Weighted Spaces of Holomorphic Functions on Balanced Domains

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Dedicated to Professor George Maltese on the occasion of his 60th birthday

Weighted spaces $HV_0(G)$ and $HV(G)$ of holomorphic functions as well as weighted inductive limits $\mathcal{V}_0H(G)$ and $\mathcal{V}H(G)$ of spaces of holomorphic functions arise naturally in various applications of functional analysis, for example, in partial differential equations and convolution equations, complex and Fourier analysis, distribution theory, and spectral theory. However, many (structural) properties of these spaces which would be very helpful in concrete analytical investigations are rather hard to prove in a general context. In the present article, attention is restricted to (increasing systems V and decreasing sequences $\mathcal{V}$ of) *radial* weights on *balanced* domains $G \subset \mathbf{C}^N$ ($N \geq 1$), which makes it possible to apply arguments involving the Taylor series of holomorphic functions about zero. In this way, we obtain some progress, which is even interesting in the case of Banach spaces of this type.

More specifically, in Section 1 we use (Fejér's result on) the contractive properties of the Cesàro means of the Taylor series of functions in the disk algebra to derive remarkable consequences for spaces of holomorphic functions on arbitrary balanced open sets $G \subset \mathbf{C}^N$. Our method leads to simple proofs that the spaces $HV_0(G)$ and $\mathcal{V}_0H(G)$ have the bounded approximation property whenever they contain the polynomials, and that then the polynomials are dense. Similar results are also true for the larger spaces $HV(G)$ and $\mathcal{V}H(G)$, but only under certain (quite natural) weaker topologies. The bidualities $((HV_0(G))_b')_b' = HV(G)$ and $((\mathcal{V}_0H(G))_b')_i' = \mathcal{V}H(G)$ (which were established in [5] under slightly stronger hypotheses) actually hold in the present generality. The results of Section 1 serve as a basis for the developments in the subsequent sections.

Section 2 is devoted to a problem which had already been raised in [6] and [7] (and is part of a more general problem that has interested various authors): Can one interchange the inductive limit $\mathcal{V}_0H(G) = \text{ind}_n\, H(v_n)_0(G)$ and the ϵ-(tensor) product with an arbitrary Banach space X, in particular if $\mathcal{V}$ satisfies the condition that for each $n \in \mathbf{N}$ there is $m > n$ such that v_m/v_n vanishes at infinity on G? This is closely related to the important question of whether one can obtain a *projective description* $\mathcal{V}_0H(G,E) = H\bar{V}_0(G,E)$

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for spaces of vector-valued holomorphic functions. Rather surprisingly, for a (DFS)-space $\mathcal{V}_0 H(G)$, such a problem is equivalent to asking if Grothendieck's "problème des topologies" has a positive solution for every pair $((\mathcal{V}_0 H(G))'_b, X)$, X Banach. Since Taskinen's counterexamples [34] to Grothendieck's (general) problem, some positive results have been obtained (see [11] and [36]), but they involve the bounded approximation property. Indeed, it was with this application in mind that we proved our results in Section 1, permitting us now to deduce that the topological equalities

$$\mathcal{V}_0 H(G) \epsilon E = \operatorname{ind}_n (H(v_n)_0(G) \epsilon E) \quad \text{and} \quad \mathcal{V}_0 H(G,E) = H\bar{V}_0(G,E)$$

hold for quasicomplete locally convex (l.c.) spaces E with the countable neighborhood property, and for (DFS)-spaces $\mathcal{V}_0 H(G)$ in the present setting (of radial weights v_n on balanced domains G).

In the first part of Section 3, we explain the relation of our results to some of the work of Kaballo and Vogt [23] and Hollstein [19] on (lifting theorems for vector-valued holomorphic functions and) tensor sequences. The articles of Shields and Williams [30; 31] and of Kaballo [22] lead to interesting examples and give rise to a nice question in Banach space theory. At the end of the paper, we establish some remarkable vector-valued generalizations of the (bi-) dualities $((HV_0(G)'_b)'_b = HV(G)$ and $((\mathcal{V}_0 H(G))'_b)'_i = \mathcal{V}H(G)$.

NOTATION. Our notation on locally convex spaces is standard; see for example Grothendieck [18], Jarchow [20], and Pérez Carreras and Bonet [29]. For a l.c. space E, E^* denotes the space of all linear functionals on E, while E' is the topological dual and E'_b the strong dual; if X is a normed space, the dual Banach space is simply denoted by X'. A l.c. space E has the *bounded approximation property* if there exists an equicontinuous net of finite rank operators on E which converges pointwise to the identity. For l.c. inductive limits and their (strong) regularity properties, see [29]; for topological tensor products, see [18].

If G is an open subset of $\mathbf{C}^N$ ($N \geq 1$), then $H(G)$ will be the space of all holomorphic functions on G endowed with the topology τ_0 of uniform convergence on the compact sets $K \subset G$. For our notation on *weighted spaces* and *weighted inductive limits*, see [7]. If V is a system of nonnegative upper semicontinuous functions on G, we define the weighted spaces of holomorphic functions by

$$HV(G) := \{f \in H(G) : \text{for each } v \in V,\ p_v(f) := \sup_{z \in G} v(z)|f(z)| < \infty\},$$

$$HV_0(G) := \{f \in H(G) : \text{for each } v \in V,\ vf \text{ vanishes at infinity on } G; \text{ that is, for each } \epsilon > 0 \text{ there is } K \subset G \text{ compact with } v(z)|f(z)| < \epsilon \text{ for } z \in G \backslash K\}.$$

The weighted topology on $HV(G)$ (and $HV_0(G)$), denoted by τ_V, is given by the system $(p_v)_{v \in V}$ of seminorms. If V is reduced to a single weight v, we sometimes write $Hv(G)$ and $Hv_0(G)$ instead of $HV(G)$ and $HV_0(G)$, respectively. If $\mathcal{V} = (V_n)_{n \in \mathbf{N}}$ is a decreasing sequence of systems V_n of weights (i.e., $V_{n+1} \leq V_n$ for each $n \in \mathbf{N}$, cf. [7]), we introduce the weighted inductive limits

$$\mathcal{V}H(G) := \text{ind}_n HV_n(G) \quad \text{and} \quad \mathcal{V}_0 H(G) := \text{ind}_n H(V_n)_0(G).$$

For a decreasing sequence $\mathcal{V} = (v_n)_{n \in \mathbf{N}}$ of weights, the *regularly decreasing* condition on $\mathcal{V}$ is discussed in [7]. We will also use the vector-valued analogs of weighted spaces and weighted inductive limits, as well as the weighted spaces of continuous functions (mainly in Sections 2 and 3).

1. Weighted Spaces and Weighted Inductive Limits of Spaces of Holomorphic Functions: Radial Weights on Balanced Domains

In this section, we exploit the properties of the (first) Cesàro means of (the partial sums of) the Taylor series of a holomorphic function about zero, in order to derive some interesting facts for weighted spaces of holomorphic functions and for the corresponding weighted inductive limits in the case that all the weights are radial on a balanced domain $G \subset \mathbf{C}^N$.

Thus, unless something else is stated explicitly, G will always be a balanced open set in $\mathbf{C}^N$ ($N \geq 1$). Then each $f \in H(G)$ has a Taylor series representation about zero,

$$f(z) = \sum_{k=0}^{\infty} p_k(z), \quad z \in G,$$

where p_k is a k-homogeneous polynomial ($k = 0, 1, \ldots$). The series converges to f uniformly on each compact subset of G. The *Cesàro means* of the partial sums of the Taylor series of f are denoted by $C_n(f)$ ($n = 0, 1, \ldots$); that is,

$$[C_n(f)](z) = \frac{1}{n+1} \sum_{l=0}^{n} \left(\sum_{k=0}^{l} p_k(z) \right), \quad z \in G.$$

Each $C_n(f)$ is a polynomial (of degree $\leq n$), and $C_n(f) \to f$ uniformly on every compact subset of G ($f \in H(G)$ arbitrary). By the Cauchy inequalities, the coefficients of the Taylor polynomials, and hence the polynomials $C_n(f)$, depend continuously on $f \in H(G)$ with respect to the compact-open topology τ_0 (on $H(G)$). (Of course, it would be possible to treat domains $G \subset \mathbf{C}^N$ which are ξ-balanced for some $\xi \in \mathbf{C}^N$, but for simplicity we shall take $\xi = 0$ here.)

The following lemma (which also motivates the definition of radial weights on balanced open sets) is well known; for example, see (the proof of) Mujica [27, Prop. 5.2]. We include it here for the convenience of the reader, together with a short and simple proof by reduction to the disk algebra.

1.1. LEMMA. *For $f \in H(G)$ and $z \in G$, we always have:*

$$|[C_n(f)](z)| \leq \max_{|\lambda|=1} |f(\lambda z)|, \quad n = 0, 1, \ldots.$$

Proof. Fix f and z. Since G is balanced, $\{\lambda z : \lambda \in \bar{D}\} \subset G$, where D is the open unit disk in $\mathbf{C}$. If we define the function $g = g_z$ of one complex variable

λ by $g(\lambda):=f(\lambda z)$ for each $\lambda \in \bar{D}$, then clearly $g \in A(\bar{D}):=\{h: \bar{D} \to \mathbf{C}$ continuous; $h|_D$ holomorphic$\}$. (In fact, g is defined and holomorphic on an open set containing $\bar{D}$.) We note that for arbitrary $\lambda \in \bar{D}$,

$$g(\lambda)=f(\lambda z)=\sum_{k=0}^{\infty} p_k(\lambda z)=\sum_{k=0}^{\infty} p_k(z)\lambda^k, \quad \text{where } f(z)=\sum_{k=0}^{\infty} p_k(z),$$

whereby the series on the right-hand side of the equality for $g(\lambda)$ is exactly the Taylor series of g about zero. At this point, the well-known inequality (e.g., cf. [25, p. 37])

$$\max_{|\lambda|=1}|[C_n(g)](\lambda)| \le \max_{|\lambda|=1}|g(\lambda)| \quad \text{for } g \in A(\bar{D}),\ n \in \mathbf{N}_0$$

implies the desired estimate

$$|[C_n(f)](z)|=|[C_n(g)](1)| \le \max_{|\lambda|=1}|g(\lambda)|=\max_{|\lambda|=1}|f(\lambda z)|. \qquad \square$$

Now let V be a system of nonnegative continuous functions on G such that the topology τ_V of $HV(G)$ is stronger than τ_0 and all $v \in V$ are *radial* in the sense that

$$v(\lambda z)=v(z) \quad \text{for all } z \in G \text{ and all } \lambda \in \mathbf{C} \text{ with } |\lambda|=1.$$

We will also assume (unless stated otherwise) that *the polynomials are contained in* $HV_0(G)$. For G bounded, this is equivalent to requiring that each $v \in V$ extends continuously to $\bar{G}$ with $v|_{\partial G} \equiv 0$, while for $G=\mathbf{C}^N$, the assumption means exactly that each $v \in V$ is *rapidly decreasing* (at ∞).

