and [Fr] that a definite, smooth, simply connected, compact 4-manifold with $b_2 > 0$ must be topologically $\mathbf{CP}^2 \# \cdots \# \mathbf{CP}^2$. If M is not definite, again because $b_2 > 0$ there exists a (nonzero) harmonic 2-form ω which is anti-self-dual; otherwise, M would be definite. Then Proposition 2.4 implies that M is Ricci-flat (because M is not definite) with F a null operator over Λ^2_-, implying that ω is parallel. Thus M is a Kähler manifold and the parallel 2-form ω is anti-self-dual. Reversing the orientation, M will be an anti-self-dual Kähler manifold and so diffeomorphic to a K3 surface (see [Hi] and [Ya]).

3. Self-Dual Manifolds with Nonnegative Curvature on Totally Isotropic 2-Planes

Let $T_xM \otimes C$ denote the complexified tangent space, and extend the Riemannian metric $\langle\, ,\rangle$ to a complex bilinear form $(\, ,)$. An element Z in $T_xM \otimes C$ is said to be *isotropic* if $(Z, Z) = 0$. A 2-plane $\sigma \subseteq T_xM \otimes C$ is *totally* isotropic if $(Z, Z) = 0$ for any Z in $T_xM \otimes C$. If σ is a totally isotropic 2-plane then there exists a basis $\{Z, W\}$ of σ such that

$$Z = e_i + \sqrt{-1}\, e_j \quad \text{and} \quad W = e_m + \sqrt{-1}\, e_k,$$

where $\{e_i, e_j, e_m, e_k\}$ is an orthonormal basis of T_xM.

(3.2) Definition. A 4-manifold has nonnegative curvature on totally isotropic 2-planes if for Z and W as above we have

$$K_{ik} + K_{im} + K_{jk} + K_{jm} - 2R_{ijkm} \geq 0.$$

The reader is referred to [MM, pp. 200–203] for the details about curvature on totally isotropic 2-planes.

(3.3) Proposition. *If a half-conformally flat manifold has nonnegative curvature on totally isotropic 2-planes, then the Weitzenböck operator F is nonnegative. In particular, the scalar curvature of M is nonnegative. Conversely, if F is a nonnegative operator then M has nonnegative curvature on totally isotropic 2-planes.*

Proof. Let ω be an eigenvector of F with corresponding eigenvalue r. As in [S2, Prop. 2], we can consider an orthonormal basis $\{e_i, e_j, e_k, e_m\}$ of T_xM such that $\omega = (\sqrt{2}/2)(e_{ij} \pm e_{km})$. If we set

$$Z = e_i \pm \sqrt{-1}\, e_j \quad \text{and} \quad W = e_k \pm \sqrt{-1}\, e_m,$$

$\{Z, W\}$ is a totally isotropic 2-plane whose curvature will be given by r, since by (2.3) we have

$$r = K_{ik} + K_{im} + K_{jk} + K_{jm} \pm 2R_{ijkm}.$$

Now, supposing $W^- = 0$, the nonnegativity of the scalar curvature follows from Proposition (2.2). To prove the converse we observe that, given a totally isotropic 2-plane $\sigma = \{Z, W\}$, there exists an orthonormal basis $\{e_i, e_j, e_k, e_m\}$ of the tangent space such that its curvature is equal to $\langle F(\omega), \omega\rangle$, where $\omega = (\sqrt{2}/2)(e_{ij} \pm e_{km})$. □

(3.4) REMARK. From the proof of the above proposition and the Weitzenböck formula, it follows that for 4-manifolds the nonnegativity of the curvature on totally isotropic 2-planes implies that harmonic 2-forms are parallel.

As is observed in [MM, p. 201], on an oriented 4-manifold the nonnegativity of the curvature on totally isotropic 2-planes is equivalent to the inequality $-W + S/6 \geq 0$. Since W is trace-free, we can state the following result.

(3.5) PROPOSITION. *Let M^4 be an oriented 4-manifold with nonnegative curvature on totally isotropic 2-planes. If the scalar curvature is identically zero then M is conformally flat.*

(3.6) PROPOSITION. *Let M^4 be a compact half-conformally flat manifold with nonnegative curvature on totally isotropic 2-planes. Then one of the following holds.*

(a) *M is conformally flat; then either the second Betti number $b_2 = 0$, or M is covered by the Euclidean space $\mathbf{R}^4$ or $S^2 \times \mathbf{H}^2$, where S^2 has constant sectional curvatures and $\mathbf{H}^2$ is the hyperbolic plane.*

(b) *M is a Kähler manifold and $b_2 = 1$.*

Proof. If M is conformally flat, the result follows from Theorem 2 in [No].

