On the Darboux–Picard Theorem in $\mathbf{C}^n$

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I. Introduction

In one complex variable we have the following Darboux–Picard theorem.

THEOREM. *Let D be an open disc, and let $f:\bar{D}\to\mathbf{C}$ be continuous and satisfy:*

(i) *f is holomorphic in D, and*
(ii) *f is one-to-one on bD.*

Then f is one-to-one throughout $\bar{D}$ and $f(D)$ is the inside of the Jordan curve $f(bD)$.

For a proof see, for instance, Burckel [1, p. 310].

In this note we will show that the same result still holds if D is sitting in $\mathbf{C}^n$ for $n\geq 2$. But first, a simple example shows that if we map the unit disc in $\mathbf{C}$ into some $\mathbf{C}^n$ with $n>1$, then in general the conclusion does not hold.

EXAMPLE. Let U be the unit disc in $\mathbf{C}$. Define $G:\bar{U}\to\mathbf{C}^2$ by

$$z\mapsto(z(z-\tfrac{1}{2})(z+\tfrac{1}{2}),2(z-\tfrac{1}{2})(z+\tfrac{1}{2})).$$

Then we have $G(\frac{1}{2})=G(-\frac{1}{2})=(0,0)$ and G is one-to-one on bU.

Here is our main result.

MAIN THEOREM. *Let $D\subseteq\mathbf{C}^n$, $n\geq 2$, be a bounded domain with bD homeomorphic to S^{2n-1}, and let $f=(f_1,\dots,f_n):\bar{D}\to\mathbf{C}^n$ be a mapping such that $f_j\in H(D)\cap C^0(\bar{D})$ for $j=1,\dots,n$. Suppose that f is one-to-one on bD; then f is one-to-one throughout $\bar{D}$.*

Some related problems were considered in Globevnik and Stout [2].

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II. Proof of the Main Theorem

We first note that the restriction of f to the boundary bD is a homeomorphism. Hence, by our hypotheses, $f(bD)$ is homeomorphic to S^{2n-1} and $f(bD)$ separates $\mathbf{C}^n$ into two components, a bounded component called Ω and an unbounded component called Ω_∞.

We claim that $f(D) \subseteq \bar{\Omega} = \Omega \cup f(bD)$. If not, since $f(\bar{D})$ is compact, there exists a point $x \in D$ such that $f(x) = y \in \Omega_\infty \cap b(f(\bar{D}))$ and

$$\begin{aligned} \operatorname{dist}(y, \bar{\Omega}) &= \max\{\operatorname{dist}(z, \bar{\Omega}) \mid z \in f(\bar{D})\} \\ &= \delta. \end{aligned}$$

Hence there exists a point $y_1 \in \mathbf{C}^n - f(\bar{D})$ as well as a small $\epsilon > 0$ such that $\overline{B(y_1, \epsilon)} \subseteq (\mathbf{C}^n - f(\bar{D})) \cap B(y, \delta/3)$. Note that $f(bD) \cap B(y, \delta/3) = \emptyset$.

Now consider the complex line $\mathbf{L}$ through the points y and y_1. $\mathbf{L}$ is the intersection of $n-1$ hyperplanes $a_{j1}w_1 + \cdots + a_{jn}w_n = c_j$ for $j = 1, \ldots, n-1$ in $\mathbf{C}^n$. Then the pullback of $\mathbf{L}$, that is,

$$f^{-1}(\mathbf{L}) = \{z \in \bar{D} \mid a_{j1}f_1(z) + \cdots + a_{jn}f_n(z) = c_j, j = 1, \ldots, n-1\},$$

defines a complex subvariety V in D with $\dim_{\mathbf{C}} V \geq 1$. Obviously, $x \in V$ and $x \notin bD \cap f^{-1}(\mathbf{L})$. Also one can find an $M > 0$ such that

$$f(bD) \cap \mathbf{L} \subseteq B(y_1, M) \cap \mathbf{L}.$$

Therefore, by Runge's approximation theorem there exists a rational function $h(w)$ defined on $\mathbf{L}$ with exactly one pole at y_1 with $|h(y)| \geq 10$ and

$$\max|h(w)| \leq 1 \quad \text{for } w \in (\overline{B(y_1, M)} - B(y, \delta/3)) \cap \mathbf{L}.$$

This implies that $h \circ f(z)$ is a holomorphic function on $\bar{V}$ and that $|h \circ f(z)|$ attains its maximum at some interior point. This contradicts the maximum principle (i.e., see Gunning and Rossi [3]). So we have $f(D) \subseteq \bar{\Omega}$.