1.2. PROPOSITION. *Let V be a system of nonnegative continuous radial functions on a balanced open set $G \subset \mathbf{C}^N$. We assume that τ_V is stronger than τ_0 and that $HV_0(G)$ contains all the polynomials. Then the sequence $(C_n)_{n \in \mathbf{N}_0}$, $C_n: f \to C_n(f)$ for $f \in HV(G)$ and $n \in \mathbf{N}_0$, has the following properties:*

(a) *Each C_n is a continuous linear operator of finite rank from $HV(G)$ into $HV_0(G)$.*

(b) *For all $n \in \mathbf{N}_0$, $v \in V$, and $f \in HV(G)$, we have*

$$\sup_{z \in G} v(z)|[C_n(f)](z)| \le \sup_{z \in G} v(z)|f(z)|,$$

whence $(C_n)_n$ is equicontinuous in $\mathcal{L}(HV(G), HV_0(G))$. (If V is reduced to a single weight v, then $(C_n)_n$ belongs to the unit ball of the space $\mathcal{L}(Hv(G), Hv_0(G))$ of bounded operators.)

(c) *There exists a basis $\mathcal{B}$ of absolutely convex bounded sets in $HV(G)$ such that for each $B \in \mathcal{B}$ and $n=0,1,\dots,$*

$$C_n(B) \subset B \cap HV_0(G).$$

(d) *Let $\bar{\tau}$ be the strongest locally convex topology on $HV(G)$ which coincides with the compact-open topology τ_0 on all bounded sets. Then $(C_n)_n$ is also equicontinuous in the space*

$$\mathcal{L}((HV(G), \bar{\tau}), (HV_0(G), \bar{\tau}|_{HV_0(G)})).$$

(e) *For each $f \in HV_0(G)$, $C_n(f) \to f$ holds in the weighted topology τ_V of this space; for each $f \in HV(G)$, $C_n(f) \to f$ with respect to $\bar{\tau}$.*

Proof. (a) Clearly, C_n is a linear operator of finite rank, and our assumption on the polynomials implies that $C_n(HV(G)) \subset HV_0(G)$, $n = 0, 1, \ldots$. Continuity is clear since the topology of $HV(G)$ is stronger than τ_0.

For (b), fix $f \in HV(G)$, $v \in V$, and $n \in \mathbf{N}_0$. Lemma 1.1 gives

$$\sup_{z \in G} v(z)|[C_n(f)](z)| \le \sup_{z \in G} v(z)(\max_{|\lambda|=1} |f(\lambda z)|) = \sup_{z \in G} v(z)|f(z)|$$

because v is radial.

Concerning (c), we first note that the sets $B = B((M_v)_v)$,

$$B = \bigcap_{v \in V} \{f \in HV(G): v(z)|f(z)| \le M_v \text{ for all } z \in G\},$$

yield a basis $\mathcal{B}$ of absolutely convex bounded sets in $HV(G)$ as $(M_v)_{v \in V}$ runs through all systems of positive numbers (indexed by V). In view of Lemma 1.1 and the fact that each $v \in V$ is radial, we get for arbitrary $B \in \mathcal{B}$, $f \in B$, $v \in V$, and $n \in \mathbf{N}_0$,

$$v(z)|[C_n(f)](z)| \le v(z)(\max_{|\lambda|=1} |f(\lambda z)|) = \max_{|\lambda|=1} v(\lambda z)|f(\lambda z)| \le M_v \quad \text{for all } z \in G;$$

that is, $C_n(B) \subset B \cap HV_0(G)$, $n = 0, 1, \ldots$, as claimed.

(d) $\bar{\tau}$ is a *generalized inductive limit topology;* the sets $\Gamma(\bigcup_{B \in \mathcal{B}} U_B \cap B)$, where each U_B ($B \in \mathcal{B}$, $\mathcal{B}$ as in (c)) is a 0-neighborhood in $HV(G)$ for the compact-open topology τ_0 (and Γ denotes the absolutely convex hull), form a basis of 0-neighborhoods for $\bar{\tau}$. Since $(C_n)_n$ is clearly equicontinuous on $HV(G)$ with respect to τ_0 (see e.g. Lemma 1.1), the desired equicontinuity easily follows from this description of a 0-neighborhood basis for $\bar{\tau}$ and from (c).

To see the first part of (e), fix $f \in HV_0(G)$, $v \in V$, and $\epsilon > 0$. There is a balanced compact subset K of G with $v(z)|f(z)| \le \epsilon/2$ for all $z \in G \backslash K$. Take $M > 0$ such that $\max_{z \in K} v(z) \le M$. Since $C_n(f) \to f$ uniformly on K, we can choose $N \in \mathbf{N}$ large enough so that $\max_{z \in K}|f(z) - [C_n(f)](z)| \le \epsilon/M$ for each $n \ge N$. For any such natural number n, we have

$$\begin{aligned}
\sup_{z \in G} v(z)|f(z) - [C_n(f)](z)| &\le \max(\max_{z \in K} v(z)|f(z) - [C_n(f)](z)|, \sup_{z \in G \backslash K} \ldots) \\
&\le \max\left(M\frac{\epsilon}{M}, \frac{\epsilon}{2} + \sup_{z \in G \backslash K} v(z)|[C_n(f)](z)|\right) \\
&\le \max\left(\epsilon, \frac{\epsilon}{2} + \sup_{z \in G \backslash K} v(z)[\max_{|\lambda|=1} |f(\lambda z)|]\right) \\
&\le \max\left(\epsilon, \frac{\epsilon}{2} + \sup_{z \in G \backslash K} v(z)|f(z)|\right) \le \epsilon,
\end{aligned}$$

where we have again utilized Lemma 1.1 and v radial, as well as the fact that

$$z \in G \backslash K \Rightarrow \lambda z \in G \backslash K \text{ for each } |\lambda| = 1,$$

which holds because K is balanced. The second assertion of (e) is obvious since $C_n(f) \to f$ in τ_0 for each $f \in HV(G)$, and the set $\{C_n(f): n \in \mathbf{N}_0\}$ is always bounded in $HV_0(G)$ by (b). □

The definition of the topology $\bar{\tau}$ and the description of a 0-neighborhood basis for this topology which is used in the proof of Proposition 1.2(d) remain valid for "arbitrary" systems V of (not necessarily radial) weights on arbitrary (not necessarily balanced) open sets $G \subset \mathbf{C}^N$. The next definition, which is prompted by the classical case of the strict topology on $H^\infty(G)$ and [8, §3], and the subsequent remarks also work in this (more general) context.

Let $B_0^+(G)$ denote the set of all nonnegative upper semicontinuous (u.s.c.) functions on G that vanish at infinity on G; each such function is bounded. If G is balanced, and hence has a basis of compact sets which are also balanced, then each $\varphi \in B_0^+(G)$ is certainly dominated by a radial function in $B_0^+(G)$, and hence it will suffice to consider the radial elements of $B_0^+(G)$. In general, every $\varphi \in B_0^+(G)$ is dominated by a continuous $\psi \in B_0^+(G)$, and it is enough to consider continuous functions in $B_0^+(G)$. Now put

$$W := B_0^+(G)V = \{\varphi v : \varphi \in B_0^+(G),\ v \in V\}.$$

Considering the weighted spaces $HW(G), HW_0(G)$ of holomorphic functions associated with the system W, it is quite easy to see (e.g., cf. Ernst [17, Prop. 1 and 2] for the corresponding spaces of continuous functions) that $HW(G) = HW_0(G)$ holds (algebraically, and hence topologically), and that $HV(G)$ is continuously embedded into this space and equals it algebraically; moreover, the two spaces even have the same bounded sets on which the weighted topology τ_W coincides with τ_0. Now we can clearly conclude that $\tau_W \le \bar{\tau}$.

Each bounded subset of $HV(G)$ is τ_0-compact, and hence $\bar{\tau}$-compact, by Montel's theorem; it is *a fortiori* τ_W-compact. Conversely, it was already stated that every τ_W-bounded set is bounded in $HV(G)$. Thus, τ_V, $\bar{\tau}$, and τ_W have the same bounded sets. $\bar{\tau}$ and τ_W are semi-Montel topologies. τ_W is stronger than τ_0 and complete. On the other hand, if $HV(G)$ is bornological, then $\bar{\tau}$ must also be weaker than the original topology τ_V of $HV(G)$.

Some justification for the introduction of the topology $\bar{\tau}$ comes from Mujica's duality (see [26] together with [16, p. 417]). In fact, we can now show the following.

1.3. PROPOSITION. *Let V be a system of nonnegative continuous functions on an open subset G of $\mathbf{C}^N$ such that τ_V is stronger than τ_0. If $HV(G)$ is bornological, then it is the strong dual of the complete barrelled space*

$$F := (HV(G), \bar{\tau})'_b = (HV(G), \bar{\tau})'_c.$$

Proof. We utilize methods analogous to [5, Example 3.A]: $HV(G)$ clearly has a basis of absolutely convex 0-neighborhoods which are τ_0-closed (in $HV(G)$). As one can see from the beginning of the proof of 1.2(c) (where the stronger assumptions of that proposition were not yet needed), each bounded set in $HV(G)$ is contained in an absolutely convex bounded set B which is even closed in $(H(G), \tau_0)$ (or, for that matter, closed in $H(G)$ with respect to the topology of pointwise convergence on G). Since the topology of $HV(G)$ is stronger than τ_0, Montel's theorem implies that B is τ_0-compact (in $H(G)$ or) in $HV(G)$. Since τ_V and $\bar{\tau}$ have the same bounded sets and $\bar{\tau}$ is semi-Montel, we may now apply [5, Thm. 1 and Cor. 2] to conclude. □

If $HV(G)$ is a Fréchet space (e.g., $V=(v_n)_{n\in\mathbf{N}}$ is a sequence of positive functions), then F in Proposition 1.3 is a (complete barrelled) (DF)-space, and [5, Cor. 5] shows how additional topological properties of $HV(G)$ are reflected in topological properties of F. In particular, for quasinormable Fréchet spaces $HV(G)$, F must be a boundedly retractive (LB)-space (and hence bornological).

In the (simple) case that $V=\{v\}$ for a single (strictly positive continuous) function v, one can prove that $HW(G)=HW_0(G)=(Hv(G),\bar{\tau})$ holds topologically and is a complete semi-Montel (gDF)-space (see [8, §3]).

Let us now return to systems of radial weights on balanced domains. Proposition 1.2(e) and $\tau_W \leq \bar{\tau}$ imply the first part of the following remark. The second part becomes obvious by looking at the proof of 1.2(b) (and making use of the fact that one may restrict attention to radial functions in $B_0^+(G)$).

1.4. Remark. Under the hypotheses of Proposition 1.2, $C_n(f)$ converges to f in $HW(G)$ for every $f\in HW(G)$, and also $(C_n)_n$ is equicontinuous in $\mathcal{L}(HW(G), HW(G))$.

We are now ready for the main theorems of this section.