If M is half-conformally flat but not conformally flat, then the first Pontrjagin number and the signature τ are nonzero (see [AHS, p. 428]). Since $\tau = b_2^+ - b_2^-$ and the second Betti number $b_2 = b_2^+ + b_2^-$ (where b_2^+ and b_2^- denote the dimensions of the subspaces of harmonic 2-forms which are self-dual and anti–self-dual, respectively), we conclude that b_2 is nonzero. Therefore, let ω be an harmonic 2-form. Since we can suppose that $W^- = 0$, and since by Proposition (3.5) M has a point of positive scalar curvature, ω is a self-dual 2-form. It follows by (3.4) that ω is parallel and, because M is oriented, M is a Kähler manifold. Also, this implies that F has at least one null eigenvalue. We claim that the only harmonic 2-forms on M are of the type $c\omega$, $c \in \mathbf{R}$. In fact, since all harmonic 2-forms must be parallel and self-dual, all we need to prove is that—at the point where the scalar curvature S is nonnull—the operator F restricted to Λ_+^2 has only one null eigenvalue. Then the same arguments used to prove Theorem 3 in [S1] will conclude the proposition. For that, consider an orthonormal basis $\{e_1, e_2, e_3, e_4\}$. The

2-forms $e_{12}, e_{34}, e_{13}+e_{24}, e_{14}-e_{23}$ span the unitary algebra $u(2)$. Because M is a Kähler manifold, its holonomy group is a subgroup of the unitary group $U(2)$, implying that the range of the curvature operator $\mathfrak{R}$ lies inside the algebra $u(2)$. Observe that $u(2)$ contains all anti-self-dual 2-forms and the self-dual form $e_{12}+e_{34}$. Therefore the self-dual 2-forms orthogonal to $e_{12}+e_{34}$ are in the kernel of $\mathfrak{R}$, and thus we have

$$\begin{aligned} \langle \mathfrak{R}(e_{13}-e_{24}), e_{13}-e_{24}\rangle &= K_{13}+K_{24}+2R_{1324}=0, \\ \langle \mathfrak{R}(e_{14}+e_{23}), e_{14}+e_{23}\rangle &= K_{14}+K_{23}-2R_{1423}=0, \end{aligned} \tag{3.7}$$

which together with (2.1) imply

$$K_{13}+K_{24}=-2R_{1324}=K_{14}+K_{23}=2R_{1423}=S/12$$

and hence $K_{12}+K_{34}=S/3$. Now, with the same notation used in Section 2, let α_1 be an eigenvector of F with corresponding eigenvalue 0. Using (3.7), for the other eigenvectors we obtain

$$\langle F(\alpha_2), \alpha_2\rangle = K_{12}+K_{34}+K_{14}+K_{23}-2R_{1324}=S/2,$$

$$\langle F(\alpha_3), \alpha_3\rangle = K_{12}+K_{34}+K_{13}+K_{24}+2R_{1423}=S/2,$$

proving that they are nonnull. □

We notice that the last part of the above proof implies that a self-dual Kähler manifold with nonnegative scalar curvature has nonnegative curvature on totally isotropic 2-planes. Since self-dual compact Kähler manifolds (with respect to the natural orientation) are locally symmetric spaces (see [De, Thm. 1]), they are manifolds with constant positive scalar curvature. It follows by [Bo, Prop. 9.3] that M is isometric to $\mathbf{CP}^2$ with its standard metric. Thus we conclude the following.

(3.8) Proposition. *The complex projective space $\mathbf{CP}^2$ with its standard metric is the only self-dual compact Kähler manifold which has nonnegative curvature on totally isotropic 2-planes.*

In [Po], Poon defined a Riemannian metric with positive scalar curvature and self-dual Weyl tensor on $\mathbf{CP}^2 \# \mathbf{CP}^2$. Therefore, on half-conformally flat manifolds, the nonnegativity of scalar curvature does not imply the nonnegativity of the curvature on totally isotropic 2-planes. We will finish this section by proving Theorem 2, which gives a sufficient condition in terms of sectional curvatures for such an implication.