Next we show that $f(D) \subseteq \Omega$. Once this is proved, it is easy to see that $f: D \to \Omega$ is a proper map. This claim will be proved using a similar argument. Suppose that there exists $p \in bD$ and $p_0 \in D$ such that $f(p) = f(p_0) = q_0 \in f(bD)$.Consider the subvariety with boundary $\bar{V}_0$ of $\bar{D}$ given by

$$\bar{V}_0 = \{z \in \bar{D} \mid f(z) = q_0\}.$$

Then we must have $\dim_{\mathbf{C}} V_0 = 0$. For suppose that $\dim_{\mathbf{C}} V_0 \geq 1$. Since f is one-to-one on the boundary, we see that $\bar{V}_0 \cap bD = \{p\}$. Therefore, the maximum principle shows that the restrictions of both $|e^{z_j}|$ and $|e^{-z_j}|$ to $\bar{V}_0$ attain their maxima at p for $j = 1, \ldots, n$. Hence $x_j = \operatorname{Re} z_j$ is constant on connected components of $\bar{V}_0$. Similarly, $y_j = \operatorname{Im} z_j$ is also constant on any connected components of $\bar{V}_0$. This gives the desired contradiction. Hence $\dim_{\mathbf{C}} V_0 = 0$.

It follows that p_0 is an isolated point of V_0. Now choose a polydisc $\Delta^n = \Delta^n(p_0; r) \subset\subset D$ centered at p_0 with the same radius r in each direction so that

$$\bar{V}_0 \cap \overline{\Delta^n(p_0; r)} = \{p_0\}.$$

Since the boundary $b\Delta^n$ is a compact subset, we see that $f(b\Delta^n)$ is a compact subset of $\bar{\Omega}$ and $q_0 \notin f(b\Delta^n)$. Put $d = \operatorname{dist}(q_0, f(b\Delta^n))$. Since q_0 is a boundary point of $\bar{\Omega}$, there exists a point q_1 in $\Omega_\infty \cap B(q_0, d/10)$. Let $\mathbf{L}$ be the complex line through q_0 and q_1, and let $\bar{W} = f^{-1}(\mathbf{L})$ be the complex subvariety of $\bar{D}$ defined as before. Then there exists an $M > 0$ such that

$$f(W \cap (b\Delta^n)) \cap \mathbf{L} \subseteq (B(q_0, M) - B(q_0, d/2)) \cap \mathbf{L}.$$

Hence (again by Runge's approximation theorem) there exists a rational function $g(w)$ on $\mathbf{L}$ with exactly one pole at q_1 with $|g(q_0)| \geq 10$ and

$$\max|g(w)| \leq 1 \quad \text{for } w \in f(W \cap (b\Delta^n)) \cap \mathbf{L}.$$

It follows that the modulus of the nonconstant holomorphic function $g \circ f|_{\overline{W \cap \Delta^n}}$ attains its maximum at an interior point of $W \cap \Delta^n$. This is impossible. Hence $f^{-1}(q_0) = \{p\}$ and $f(D) \subseteq \Omega$.

Thus we have shown that $f: D \to \Omega$ is a proper map. Let $J = \det(f')$ be the Jacobian associated with f, and let $Z = Z(J)$ be the zeros of J in D. Denote by S the subvariety $f(Z)$ of Ω. Then a standard result (i.e., see Rudin [4]) shows that

$$f: D - f^{-1}(S) \to \Omega - S$$

is an m-to-one covering map for some $m \geq 1$. Here m is called the *multiplicity* of f. The following lemma is proved in Rudin [5].

LEMMA. *If $f: D \to \Omega$ is a proper holomorphic map with multiplicity m, then there exist m pairwise f-related sequences $\Sigma_1, \ldots, \Sigma_m$ in D that converge to distinct points $\zeta_1, \ldots, \zeta_m$ of bD.*

Here two sequences $\{a_i\}$ and $\{b_i\}$ in the domain of f are said to be f-related if $f(a_i) = f(b_i)$ for all i. Since, by our hypotheses, f is continuous up to the boundary and one-to-one on the boundary, we see that $m = 1$. This then implies that f is one-to-one throughout $\bar{D}$, and the proof of the main theorem is now completed. □

Finally, we remark that the proof given here also works for the 1-dimensional case.

References

[1] R. B. Burckel, *An introduction to classical complex analysis,* vol. 1, Academic Press, New York, 1979.

[2] J. Globevnik and E. L. Stout, *The ends of discs,* Preprint series, Dept. Math., Univ. Ljubljana, 23, No. 129 (1985), 201–219.

[3] R. C. Gunning and H. Rossi, *Analytic functions of several complex variables,* Prentice-Hall, Englewood Cliffs, NJ, 1965.

[4] W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer, New York, 1980.
[5] ———, *Proper holomorphic maps and finite reflections groups,* Indiana Univ. Math. J. 31 (1982), 701–720.

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