1.5. Theorem. *Under the hypotheses of Proposition 1.2, the following assertions hold:*

(a) *$HV_0(G)$ has the bounded approximation property (and even the metric approximation property in the case of a Banach space; i.e., if $V=\{v\}$ for a single function v). Moreover, the polynomials are dense in $HV_0(G)$.*
(b) *$(HV(G),\bar{\tau})$ has the bounded approximation property, and the polynomials are dense in this space, too. The same result holds for $HW(G)=HW_0(G)$ instead of $(HV(G),\bar{\tau})$.*
(c) *For each bounded set $B\subset HV(G)$, there is an absolutely convex bounded set C in $HV_0(G)$ such that B is contained in the τ_0-closure of C in $HV(G)$. (For $V=\{v\}$, the unit ball of $Hv(G)$ is contained in the τ_0-closure of the unit ball of $Hv_0(G)$.)*
(d) *If $HV(G)$ is bornological (which holds, for example, if V is a sequence $(v_n)_{n\in\mathbf{N}}$), then $((HV_0(G))_b')_b'$ is canonically isomorphic to $HV(G)$ (and the biduality holds isometrically for $V=\{v\}$). In this case, $HV_0(G)$ must be quasibarrelled and distinguished, and an algebraic identity $HV(G)=HV_0(G)$ implies that $HV_0(G)$ is reflexive.*

(e) *If $HV(G)$ is bornological then we also have the following canonical dualities:*

$$(HV_0(G))'_b = (HV(G), \bar{\tau})'_b = (HV(G), \bar{\tau})'_c \text{ and } ((HV(G), \bar{\tau})'_b)'_b = HV(G).$$

Proof. (a) is a simple consequence of Proposition 1.2(b) and (e), since the equicontinuous sequence $(C_n|_{HV_0(G)})_n$ of linear operators of finite rank converges pointwise to the identity of $HV_0(G)$, and each $C_n(f)$ is a polynomial ($f \in HV_0(G)$, $n \in \mathbf{N}_0$). (b) follows similarly from Proposition 1.2(d) and (e) (and from Remark 1.4). Keeping in mind that $C_n(f)$ converges to $f \in H(G)$ in the compact-open topology, one sees that by (the proof of) 1.2(c), (c) must be true (and the proof in the Banach space case is simpler).

To verify (d) and (e), one looks at the proof of Proposition 1.3, utilizes (c) (and the argument in the proof of [5, Prop. 10] whereby condition (ii) of [5, Thm. 6] is always satisfied) and then applies [5, Thm. 6 and Remark] (resp. [8, Cor. 1.2]) to conclude. (The last equation in (e) holds "in general"—see Proposition 1.3—and is included here only for completeness; however, the first duality requires the property stated in (c) and is even equivalent to this property.) □

Proposition 1.3 and parts (d) and (e) of Theorem 1.5 generalize and improve [5, Example 3.A] and [8, §§2 and 3] to some extent.

We next turn to the weighted inductive limits (of spaces of holomorphic functions) $\mathcal{V}H(G) = \text{ind}_n HV_n(G)$ and $\mathcal{V}_0H(G) = \text{ind}_n H(V_n)_0(G)$, where $\mathcal{V} = (V_n)_{n\in\mathbf{N}}$ is a decreasing sequence of systems of nonnegative continuous functions on an open set $G \subset \mathbf{C}^N$, and where it will be assumed that each $HV_n(G)$ has a stronger topology than τ_0. For each $n \in \mathbf{N}$, $\bar{\tau}_n$ denotes the strongest l.c. topology on $HV_n(G)$ which coincides with τ_0 on all bounded sets, and $W_n := B_0^+(G)V_n$, $n = 1, 2, \ldots$. Then $(HV_n(G), \bar{\tau}_n)_n$ and $(HW_n(G) = H(W_n)_0(G))_n$ are inductive sequences; we write $(\bar{\mathcal{V}}H(G), \bar{\tau})$, $(\mathcal{W}H(G), \tau_{\mathcal{W}})$ for the l.c. inductive limits and their topologies, respectively. The algebraic equality $\mathcal{V}H(G) = \bar{\mathcal{V}}H(G) = \mathcal{W}H(G)$ holds, and we have $\tau_0 \le \tau_{\mathcal{W}} \le \bar{\tau}$, while $\bar{\tau}$ is weaker than the original topology of $\mathcal{V}H(G)$ if all $HV_n(G)$ are bornological, $n = 1, 2, \ldots$ (see the remarks before Proposition 1.3).

1.6. THEOREM. *Let $\mathcal{V} = (V_n)_{n\in\mathbf{N}}$ be a decreasing sequence of systems V_n of nonnegative continuous and radial functions on a balanced open set $G \subset \mathbf{C}^N$. We suppose that each $HV_n(G)$ has a topology which is stronger than τ_0 and that each $H(V_n)_0(G)$ contains the polynomials, $n = 1, 2, \ldots$. Then we get:*

(a) *$\mathcal{V}_0H(G)$ has the bounded approximation property, and the polynomials are dense in $\mathcal{V}_0H(G)$.*

(b) *Both $\bar{\mathcal{V}}H(G)$ and $\mathcal{W}H(G)$ have the bounded approximation property, and the polynomials are also dense in these spaces.*

For (c) *and* (d) *below, we take each V_n to consist of a single* (*strictly positive continuous and radial*) *function v_n, $n = 1, 2, \ldots$; i.e., $\mathcal{V}H(G) = \text{ind}_n Hv_n(G)$ as well as $\mathcal{V}_0H(G) = \text{ind}_n H(v_n)_0(G)$ are* (*LB*)*-spaces. In this setting, we also get:*

(c) $\mathcal{V}_0 H(G)$ *has a total bounded set (viz., the unit ball of* $H(v_1)_0(G)$).

(d) $((\mathcal{V}_0 H(G))'_b)'_i = \mathcal{V}H(G)$ *holds canonically, and* $\mathcal{V}_0 H(G)$ *is a topological subspace of* $\mathcal{V}H(G)$. *(In particular, an algebraic equality* $\mathcal{V}_0 H(G) = \mathcal{V}H(G)$ *already implies that* $\mathcal{V}_0 H(G)$ *is reflexive.) If* $\mathcal{V}$ *is regularly decreasing (in the sense of* [7]), *then we even have* $((\mathcal{V}_0 H(G))'_b)'_b = \mathcal{V}H(G)$.

Proof. (a) By Proposition 1.2(b) and (e), $(C_n)_n$, which is now defined by $C_n: f \to C_n(f)$ for all $f \in \mathcal{V}H(G)$, is an equicontinuous sequence of linear operators of finite rank from $\mathcal{V}H(G)$ into $\mathcal{V}_0 H(G)$, and the restrictions of C_n to $\mathcal{V}_0 H(G)$ converge pointwise to the identity.

(b) follows from Proposition 1.2(d) and (e) in exactly the same way. It is also clear that (a) ⇒ (c) (under the more restrictive assumptions of this part). Incidentally, up to this point, it would not even have been necessary to deal only with *countable* inductive limits (as we have done here, for simplicity).

(d) is a direct consequence of [5, 13 (see also 15)] and Theorem 1.5. On the other hand, it improves [5, Example 3.B] slightly. □

Keeping the notation introduced before Theorem 1.6, we can consider the following *associated system* $\bar{V} = \bar{V}(\mathcal{V})$ of weights (cf. [7, 0.2]):

$$\bar{V} := \{\bar{v} \geq 0 \text{ u.s.c. on } G\text{: for each } n \in \mathbf{N} \text{ there are } v_n \in V_n \text{ and } \alpha_n > 0 \text{ with } \bar{v} \leq \inf_n \alpha_n v_n\}.$$

In our case (where G is locally compact and σ-compact), the proof of [7, 0.2, Prop.] shows that each function $\bar{v} \in \bar{V}$ is dominated by some $\tilde{v} \in \bar{V}$ which, on each compact set $K \subset G$, is actually a *finite* infimum of positive multiples of elements $v_n \in V_n$ $(n = 1, 2, \ldots)$. It now suffices to consider only continuous weights in $\bar{V}$, and if all systems V_n consist of radial functions on a balanced open set $G \subset \mathbf{C}^N$, then it is also enough to restrict attention to the radial elements of $\bar{V}$. From Theorem 1.5(a) and (b), we obtain the next corollary.

1.7. Corollary. *Under the hypotheses of (the first part of) Theorem 1.6, the three spaces* $H\bar{V}_0(G)$, $(H\bar{V}(G), \bar{\tau})$ *(where* $\bar{\tau}$ *is the strongest l.c. topology on* $H\bar{V}(G)$ *which coincides with* τ_0 *on all bounded sets), and* $H\bar{W}(G) = H\bar{W}_0(G)$, *where* $\bar{W} := B_0^+(G)\bar{V}$, *have the bounded approximation property and contain the polynomials as a dense subspace.*

While it has not been our main concern here to study the relations between the various spaces of holomorphic functions which were introduced before Theorem 1.6 and Corollary 1.7, we will now state some of the relevant results. (This may also serve for the orientation of the reader.) In the sequel, it is not necessary to assume that G is balanced and that the weights are radial.

First, it is immediate that $\mathcal{V}H(G)$ (resp., $\mathcal{V}_0 H(G)$) is continuously embedded into the weighted space $H\bar{V}(G)$ (resp., $H\bar{V}_0(G)$) and that the topology $\tau_{\bar{V}}$ of this space is stronger than τ_0. (Actually, these facts were already used in the proof of Corollary 1.7.) If each V_n is reduced to a single (strictly positive and continuous) function v_n, $n = 1, 2, \ldots$, then $H\bar{V}(G) = \mathcal{V}H(G)$

algebraically, the two spaces have the same bounded sets, and $\mathcal{V}H(G)$ is a regular inductive limit (see [7], where a sufficient condition for $\mathcal{V}H(G) = H\bar{V}(G)$ is given). Since $(H\bar{V}(G), \bar{\tau})$ is semi-Montel (by the remarks before Proposition 1.3) and the corresponding space of continuous functions coincides with the weighted space analogous to $H\bar{W}(G)$ (see Ernst [17, Prop. 3 and 4]), we can apply the Baernstein lemma (cf. [7, 0.4]) to see that

$$(H\bar{V}(G), \bar{\tau}) = H\bar{W}(G) = H\bar{W}_0(G)$$

holds topologically and is a complete semi-Montel (gDF)-space.

Next, as $\bar{V} = \bar{V}(\mathcal{V})$ is associated with $\mathcal{V} = (V_n)_n$, the following system $\bar{V}(\mathcal{W})$ is associated with $\mathcal{W} = (W_n = B_0^+(G)V_n)_n$:

$$\bar{V}(\mathcal{W}) := \{\bar{w} \geq 0 \text{ u.s.c. on } G\text{: for each } n \in \mathbf{N} \text{ there are } w_n = \varphi_n v_n \in W_n \text{ and } \alpha_n > 0 \text{ with } \bar{w} \leq \inf_n \alpha_n w_n = \inf_n \alpha_n \varphi_n v_n\}.$$

Since obviously, in the notation of Corollary 1.7, $\bar{W} = B_0^+(G)\bar{V} \leq \bar{V}(\mathcal{W}) \leq \bar{V}$ holds, we have the following diagram (in which arrows represent continuous embeddings):

$$\begin{array}{ccccc} \mathcal{V}H(G) & \rightarrow & \mathcal{W}H(G) & & \\ \downarrow & & \downarrow & & \\ H\bar{V}(G) & \rightarrow & H(\bar{V}(\mathcal{W}))(G) & \rightarrow & H\bar{W}(G). \end{array}$$

As a consequence, we get $H\bar{V}(G) = H(\bar{V}(W))(G) = H\bar{W}(G)$ algebraically, and the three spaces have the same bounded sets.