(3.9) Proof of Theorem 2. We will prove first that our hypotheses imply that F is a nonnegative operator, which by Proposition (3.3) implies that M has nonnegative curvature on totally isotropic 2-planes. Proposition (2.2) implies that the operator F restricted to Λ^2_- is nonnegative. Let r_1, r_2, r_3 be the eigenvalues of F^+ with corresponding eigenvectors $\alpha_1, \alpha_2, \alpha_3$. Consider an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of T_xM such that

$$\alpha_1=\frac{\sqrt{2}}{2}(e_{12}+e_{34}),\quad \alpha_2=\frac{\sqrt{2}}{2}(e_{13}-e_{24}),\quad \alpha_3=\frac{\sqrt{2}}{2}(e_{14}+e_{23}).$$

From the definition of F and the first Bianchi identity, we have

$$r_1=K_{13}+K_{24}+2R_{1324}+K_{14}+K_{23}-2R_{1423}.$$

But from (2.1) and again by the first Bianchi identity we get

$$K_{13}+K_{24}-2R_{1423}=K_{14}+K_{23}+2R_{1324}=S/6+2R_{1234}.$$

Therefore $r_1=2(S/6+2R_{1234})$. Using (2.1) once more, we conclude that

$$r_1=2(S/3-K_{12}-K_{34}).$$

Similarly, we obtain

$$r_2=2(S/3-K_{13}-K_{24}),$$
$$r_3=2(S/3-K_{14}-K_{23}).$$

Now the hypothesis about sectional curvatures implies that the eigenvalues are nonnegative. To prove the converse, consider an orthonormal basis $\{e_i, e_j, e_m e_k\}$ of T_xM and a self-dual 2-form $\alpha=(\sqrt{2}/2)(e_{ij}+e_{mk})$. In a similar manner we can prove that the totally isotropic 2-plane $\sigma=\{Z, W\}$, where $Z=e_i+\sqrt{-1}\,e_j$ and $W=e_m+\sqrt{-1}\,e_k$, has curvature given by

$$\langle F(\alpha), \alpha\rangle=2(S/3-K_{ij}-K_{mk}),$$

implying that $K_{ij}+K_{mk}\leq S/3$. □

The next corollary follows from (2.5), (2.6), (3.8), and Theorem 2.

(3.10) COROLLARY. *Let M^4 be a compact half-conformally flat manifold with nonnegative Ricci curvature. Suppose that for any orthonormal basis $\{e_i, e_j, e_m, e_k\}$ of the tangent plane we have $K_{ij}+K_{mk}\leq S/3$. Then one of the following holds.*

(a) *M is conformally flat, and is either conformally equivalent to S^4 or is a quotient of $\mathbf{R}^4$ or $S^3\times\mathbf{R}$ by a group of fixed-point free isometries in the standard metrics.*
(b) *M is the complex projective space $\mathbf{CP}^2$ with its standard metric.*

4. Proof of Theorem 1

Let G be the holonomy group of M. If M is irreducible then so is G. Recall that Berger [Be] proved that if for some $x\in M$, G acts irreducibly on T_xM, then either M is locally symmetric or G is one of the following standard subgroups of $SO(4)$: $SO(4)$, $U(2)$, or $SU(2)$.

If M is locally symmetric then M is an analytic Riemannian manifold (see [He, p. 187, Prop. 5.5]); then the fact that M is irreducible implies that M is locally irreducible. By Corollary 4 in [De] we have that M is half-conformally flat. Proposition (3.6) implies that M is either conformally flat or Kähler.

Irreducible locally symmetric spaces that are conformally flat have constant sectional curvatures, since they are Einstein. This and Proposition (3.8) imply that if M is locally symmetric then M is isometric either to the sphere S^4 or to the complex projective space $\mathbf{CP}^2$ with their standard metrics.

If $G = SU(2)$, Berger also proved that M is Ricci-flat. By Proposition (3.5) M is conformally flat, and this together with the fact that it is Ricci-flat implies that the sectional curvatures vanish, which contradicts that M is simply connected.

We were left with two possibilities for G: $SO(4)$ and $U(2)$. As we already observed, the nonnegativity of the curvature on totally isotropic 2-planes implies that harmonic 2-forms are parallel (Remark 3.4). This implies, by the holonomy principle, that if $G = SO(4)$ then M has the real cohomology of S^4, and if $G = U(2)$ then M has the real cohomology of $\mathbf{CP}^2$ and is a Kähler manifold. In the former case the second Betti number $b_2 = 0$. Since M is simply connected, this fact implies that $H_2(M, \mathbf{Z}) = 0$. Now, the solution of the Poincaré conjecture for dimension 4 [Fr] implies that in this case M is homeomorphic to S^4. In the latter case we have $b_2 = 1$ and $H_2(M, \mathbf{Z}) = \mathbf{Z}$ and, since the intersection form is ± 1, by a result of Whitehead [Wh] M is homotopy equivalent to $\mathbf{CP}^2$. A result of Yau [Ya] implies that a Kähler manifold homotopy equivalent to $\mathbf{CP}^2$ is biholomorphic to $\mathbf{CP}^2$. □

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