Stronger results can be obtained if each V_n consists of only one function v_n. Then we have $\mathcal{V}H(G) = \mathcal{W}H(G) = H\bar{V}(G) = H\bar{W}(G)$ algebraically, and all four spaces must have the same bounded sets—namely, the subsets of positive multiples of the unit balls in $Hv_n(G)$, $n = 1, 2, \ldots$. Hence $\mathcal{W}H(G)$ is a regular inductive limit, too. Moreover, our previous considerations (before Remark 1.4) show that $\bar{\mathcal{V}}H(G) = \mathcal{W}H(G)$ holds topologically, and that this is a complete semi-Montel (gDF)-space. (One can also deduce directly from [29, 8.1.7] that $\bar{\mathcal{V}}H(G)$ must now carry the finest l.c. topology that coincides with τ_0 on the bounded sets.) At this point, using $\mathcal{W}H(G) = \operatorname{ind}_n H(W_n)_0(G)$ as well as $H\bar{W}(G) = H\bar{W}_0(G)$, [8, §3], [7, Thm. 1.3(a)], and (again) the Baernstein lemma, it follows that also $\mathcal{W}H(G) = H\bar{W}(G)$ holds topologically.

Some of the preceding definitions and results can be found in Napalkov [28] (with slightly different proofs). In [28], it is proved that for a semi-Montel space $\mathcal{V}H(G)$, the following additional topological equalities hold:

$$\mathcal{V}H(G) = H\bar{V}(G) = \mathcal{W}H(G).$$

Actually, this can now be deduced simply by another application of the (ubiquitous) Baernstein lemma. (Under a slightly more restrictive hypothesis, the first equation had already been proved in [7, Thm. 1.6].)

This is the appropriate point to discuss Mujica's duality [26] for the (LB)-space $\mathcal{V}H(G)$.

1.8. Proposition. *Let $\mathcal{V} = (v_n)_n$ be a decreasing sequence of strictly positive continuous functions on an open subset G of $\mathbf{C}^N$.*

(a) *Then $\mathcal{V}H(G)$ is the inductive dual F_i' of the Fréchet space*

$$F := (\bar{\mathcal{V}}H(G), \bar{\tau})_b' = (\bar{\mathcal{V}}H(G), \bar{\tau})_c' = (\mathcal{W}H(G), \tau_{\mathcal{W}})_b' = (\mathcal{W}H(G), \tau_{\mathcal{W}})_c'$$

and hence complete. If $\mathcal{V}$ is regularly decreasing, F is quasinormable and $\mathcal{V}H(G) = F_b'$.

(b) *The strong topology of F_b' is stronger than the weighted topology $\tau_{\bar{V}}$ of $H\bar{V}(G)$.*

(c) *Consider the following assertions:*
(1) *$\mathcal{V}H(G) = H\bar{V}(G)$ holds topologically;*
(2) *$F_b' = H\bar{V}(G)$ holds topologically;*
(3) *$H\bar{V}(G)$ is a (DF)-space.*
In general, (1) ⇒ (2) *and* (2) ⇒ (3); *if the bounded subsets of $H\bar{V}(G)$ are metrizable* (*which is true if $\mathcal{V} = (v_n)_n$ satisfies condition* (D) *of* [3, II]), *then also* (3) ⇒ (1), *and* (1), (2), *and* (3) *are equivalent.*

Proof. (a) is a simple consequence of [5, Cor. 3(a)] (due to Mujica) and Remark 1.4.

(b) $H\bar{V}(G)$ has a basis of absolutely convex τ_0-closed 0-neighborhoods (cf. the proof of Proposition 1.3), which must be 0-neighborhoods in F_b' as one can see from the proof of [5, Cor. 2].

(c) By (a) and (b), (1) ⇒ (2) is obvious, as is (2) ⇒ (3). Now let $H\bar{V}(G)$ be a (DF)-space whose bounded subsets are metrizable. By the proof of Proposition 1.3, $H\bar{V}(G)$ has a basis of absolutely convex bounded sets which are τ_0-compact. Invoking [3, I, 1.5(a), 1.2(b), and 1.4(c)], we get that $H\bar{V}(G)$ is bornological, which clearly suffices to conclude that $\mathcal{V}H(G) = H\bar{V}(G)$ topologically. □

Returning to a sequence of radial functions on a balanced domain, we have the following.

1.9. Consequence. Under the hypotheses of Theorem 1.6(c) and (d), the following dualities hold:

$$(\mathcal{V}_0H(G))_b' = (\bar{\mathcal{V}}H(G), \bar{\tau})_b' = (\bar{\mathcal{V}}H(G), \bar{\tau})_c' \quad \text{and} \quad ((\bar{\mathcal{V}}H(G), \bar{\tau})_b')_i' = \mathcal{V}H(G).$$

The second equation holds "in general" (see Proposition 1.8(a)), but the first one is actually equivalent to a condition like (I) in [5, Thm. 7(b)].

2. The ϵ-Tensor Product of a (DFS)- and a Banach Space; Applications to Spaces of Holomorphic Functions

Let $E = \text{ind}_n E_n$ be an injective inductive limit of a sequence of Banach spaces with compact linking maps, that is, a *(DFS)-space*. (Recall that the strong dual E'_b of E is a Fréchet Schwartz space and that $E=(E'_b)'_b$.) In the first part of this section, we investigate under which conditions the topological equality

$$E\otimes_\epsilon X=(\mathrm{ind}_n E_n)\otimes_\epsilon X=\mathrm{ind}_n(E_n\otimes_\epsilon X) \qquad (*)$$

holds for all Banach spaces X. Applications to weighted inductive limits of spaces of holomorphic functions are given in the second part.

It turns out that (*) is related to a "dual" problem on the projective tensor product of Fréchet spaces—namely, the famous "problème des topologies" of A. Grothendieck. The following notation is due to Taskinen [34]: A pair (F,G) of Fréchet spaces is said to have *property* (*BB*) if every bounded subset B of $F\hat{\otimes}_\pi G$ is contained in the closure of the absolutely convex hull of the tensor product $C\otimes D$ of a bounded set C in F and a bounded set D in G. With this notation, Grothendieck's problème des topologies [18] asked if each pair of Fréchet spaces satisfies (BB). After a long time, Taskinen [34] finally gave the first counterexamples. In fact, as we show below in Proposition 2.1, question (*) is equivalent to asking if the pair (E'_b,X) has property (BB) for each Banach space X.

Proposition 2.1 combines a number of results which had been scattered in the literature. In its statement, C_p $(1<p<\infty)$ denotes the *Johnson spaces* as defined in, for example, Jarchow [20], except that these spaces will here be selected in such a way that $C'_p=C_q$ for $1/p+1/q=1$. (This amounts to choosing a sequence $(G_n)_{n\in\mathbf{N}}$ of finite-dimensional Banach spaces which is "dense in all finite-dimensional Banach spaces" and letting C_p be the l_p-direct sum of $\bigoplus_n G_n\times\bigoplus_n G'_n$.) Moreover, writing "$C_p$" in Proposition 2.1, (1′)–(3′), and (3″), we really mean that the conditions are supposed to hold "for every C_p, $1<p<\infty$, or, equivalently, for just one such space." For the definition of Schwartz's ϵ-product, see [7, §3].

2.1. PROPOSITION. *For a* (*DFS*)*-space* $E=\mathrm{ind}_n E_n$, *the following conditions are equivalent:*

(1) (E'_b,X) *has property* (BB) *for every Banach space* X,
 (1′) (E'_b,C_p) *satisfies* (BB);
(2) $\mathcal{L}_b(E'_b,X')$ *is a bornological* (*DF*)*-space for every Banach space* X,
 (2′) $\mathcal{L}_b(E'_b,C_p)$ *is a bornological* (*DF*)*-space*;
(3) $E\otimes_\epsilon X=\mathrm{ind}_n(E_n\otimes_\epsilon X)$ *holds topologically for every Banach space* X; *that is,* $E=\mathrm{ind}_n E_n$ *is an inductive limit with local partition of unity in the sense of Hollstein* [19, 2.2, 3.2],
 (3′) $E\otimes_\epsilon C_p=\mathrm{ind}_n(E_n\otimes_\epsilon C_p)$ *holds topologically,*
 (3″) $E\check{\otimes}_\epsilon C_p=\mathrm{ind}_n(E_n\check{\otimes}_\epsilon C_p)$ *holds topologically.*

These equivalent conditions are implied by:

(4) $E\epsilon X=\mathrm{ind}_n(E_n\epsilon X)$ *holds* (*algebraically and*) *topologically for every Banach space* X;

conversely, (1)–(3) *imply* (4) *if* E *has the approximation property.* (*In any case,* (1)–(3) *imply condition* (4) *for every Banach space* X *with the a.p.*)

Proof. (1) ⇒ (1′) is trivial. Since E'_b clearly has the density condition (cf. [2]) and every Banach space is strongly representable in C_p (see [10, §1]), a look at [10, 2.2(2)] yields (1′) ⇒ (1).

It is well known that, in the presence of (1),

$$\mathcal{L}_b(E'_b, X') = (E'_b \hat{\otimes}_\pi X)'_b$$

topologically for every Banach space X. By [4, Cor. 1.7], the Fréchet space $E'_b \hat{\otimes}_\pi X$ also has the density condition, whence its strong dual must be bornological, and we get (2). Next, while (2) ⇒ (2′) is (again) trivial, (2′) ⇒ (1′) follows directly from Taskinen [35, Prop. 1] (see also [12, 1.2 and 1.3]) since E'_b and C_p are separable and $C'_p = C_q$ $(1 < p < \infty,\ 1/p + 1/q = 1)$.

The equivalence (3) ⇔ (3′) is due to Hollstein [19, 3.2]. To see that (3′) ⇔ (3″), observe first that since C_p has the approximation property $(1 < p < \infty)$, $\text{ind}_n(E_n \check{\otimes}_\epsilon C_p)$ is an injective inductive limit. Then the density of $E_n \otimes_\epsilon C_p$ in $E_n \check{\otimes}_\epsilon C_p$ for each $n = 1, 2, \ldots$ implies the desired result (use [29, 6.3.1] and [7, 1.2]).

For (2′) ⇔ (3″), we note that

$$\mathcal{L}_b(E'_b, C_p) = E\epsilon C_p = E \check{\otimes}_\epsilon C_p$$

since E'_b is Fréchet Schwartz and hence equals E'_c (and C_p has the approximation property). As $\text{ind}_n E_n$ is compactly regular, it follows from [6, 3.13] that the spaces $E\epsilon C_p$ and $\text{ind}_n(E_n \epsilon C_p) = \text{ind}_n(E_n \check{\otimes}_\epsilon C_p)$ are equal algebraically and have the same bounded sets. Thus (3″) holds if and only if $\mathcal{L}_b(E'_b, C_p) = E \check{\otimes}_\epsilon C_p$ is a bornological (DF)-space, but this means exactly (2′).

Certainly, if (4) is true then

$$\mathcal{L}_b(E'_b, X) = E\epsilon X = \text{ind}_n(E_n \epsilon X)$$

must be a bornological (DF)-space for every Banach space X; in particular, (2) holds. On the other hand, for any Banach space X, we know (say, from [7, 5.10]) that $E\epsilon X$ and $\text{ind}_n(E_n \epsilon X)$ coincide algebraically, have the same bounded sets, and even induce the same topology on these bounded sets. If E or X has the approximation property, then $E \otimes_\epsilon X$ is dense in $E\epsilon X$; thus, in the presence of (3), $E\epsilon X$ must at least be a (DF)-space which, however, (by what we have just said) suffices to conclude (4). □

The proof of Proposition 2.1 suggests several variants of the properties (1)–(4). We add that, in view of Hollstein [19, 3.4 and 3.2], (3′) is also equivalent to the weaker condition that $E \otimes_\epsilon C_p$ is only quasibarrelled, while in (3) the class of Banach spaces can even be replaced by the (much larger) class of all l.c. spaces with the countable neighborhood property. Moreover, it is clear that some of the implications in Proposition 2.1 remain valid if $E = \text{ind}_n E_n$ is only an (LB)-space which satisfies certain weaker hypotheses (than compactness of the linking maps).

By Taskinen's construction [34] and its modifications (e.g., see [13]), there are examples of Fréchet Schwartz spaces F and *nonnormable* Fréchet spaces

G such that (F, G) does not enjoy property (BB) and $F_b' \otimes_\epsilon G_b'$ is not even a (gDF)-space. Thus, we are led to an open question as follows.

QUESTION (Taskinen [36]). If F is a Fréchet Schwartz space, must then (F, X) necessarily have property (BB) for each Banach space X?

Because of Proposition 2.1, this is an equivalent form of the known problem (Hollstein [19, end of §3]) of whether the ϵ-tensor product of a (DFS)-space and a Banach space must always be bornological (or, equivalently, whether each (DFS)-space has a local partition of unity). None of the variants of Taskinen's construction of counterexamples to Grothendieck's problème des topologies seems to be applicable here; an answer to the question would probably require new methods.

On the other side, if one is willing to admit additional hypotheses (in the form of "strong" approximation properties), then the situation is not so bad. Some positive results are indeed known (which will be helpful in the context of weighted spaces of holomorphic functions below).

2.2. PROPOSITION. (a) (Taskinen [36, p. 344]) *Let F be a Fréchet Schwartz space which is the reduced projective limit* $\text{proj}_n F_n$ *of a sequence of Banach spaces F_n with the bounded approximation property. Then (F, X) has property* (BB) *for all Banach spaces X.*

(b) (Bonet and Díaz [11, Props. 4, 11, 12]) *If $E = \text{ind}_n E_n$ is a (DFS)-space with the bounded approximation property which has a total bounded set, then*

$$E \otimes_\epsilon X = \text{ind}_n (E_n \otimes_\epsilon X) \quad \textit{and} \quad E \epsilon X = \text{ind}_n (E_n \epsilon X)$$

holds algebraically and topologically for each Banach space X.

Concerning (b), note that a Fréchet Schwartz space E has the bounded approximation property if and only if its strong dual E_b' has the bounded approximation property. (This is quite obvious from the definition.)

As in the case of Proposition 2.1, it is also possible to extend 2.2(b) (from Banach spaces X to larger classes of bornological (DF)-spaces). For instance, we have the following.

Let $E = \text{ind}_n E_n$ be a (DFS)-space with (one of) the equivalent conditions (1)–(3) *of Proposition 2.1 (e.g., in view of Proposition 2.2(b), we may take E to be a (DFS)-space with the bounded approximation property and a total bounded set), and let $X = \text{ind}_m X_m$ be an injective (LB)-space. Then*

$$E \otimes_\epsilon X = \text{ind}_n (E_n \otimes_\epsilon X_n)$$

holds topologically if one of the following additional conditions is satisfied:

(i) *each X_n is a Hilbert space*;
(ii) *X is a π-space (cf. Hollstein* [19]); *or*
(iii) *X is a strong (DFG)-space (cf.* [12, 2.2]).

(In fact, it suffices to show that $E\otimes_\epsilon X$ is a bornological (DF)-space. For (i) and (ii), apply Proposition 2.1 and Hollstein [19, 3.3]; for (iii), again use Proposition 2.1 plus [12, 4.10(b)].)

If, moreover, $X = \mathrm{ind}_m X_m$ is a compactly regular inductive limit and E or X has the approximation property, the following topological equalities hold:

$$E\epsilon X = \mathrm{ind}_m(E\epsilon X_m) = \mathrm{ind}_n(E_n\epsilon X) = \mathrm{ind}_{n,m}(E_n\epsilon X_m) = \mathrm{ind}_n(E_n\epsilon X_n).$$

(By [7, 5.10], these spaces coincide algebraically, have the same bounded subsets, and induce the same topology on the bounded sets; they are all continuously embedded into $E\epsilon X$. If E or X has the approximation property, $E\otimes_\epsilon X$ is a dense subspace of $E\epsilon X$; now $E\epsilon X$ must be a (DF)-space, and we get the desired conclusion.)

Finally, we recall that, without any further assumption, *the ϵ-product $E\epsilon X$ of two (DFS)-spaces E and X is always again a (DFS)-space* (see [6, 4.3]). At this point, we can turn to the consequences of the preceding results for spaces of holomorphic functions.

2.3. PROPOSITION. *Let $V = (v_n)_{n\in\mathbf{N}}$ be an increasing sequence of positive continuous and radial functions v_n on a balanced open set $G \subset \mathbf{C}^N$ such that $HV_0(G)$ contains the polynomials and*

> *for each $n\in\mathbf{N}$, there exists $m > n$ such that v_n/v_m vanishes at infinity on G.*

Then $HV(G) = HV_0(G)$ is a Fréchet Schwartz space and, for each Banach space X,

(a) *$(HV_0(G), X)$ has property* (BB);
(b) *$\mathfrak{L}_b(HV_0(G), X) = (HV_0(G))'_b\epsilon X = \mathrm{ind}_n((H(v_n)_0(G))'_b\epsilon X)$ is a bornological (DF)-space.*

Proof. By the condition on V, $HV(G) = HV_0(G)$ is obviously a Fréchet Schwartz space, and can be written as the reduced (cf. 1.5(a)) projective limit $\mathrm{proj}_n H(v_n)_0(G)$ whose strong dual coincides with $\mathrm{ind}_n(H(v_n)_0(G))'_b$. By Theorem 1.5(a), all the generating Banach spaces $H(v_n)_0(G)$ have the bounded approximation property. Now (a) and (b) follow from Propositions 2.2(a) and 2.1 above. □

2.4. PROPOSITION. *Let $\mathcal{V} = (v_n)_{n\in\mathbf{N}}$ be a decreasing sequence of positive continuous and radial functions v_n on a balanced open set $G \subset \mathbf{C}^N$ such that $H(v_1)_0(G)$ contains the polynomials and*

(V) *for each $n\in\mathbf{N}$, there is $m > n$ such that v_m/v_n vanishes at infinity on G.*

If $\bar{V} = \bar{V}(\mathcal{V})$ denotes the associated system of weights and X is an arbitrary Banach space, then we have the following topological equalities:

$$\mathcal{V}_0H(G, X) = \mathcal{V}H(G, X) = H\bar{V}_0(G, X) = H\bar{V}(G, X).$$

Proof. Let X be an arbitrary Banach space. By condition (V), we clearly have the identity $\mathcal{V}H(G,X)=\mathcal{V}_0H(G,X)$ and (as $\mathcal{V}H(G,X)=H\bar{V}(G,X)$ algebraically by [7, p. 125]) also $H\bar{V}(G,X)=H\bar{V}_0(G,X)$. Moreover, by [1, Cor. 30],

$$\mathcal{V}_0H(G,X)=\mathrm{ind}_n\, H(v_n)_0(G,X)=\mathrm{ind}_n(H(v_n)_0(G)\epsilon X)$$

and

$$H\bar{V}_0(G,X)=H\bar{V}_0(G)\epsilon X$$

is true topologically. By Theorem 1.6(a) and (c), $\mathcal{V}_0H(G)$ has the bounded approximation property and a total bounded set. Thus, we can apply the second part of Proposition 2.2(b) and [7, Thm. 1.6] to conclude

$$\begin{aligned}\mathcal{V}_0H(G,X)&=\mathrm{ind}_n(H(v_n)_0(G)\epsilon X)=\mathcal{V}_0H(G)\epsilon X\\&=H\bar{V}_0(G)\epsilon X=H\bar{V}_0(G,X).\end{aligned}$$ □

Under the hypotheses of Proposition 2.4, it follows that $\mathcal{V}_0H(G,X)=\mathcal{V}H(G,X)$ is a topological subspace of the corresponding space of continuous functions, $\mathcal{V}_0C(G,X)=\mathcal{V}C(G,X)$. Proposition 2.4 provides a (partial) positive answer to a question raised in [7, pp. 115, 137]. The remarks after Proposition 2.2 can help to extend 2.4 from Banach spaces X to l.c. spaces with the countable neighborhood property.

2.5. COROLLARY.

$$\mathcal{V}_0H(G,X)=\mathcal{V}H(G,X)=H\bar{V}_0(G,X)=H\bar{V}(G,X)$$

also holds topologically (and $\mathcal{V}_0H(G,X)$ is a topological subspace of $\mathcal{V}_0C(G,X)$) if X is only a l.c. space with the countable neighborhood property (and $\mathcal{V}$ is as in Proposition 2.4).

Proof. While $\mathcal{V}H(G,X)=\mathcal{V}_0H(G,X)$ is still clear, the algebraic identity of the spaces $\mathcal{V}H(G,X)$ and $H\bar{V}(G,X)$ now follows from Bonet [9, Thm. 8]. The proof of Proposition 2.4 actually shows that $\mathcal{V}_0H(G)$ is an inductive limit with local partition of unity, and Hollstein [19, 3.2] implies

$$\mathcal{V}_0H(G)\otimes_\epsilon X=\mathrm{ind}_n(H(v_n)_0(G)\otimes_\epsilon X) \qquad (*)$$

for each l.c. space X with the countable neighborhood property.

At this point, utilizing [7, 1.6] and [1, 30] (and working with the completion $\hat{X}$ of X, if necessary), we can easily see that the left side of (*) is a topological subspace of $H\bar{V}_0(G,X)$. Now, in view of (*) (and of [1, 30]), $H\bar{V}_0(G,X)$ and $\mathcal{V}_0H(G,X)$ must induce the same topology on $\mathcal{V}_0H(G)\otimes X$, and this space is dense in $\mathcal{V}_0H(G,X)$ since all $H(v_n)_0(G)$ have the approximation property by Theorem 1.5(a), $n=1,2,\dots$. Then [7, 1.2] can serve to conclude that $\mathcal{V}_0H(G,X)=H\bar{V}_0(G,X)$ topologically. □

However, Corollary 2.5 need not imply that, for a bornological (DF)-space X, the locally convex space $\mathcal{V}_0H(G,X)=H\bar{V}_0(G,X)$ must also be bornological (DF). (In some particular cases, positive answers to such a question would follow from our remarks after Proposition 2.2.)

3. Tensor Sequences and Vector-Valued Duality

In the beginning of this section, let v be a strictly positive continuous function on an open set $G \subset \mathbf{C}^N$. For the Banach spaces $Hv_0(G)$ and $Hv(G)$, the existence of a sequence $(C_n)_n$ with the properties exhibited in Proposition 1.2(b) and (e), as well as the biduality $Hv_0(G)'' = Hv(G)$, is related to a lifting problem for vector-valued functions. To discuss this relation, we list the following conditions:

(1) $0 \to Hv_0(G) \xrightarrow{i} Hv(G) \xrightarrow{\pi} Hv(G)/Hv_0(G) \to 0$
is a $\otimes$-sequence in the sense of Kaballo and Vogt [23];
(2) $\mathrm{id} \otimes_\epsilon \pi : F \otimes_\epsilon Hv(G) \to F \otimes_\epsilon (Hv(G)/Hv_0(G))$
is a (surjective) topological homomorphism for each l.c. space F;
(3) $\mathrm{id} \otimes_\pi i : F \otimes_\pi Hv_0(G) \to F \otimes_\pi Hv(G)$
is a topological embedding for each l.c. space F;
(4) $0 \to F \check{\otimes}_\epsilon Hv_0(G) \xrightarrow{\mathrm{id} \check{\otimes}_\epsilon i} F \check{\otimes}_\epsilon Hv(G) \xrightarrow{\mathrm{id} \check{\otimes}_\epsilon \pi} F \check{\otimes}_\epsilon (Hv(G)/Hv_0(G)) \to 0$
is an exact sequence for each Fréchet space F;
(5) $0 \to F \hat{\otimes}_\pi Hv_0(G) \xrightarrow{\mathrm{id} \hat{\otimes}_\pi i} F \hat{\otimes}_\pi Hv(G) \xrightarrow{\mathrm{id} \hat{\otimes}_\pi \pi} F \hat{\otimes}_\pi (Hv(G)/Hv_0(G)) \to 0$
is an exact sequence for each Fréchet space F;
(6) $0 \to (Hv(G)/Hv_0(G))' = Hv_0(G)^\perp \xrightarrow{{}^t\pi} Hv(G)' \xrightarrow{{}^t i} Hv_0(G)' \to 0$
(that is, the dual sequence of (1)) *splits;* that is, $Hv_0(G)^\perp$ is complemented in $Hv(G)'$;
(7) $i : Hv_0(G) \to Hv(G)$ is *approximately left invertible;* that is, there are $C > 0$ and $(T_\alpha)_\alpha \subset \mathcal{L}(Hv(G), Hv_0(G))$ with $\|T_\alpha\| \leq C$ for all α such that $(T_\alpha \circ i)(f) \to f$ in $Hv_0(G)$ for each $f \in Hv_0(G)$;
(8) $\mathrm{id} \check{\otimes}_\epsilon ({}^t i) : Hv_0(G) \check{\otimes}_\epsilon Hv(G)' \to Hv_0(G) \check{\otimes}_\epsilon Hv_0(G)'$ is surjective;
(9) $Hv_0(G)'' = Hv(G)$ holds topologically.

According to Kaballo and Vogt (see [23, 1.1, 1.5, and 1.8]):

(1) *to* (6) *are all equivalent* (*actually,* (1) $\Leftrightarrow$ (2) *could be considered as the definition of a* $\otimes$*-sequence*), *and* (7) *or* (9) $\Rightarrow$ (1)–(6) $\Rightarrow$ (8). *Moreover,* (8) $\Rightarrow$ (7) *and* (9) $\Rightarrow$ (7) *are true whenever* $Hv_0(G)$ *has the bounded approximation property.*

(For the last implication, see Kaballo [21, 2.4(iii)]. In [21, 2.8] it is also mentioned that, in the presence of (1),

$$\mathrm{id} \check{\otimes}_\epsilon \pi : F \check{\otimes}_\epsilon Hv(G) \to F \check{\otimes}_\epsilon (Hv(G)/Hv_0(G))$$

must be surjective as well for, say, each complete l.c. space F which is a compactly regular inductive limit $\mathrm{ind}_\alpha F_\alpha$ of Fréchet spaces F_α with the approximation property.)

Now, if v is radial on a balanced $G \subset \mathbf{C}^N$ and $Hv_0(G)$ contains the polynomials, Proposition 1.2(b) and (e) clearly yield (7) (for the sequence $(C_n)_n$, and with $C=1$) as well as the bounded approximation property of $Hv_0(G)$ (see Theorem 1.5(a)), while (9) follows from Theorem 1.5(d). Hence we obtain the following result.

3.1. PROPOSITION. *If v is a strictly positive continuous and radial function on a balanced open set $G \subset \mathbf{C}^N$ and $Hv_0(G)$ contains the polynomials, then all the conditions* (1)–(9) *hold.*

In this case, a glance at [1, 30] shows that for any complete l.c. space F, we have the following representations of the complete ϵ-tensor products of (4) as spaces of holomorphic F-valued functions:

$$F \check{\otimes}_\epsilon Hv_0(G) = Hv_0(G,F) \quad \text{and} \quad F \check{\otimes}_\epsilon Hv(G) = Hv^P(G,F),$$

and thus we can rewrite the space $F \check{\otimes}_\epsilon (Hv(G)/Hv_0(G))$ in (4) as

$$Hv^P(X,F)/Hv_0(G,F).$$

More generally, if V is a sequence $(v_n)_n$ of functions as in Proposition 3.1 (so that $HV_0(G)$ and $HV(G)$ are Fréchet spaces), then, applying the Mittag-Leffler technique as indicated in the proof of [21, Thm. 2.9], it is possible to prove that (4) holds again, which yields the following proposition.

3.2. PROPOSITION. *If $V=(v_n)_{n \in \mathbf{N}}$ is an increasing sequence of positive continuous and radial functions on a balanced open set $G \subset \mathbf{C}^N$ such that $HV_0(G)$ contains the polynomials, then*

$$0 \to HV_0(G) \to HV(G) \to HV(G)/HV_0(G) \to 0$$

is a $\otimes$-sequence.

Hence, by [23, 1.1 and 1.5], (1) to (5) above (and the remarks after Proposition 3.1) remain valid in this more general context.

As usual with $\otimes$-sequences, the passage to inductive limits presents some difficulties. It is remarkable that the following result can be proved *in full generality,* but our proof is nontrivial.

3.3. PROPOSITION. *Let $\mathcal{V}=(v_n)_{n \in \mathbf{N}}$ be a decreasing, regularly decreasing sequence of positive continuous and radial functions v_n on a balanced open set $G \subset \mathbf{C}^N$ such that $H(v_1)_0(G)$ contains the polynomials. Then*

$$0 \to \mathcal{V}_0 H(G) \to \mathcal{V}H(G) \to \mathcal{V}H(G)/\mathcal{V}_0 H(G) \to 0$$

is a $\otimes$-sequence.

Proof. In view of Theorem 1.6(d), $\mathcal{V}_0 H(G)$ is a topological subspace of $\mathcal{V}H(G)$. $\mathcal{V}$ regularly decreasing implies that $\mathcal{V}_0 H(G) = \operatorname{ind}_n H(v_n)_0(G)$ and $\mathcal{V}H(G) = \operatorname{ind}_n Hv_n(G)$ are boundedly retractive (LB)-spaces. Thus $\mathcal{V}_0 H(G)$ is complete, hence a closed subspace of $\mathcal{V}H(G)$, and $\mathcal{V}H(G)/\mathcal{V}_0 H(G)$ Hausdorff. By Hollstein [19, 1.2], it suffices to show that $\mathcal{V}_0 H(G) \otimes_\pi X$ is a topological subspace of $\mathcal{V}H(G) \otimes_\pi X$ for every Banach space X with the bounded approximation property. To do this, we fix X (with unit ball B) and abbreviate $E_n := H(v_n)_0(G)$, $F_n := Hv_n(G)$, $n=1,2,\dots$; the respective unit balls will be denoted by U_n and W_n.

It is clearly enough to verify that there is $M \geq 1$ such that for each sequence $(\alpha_n)_n$ of positive numbers

$$A := \Gamma\left(\left[\bigcup_{j=1}^{\infty}\left(\sum_{n=1}^{j} \alpha_n W_n\right)\right] \otimes B\right) \cap (\mathcal{V}_0 H(G) \otimes X) \subset \overline{M\Gamma\left(\left[\bigcup_{j=1}^{\infty}\left(\sum_{n=1}^{j} \alpha_n U_n\right)\right] \otimes B\right)} =: MC,$$

where the closure is taken in $\mathcal{V}_0 H(G) \otimes_\pi X$.

We first claim that $|\langle S, z\rangle| \leq 1$ holds for arbitrary $z \in A$ and an arbitrary continuous linear mapping $S \in \mathcal{L}(X, (\mathcal{V}_0 H(G))_b')$ of finite rank which belongs to the polar C° of C. In fact, any $z \in A$ can be written

$$z = \sum_{l=1}^{k} \gamma_l \left(\sum_{n=1}^{p_l} \alpha_n z_{l,n}\right) \otimes x_l$$

with $k \in \mathbf{N}$ and $\gamma_l \in \mathbf{C}$ such that $\sum_{l=1}^{k} |\gamma_l| \leq 1$, $x_l \in B$, $p_l \in \mathbf{N}$, as well as $z_{l,n} \in W_n$ $(l = 1, \dots, k;\ n = 1, \dots, p_l)$. Put $R := \operatorname{span}\{z_{l,n}: 1 \leq l \leq k,\ 1 \leq n \leq p_l\}$ and $N := S(X)$. For each $n \in \mathbf{N}$ and arbitrary $\epsilon > 0$, since $F_n = E_n''$ isometrically (see Theorem 1.5(d)), we may apply the principle of local reflexivity (cf. [24, II, 5.1]) to get a $(1+\epsilon)$-isomorphism $T_n: R \cap F_n \to E_n$ such that

(i) $T_n|_{R \cap E_n} = \mathrm{id}$ and

(ii) $\langle T_n(x), u|_{E_n}\rangle = \langle x, u|_{E_n}\rangle$ for all $x \in R \cap F_n$ and $u \in N$.

By (ii),

$$y := \sum_{l=1}^{k} \gamma_l \left(\sum_{n=1}^{p_l} \alpha_n T_n(z_{l,n})\right) \otimes x_l \in (1+\epsilon) C$$

now satisfies

$$\langle S, y\rangle = \sum_{l=1}^{k} \gamma_l \left(\sum_{n=1}^{p_l} \alpha_n \langle T_n(z_{l,n}), S(x_l)\rangle\right) = \sum_{l=1}^{k} \gamma_l \left(\sum_{n=1}^{p_l} \alpha_n \langle z_{l,n}, S(x_l)\rangle\right) = \langle S, z\rangle.$$

Since $S \in C^\circ$, we conclude $|\langle S, z\rangle| \leq 1 + \epsilon$, whereby $|\langle S, z\rangle| \leq 1$ (as $\epsilon > 0$ was arbitrary). This proves our claim.

Now we can still utilize the bounded approximation property of X: There are $M > 0$ and a net $(\varphi_\alpha) \subset \mathcal{L}(X, X)$ of finite rank operators with $\|\varphi_\alpha\| \leq M$ for all α and $\varphi_\alpha \to \mathrm{id}_X$ pointwise on X. At this point, if $T \in \mathcal{L}(X, (\mathcal{V}_0 H(G))_b')$ is an arbitrary element in C°, putting $S_\alpha := T \circ \varphi_\alpha$ for each α, we see that each $M^{-1} S_\alpha$ is a mapping of finite rank in $\mathcal{L}(X, (\mathcal{V}_0 H(G))_b')$ which belongs to C°, and certainly $\langle T, z\rangle = \lim_\alpha \langle S_\alpha, z\rangle$ holds for each $z \in A$. Hence, by our claim, we obtain $|\langle T, z\rangle| \leq M$, and the desired inclusion $A \subset MC$ is proved. □

It follows from [23, 1.1, 1.3] that

$$\mathrm{id} \otimes_\epsilon \pi: F \otimes_\epsilon \mathcal{V}H(G) \to F \otimes_\epsilon (\mathcal{V}H(G)/\mathcal{V}_0 H(G))$$

is a (surjective) topological homomorphism and

$$\mathrm{id} \otimes_\pi i: F \otimes_\pi \mathcal{V}_0 H(G) \to F \otimes_\pi \mathcal{V}H(G)$$

is a topological embedding for each l.c. space F (which can also be expressed by exact sequences as in (4) and (5), for *arbitrary* F, where, however, the last arrow "$\to 0$" must be deleted).

In the first part of this section, we studied when

$$0 \to Hv_0(G) \to Hv(G) \to Hv(G)/Hv_0(G) \to 0$$

is a $\otimes$-sequence, and showed that this is always true if v is a radial weight on a balanced domain G such that $Hv_0(G)$ contains the polynomials. For such weights we now discuss the interesting properties which are equivalent to the fact that

$$0 \to Hv_0(G) \to Cv_0(G) \to Cv_0(G)/Hv_0(G) \to 0$$

is a $\otimes$-sequence.

3.4. PROPOSITION. *For a positive continuous and radial function v on a balanced open set $G \subset \mathbf{C}^N$ such that $Hv_0(G)$ contains the polynomials, the following conditions are equivalent:*

(1) $0 \to Hv_0(G) \to Cv_0(G) \to Cv_0(G)/Hv_0(G) \to 0$ *is a $\otimes$-sequence;*
(2) $(Cv_0(G)/Hv_0(G)) \check{\otimes}_\epsilon E = Cv_0(G,E)/Hv_0(G,E)$ *for each complete l.c. space E;*
(3) $Hv_0(G)^\perp = (Cv_0(G)/Hv_0(G))'$ *is complemented in* $Cv_0(G)'$;
(4) $0 \to Hv(G) \to Cv(G) \to Cv(G)/Hv(G) \to 0$ *is a $\otimes$-sequence;*
(5) $Hv(G)$ *is a complemented subspace of* $Cv(G)$;
(6) $0 \to Hv(G) \to l_\infty(v,G) \to l_\infty(v,G)/Hv(G) \to 0$ *is a $\otimes$-sequence;*
(7) $Hv(G)$ *is complemented in* $l_\infty(v,G)$;
(8) $Hv_0(G)$ *is a $\mathfrak{L}_\infty$-space;*
(9) $Hv(G)$ *is a $\mathfrak{L}_\infty$-space;*
(10) $(Hv_0(G)',F)$ *has property* (BB) *for every Fréchet space F;*
(11) $\mathfrak{L}_b(Hv_0(G)',F_b') = Hv(G,F_b')$ *is a* (DF)*-space for every Fréchet space F.*

Proof. For the equivalence of the first nine conditions, it suffices to remember Theorem 1.5(d) and to quote results from Kaballo and Vogt [23]: (1)$\Leftrightarrow$(2) is a consequence of [23, 1.1 and 1.3] (use [1, 13, 30]), (1)$\Leftrightarrow$(3) follows from [23, 1.8, (3)$\Leftrightarrow$(4)]. Similarly, since $Hv(G) = Hv_0(G)''$ by Theorem 1.5(d) (and since the dual of each Banach space is complemented in the triple dual), $Hv(G)$ must be complemented in its bidual; thus we get (4)$\Leftrightarrow$(5) and (6)$\Leftrightarrow$(7) from [23, 1.8, relation between (1) and (3)]. Moreover, $Cv_0(G)$, $Cv(G)$, and $l_\infty(v,G)$ are $\mathfrak{L}_\infty$-spaces, whence (1)$\Leftrightarrow$(8) and (4)$\Leftrightarrow$(9)$\Leftrightarrow$(6) by [23, 1.9(i)]. Finally, the equivalence (8)$\Leftrightarrow$(9) is clearly implied by the biduality $Hv_0(G)'' = Hv(G)$ (cf. [24, II, 5.8(ii)]).

It now remains to deal with (10) and (11). If (8) is valid, $Hv_0(G)'$ is a $\mathfrak{L}_1$-space; thus (10) holds by Taskinen [36, 3.1.3]. On the other hand, (10)$\Rightarrow$(11) because then

$$\mathfrak{L}_b(Hv_0(G)',F_b') = (Hv_0(G)' \hat{\otimes}_\pi F)_b'$$

for every Fréchet space F. But in view of [14, Cor. 1.3], it follows from (11) that $Hv_0(G)'$ must be a $\mathfrak{L}_1$-space; that is, we get (9). Finally, the (canonical) topological identity $\mathfrak{L}_b(Hv_0(G)', F_b') = Hv(G, F_b')$ is a special case of Proposition 3.7. □

Proposition 3.4 should be seen in the light of the work of Shields and Williams [30; 31] and Kaballo [22].

3.5. EXAMPLES. Take G to be the open unit disk $D \subset \mathbf{C}$ and v to be a function on D as in Proposition 3.4; that is, v radial with $\lim_{r \to 1_-} v(r) = 0$.

(1) v is said to be *normal* if there exist $k > \epsilon > 0$ and $r_0 < 1$ such that, for $r_0 \le r$,

$$\frac{v(r)}{(1-r)^{\epsilon}} \searrow 0 \quad \text{and} \quad \frac{v(r)}{(1-r)^k} \nearrow \infty \quad \text{as } r \to 1_-.$$

Then Shields and Williams prove in [30, Thm. 1] that $Hv(D)$ is complemented in $L_\infty(v, D)$ and $Hv_0(D)$ is complemented in $Cv_0(D)$ whenever v is a normal weight on D. Thus all the equivalent conditions of Proposition 3.4 are certainly satisfied in this case.

Also see [31] for similar results on the corresponding spaces $hv_0(D)$ and $hv(D)$ of *harmonic* functions on D (which even hold in greater generality) and for a proof (using a method of Lindenstrauss and Pełczyński) that, under certain conditions, $hv_0(D)$ and $hv(D)$ are topologically isomorphic to the sequence spaces c_0 and l_∞, respectively. (In "good cases," $Hv_0(D)$ and $Hv(D)$ are complemented subspaces of $hv_0(D)$ and $hv(D)$, respectively; cf. [31, p. 263]. An illuminating discussion of related questions can be found in [32].) On the other hand, the authors of [30] conjecture (in their final section) that the main results of that paper do not hold for nonnormal weights v.

(2) In fact, Kaballo [22, 1.3(i)] proves that for (nonnormal) radial weights v on D with

$$\lim_{r \to 1_-} v(r) = 0 \quad \text{but} \quad \lim_{r \to 1_-} \left(v(r) \log \frac{1}{1-r} \right) = \infty,$$

$Hv_0(D)$ and $Hv(D)$ are not (even) $\mathfrak{L}_\infty$-spaces. Hence also conditions (1)–(7) and (10) and (11) of Proposition 3.4 cannot hold for such weights v; in particular, *there exist Fréchet spaces F such that $Hv(D, F_b') = \mathfrak{L}_b(Hv_0(D)', F_b')$ is not a (DF)-space.*

Even in the case of radial weights v on D, it remains open to describe exactly when the equivalent conditions of Proposition 3.4 are satisfied. And as far as we know, the (important) case $G = \mathbf{C}$ has not yet been investigated. Examples 3.5(1) and (2) give rise to interesting questions in Banach space theory, as follows.

PROBLEM. If v is a weight on $G \subset \mathbf{C}^N$ as in Proposition 3.4, find necessary and sufficient conditions (on v) such that (1)–(11) of 3.4 hold.

In the last part of Section 3 we prove some remarkable isomorphisms between spaces of vector-valued holomorphic functions and spaces of continuous linear operators, which can be regarded as vector-valued generalizations of the canonical (bi-) dualities

$$((HV_0(G))_b')_b' = HV(G) \quad \text{and} \quad ((\mathfrak{V}_0 H(G))_b')_i' = \mathfrak{V}H(G).$$

3.6. LEMMA. *Let V be a system of nonnegative continuous radial functions on a balanced open set $G \subset \mathbf{C}^N$ such that τ_V is stronger than τ_0, $HV_0(G)$ contains the polynomials, and $HV(G)$ is bornological. Then we have:*

(a) *The evaluation mapping $\Delta: z \to \delta_z$, $\delta_z(f) = f(z)$, belongs to $HV(G, (HV_0(G))'_b)$ (resp., to the unit ball of $Hv(G, Hv_0(G)')$ if V is just a single positive continuous function v on G).*

(b) *Each $u \in HV_0(G)'$ has a representation $u(f) = \int_G vf\,d\mu$ for all $f \in HV_0(G)$, where $v = v(u) \in V$ and $\mu = \mu(u)$ is a bounded Radon measure on G. Then the following equation holds in $(HV_0(G))'_b$:*

$$u = \int_G v\Delta\, d\mu = \int_G v(z)\delta_z\, d\mu(z),$$

and, in the duality $\langle \cdot, \cdot \rangle$ of $HV(G)$ and $HV_0(G)'$, according to Theorem 1.5(d) we also get

$$\langle g, u \rangle = \int_G vg\, d\mu \quad \text{for all } g \in HV(G).$$

Proof. First note that $HV_0(G)$ must be quasibarrelled by Theorem 1.5(d); thus the strong dual $(HV_0(G))'_b$ is quasicomplete.

(a) Clearly, $\{v(z)\delta_z : z \in G\}$ is equicontinuous on $HV_0(G)$ for every $v \in V$ (and belongs to the unit ball $Hv_0(G)'_1$ of $Hv_0(G)'$ if $V = \{v\}$). Moreover, since $((HV_0(G))'_b)'_b = HV(G)$ holds by Theorem 1.5(d), the mapping Δ is weakly holomorphic from G into the quasicomplete space $(HV_0(G))'_b$, and hence must be holomorphic.

(b) The first part is well known (and follows from an application of the Hahn–Banach and Riesz representation theorems). By (a), the integral $\int_G v\Delta\, d\mu$ is a well-defined element of $(HV_0(G))'_b$; thus the desired equation

$$u = \int_G v\Delta\, d\mu$$

follows by evaluating both sides at an arbitrary $f \in HV_0(G)$. Finally, by [5, Thm. 6], the canonical biduality $((HV_0(G))'_b)'_b = HV(G)$ requires that each $u \in HV_0(G)'$ has a *uniquely determined* extension $\hat{u} \in HV(G)^*$ whose restriction to each bounded subset of $HV(G)$ is τ_0-continuous. But $\hat{u}(g) := \int_G vg\, d\mu$ for $g \in HV(G)$ clearly defines such an extension $\hat{u}$ (cf. [8, proof of 1.1]). □

3.7. PROPOSITION. *If V and G are as in Lemma 3.6 and E is a quasicomplete l.c. space, we have the topological isomorphism*

$$\mathcal{L}_b((HV_0(G))'_b, E) = HV(G, E),$$

which must even be isometric if $V = \{v\}$ and E is a Banach space.

Proof. For $T \in \mathcal{L}((HV_0(G))'_b, E)$, we consider $T \circ \Delta: z \to T(\delta_z)$. By Lemma 3.6(a), $T \circ \Delta$ is holomorphic from G into E and, in fact, belongs to $HV(G, E)$ since

$$\{v(z)(T\circ\Delta)(z): z\in G\}=\{T(v(z)\delta_z): z\in G\}$$

is clearly bounded in E for every $z\in G$. Now $\phi: T\to T\circ\Delta$ defines a linear mapping from $\mathcal{L}_b((HV_0(G))'_b, E)$ into $HV(G,E)$ which for every continuous seminorm p on E and every $v\in V$ satisfies

$$\sup_{z\in G} v(z)p([\phi(T)](z))=\sup_{z\in G} p(T(v(z)\delta_z))\leq \sup_{u\in U_v^\circ} p(T(u)),$$

$T\in\mathcal{L}((HV_0(G))'_b, E)$, where $U_v=\{f\in HV_0(G): \sup_{z\in G} v(z)|f(z)|\leq 1\}$. Thus ϕ is also continuous (and of norm ≤ 1 if $V=\{v\}$ and E is a Banach space).

In the other direction, one has to work a bit harder; fix $F\in HV(G,E)$ and $u\in HV_0(G)'$. There always exists a (uniquely determined) element $e=e(F,u)\in E$ with

$$e'(e)=\langle e'\circ F, u\rangle \quad \text{for all } e'\in E', \tag{$*$}$$

where $\langle\cdot,\cdot\rangle$ again denotes the canonical duality of $HV(G)$ and $HV_0(G)'$ as in Theorem 1.5(d): Indeed, according to Lemma 3.6(b), u has a representation $u(f)=\int_G vf\,d\mu$, $f\in HV_0(G)$, with some $v\in V$ and some bounded Radon measure μ on G; then the integral $e=\int_G vF\,d\mu\in E$ is well defined and satisfies ($*$) by the last part of Lemma 3.6(b).

$I_F: u\to e=e(F,u)$ is linear, and to show that I_F is continuous from $(HV_0(G))'_b$ into E, we fix an equicontinuous set $L\subset E'$. Then $A:=\{e'\circ F: e'\in L\}$ is bounded in $HV(G)$. But Theorem 1.5(d) yields $HV_0(G)$ distinguished, hence $(HV_0(G))'_b$ barrelled, and it follows that A is equicontinuous on $HV_0(G)'$. The polar A° of A in $(HV_0(G))'_b$ is a 0-neighborhood in this space that satisfies $I_F(A^\circ)\subset L^\circ$ (where the last polar is taken in E) since $e'(I_F(u))=\langle e'\circ F, u\rangle$ for all $u\in HV_0(G)'$ and all $e'\in E'$ (cf. ($*$)). Now $\psi: F\to I_F$ defines a linear mapping of $HV(G,E)$ into $\mathcal{L}_b((HV_0(G))'_b, E)$ which is continuous (and even of norm ≤ 1 if $V=\{v\}$ and E is a Banach space): Since $HV_0(G)$ is quasibarrelled by Theorem 1.5(d), it suffices to fix $v\in V$ and an equicontinuous set $L\subset E'$. Taking some $u\in U_v^\circ$ with $u(f)=\int_G vf\,d\mu$ for $f\in HV_0(G)$ (cf. Summers [33, Thm. 4.5]) and $p(e):=\sup_{e'\in L}|e'(e)|$, $e\in E$, we can estimate

$$\begin{aligned} p([\psi(F)](u)) &= \sup_{e'\in L}\left|e'\left(\int_G vF\,d\mu\right)\right| = \sup_{e'\in L}\left|\int_G v(e'\circ F)\,d\mu\right| = \sup_{e'\in L}|\langle e'\circ F, u\rangle| \\ &\leq \sup_{e'\in L}\sup_{z\in G} v(z)|(e'\circ F)(z)| = \sup_{z\in G} v(z)p(F(z)). \end{aligned}$$

(Note that by Proposition 1.2(b) and (e), $e'\circ F\in HV(G)$ can always be approximated uniformly on compact subsets of G by functions $f_n\in HV_0(G)$ with

$$\sup_{z\in G} v(z)|f_n(z)|\leq \sup_{z\in G} v(z)|(e'\circ F)(z)| \quad \text{for all } n,$$

and that the restriction of u to bounded subsets of $HV(G)$ is τ_0-continuous.)

By definition, it is clear that $\phi\circ\psi=\mathrm{id}_{HV(G,E)}$ as for $F\in HV(G,E)$, $z\in G$, and $e'\in E'$,

$$e'([(\phi\circ\psi)(F)](z))=e'([\psi(F)\circ\Delta](z))=\langle e'\circ F, \delta_z\rangle=e'(F(z)).$$

But the second part of Lemma 3.6(b) helps to show that also

$$\psi\circ\phi=\mathrm{id}_{\mathcal{L}(HV_0(G)',E)}.$$

For $T\in\mathcal{L}(HV_0(G)',E)$ and $u=\int_G v\Delta\,d\mu\in HV_0(G)'$, we get

$$[(\psi\circ\phi)(T)](u)=[\psi(T\circ\Delta)](u)=\int_G v(T\circ\Delta)\,d\mu=T\Big(\int_G v\Delta\,d\mu\Big)=T(u). \qquad \square$$

For two quasibarrelled l.c. spaces E and F, restriction of transposed maps yields a canonical topological isomorphism $\mathcal{L}_b(E,F_b')=\mathcal{L}_b(F,E_b')$ (cf. Dierolf [15, 5.10]). Hence, remembering that $(HV_0(G))_b'$ is always barrelled in the situation of Proposition 3.7, we also have the following.

3.8. Corollary. *$HV(G,F_b')=\mathcal{L}_b(F,HV(G))$ holds topologically whenever V and G are as in Lemma 3.6 and F is a quasibarrelled l.c. space.*

3.9. Proposition. *Let $\mathcal{V}=(v_n)_n$ be a decreasing sequence of positive continuous radial functions on a balanced open set $G\subset\mathbf{C}^N$ such that $H(v_1)_0(G)$ contains the polynomials, and let E be a quasicomplete l.c. space which has the countable neighborhood property or a fundamental sequence of bounded sets. Then there is a continuous algebraic isomorphism of $\mathcal{V}H(G,E)$ onto $\mathcal{L}((\mathcal{V}_0H(G))_b',E)$.*

Proof. First note that by the duality of inductive and projective limits,

$$(\mathcal{V}_0H(G))_b'=(\mathrm{ind}_n\,H(v_n)_0(G))_b'=\mathrm{proj}_n\,H(v_n)_0(G)'$$

holds topologically. (Clearly, there is a continuous linear and bijective map from the space $(\mathrm{ind}_n\,H(v_n)_0(G))_b'$ onto $\mathrm{proj}_n\,H(v_n)_0(G)'$, but the map must even then be a topological isomorphism since the two spaces are Fréchet.) Next, $(H(v_n)_0(G)')_n$ is a *reduced* projective sequence because, by Theorems 1.5 and 1.6, the transposed maps

$$H(v_n)_0(G)''=Hv_n(G)\to((\mathcal{V}_0H(G))_b')_i'=\mathcal{V}H(G)$$

of $(\mathcal{V}_0H(G))_b'=\mathrm{proj}_n\,H(v_n)_0(G)'\to H(v_n)_0(G)'$ are injective, $n=1,2,\dots$. Thus, applying Dierolf [15, 5.2 and 2.6], one obtains the desired continuous algebraic isomorphism of

$$\mathrm{ind}_n\,\mathcal{L}_b(H(v_n)_0(G)',E)=\mathrm{ind}_n\,Hv_n(G,E)=\mathcal{V}H(G,E)$$

(see the special case of Proposition 3.7 where V consists of only one weight) onto

$$\mathcal{L}_b(\mathrm{proj}_n\,H(v_n)_0(G)',E)=\mathcal{L}_b((\mathcal{V}_0H(G))_b',E). \qquad \square$$

Actually, in the case of Proposition 3.9, the spaces

$$\mathcal{V}H(G,E) \quad\text{and}\quad \mathcal{L}_b((\mathcal{V}_0H(G))_b',E)$$

have the same bounded sets, and $\mathcal{V}H(G,E)=\mathrm{ind}_n\,Hv_n(G,E)$ is a regular inductive limit (cf., again, [15, 5.2]). If for each $n\in\mathbf{N}$ there is $m>n$ such that v_m/v_n vanishes at infinity on G and E has the countable neighborhood property, then one can see from Corollary 2.5 that the two spaces are also equal topologically:

$$\mathcal{L}_b((\mathcal{V}_0H(G))'_b, E) = \mathcal{L}_e((\mathcal{V}_0H(G))'_c, E) = \mathcal{V}_0H(G)\epsilon E$$
$$= H\bar{V}_0(G)\epsilon E = H\bar{V}_0(G, E) = \mathcal{V}_0H(G, E) = \mathcal{V}H(G, E).$$

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Note added in proof. Very recently, it was proved in [A. Peris, *Topological tensor products of a Fréchet-Schwartz space and a Banach space,* Studia Math. (to appear)] that a (DFS)-space with local partition of unity satisfies the compact approximation property (c.a.p.). Then, by constructing a (DFS)-space without c.a.p., Peris solved Taskinen's question (and the corresponding problem of Hollstein) mentioned after Proposition 2.1. In [K.D. Bierstedt, J. Bonet, and A. Peris, *Vector-valued holomorphic germs on Fréchet-Schwartz spaces,* Proc. Roy. Irish Acad. (to appear)], Proposition 2.2(b) was improved as follows:

Let $E = \mathrm{ind}_n E_n$ be a (DFS)-space such that the linking maps are approximable. Then

$$E \otimes_\epsilon X = \mathrm{ind}_n(E_n \otimes_\epsilon X) \quad \textit{and} \quad E\epsilon X = \mathrm{ind}_n(E_n \epsilon X)$$

holds algebraically and topologically for each complete locally convex space X with the countable neighborhood property.

Finally, for $G = D$ resp. $G = \mathbf{C}$, more information on the problem after Examples 3.5 can be found in the articles [W. Lusky, *On the structure of $Hv_0(D)$ and $hv_0(D)$*, Math. Nachr. 159 (1992), 279–289], [W. Lusky, *On weighted spaces of harmonic and holomorphic functions* (to appear)] resp. [A. Galbis, *Weighted Banach spaces of entire functions,* Arch. Math. (Basel) (to appear)].

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