On the Dimension of Harmonic Measure of Cantor Repellers

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1. Introduction

Let J be a Cantor repeller for a conformal map f. We prove that the harmonic measure of J has dimension strictly less than the Hausdorff dimension of J for J lying on a line.

Harmonic measure plays an important part in 1-dimensional complex analysis. Recently, the structure of harmonic measure of rather general plane sets has become much more comprehensible due to works of Carleson [C1; C2], Makarov [Ma1], and Jones and Wolff [JW1]. The deep analogy between the behaviour of sums of (almost) independent random variables and the behaviour of the Green function of a domain plays a crucial part in this subject. We refer the reader to [Ma2] for background. This analogy becomes still more conspicuous if a domain for which the harmonic measure is investigated has a regular self-similar structure. As Carleson [C2] showed, the methods of the ergodic theory turn out to be relevant in this case. This approach was used also in [P], [Z], and [MV]. In this work we will also use extensively ideas from [C2]. In some sense we continue here the study that was undertaken in [C2] and [MV].

It is worth noting that harmonic measure in a dynamical context appeared for the first time in Brolin's paper [Br], where it was established that backward orbits of a polynomial f are equidistributed with respect to the harmonic measure w of the unbounded component of the Julia set $J(f)$. Later this result was interpreted as the coincidence of w and the unique measure of maximal entropy of f ("maximal measure") in [L] and [M]. The Julia set of the polynomial f can be very complicated, but in one particular case it is simply a Cantor-like set—namely, when the orbits of all critical points $c_1, \dots, c_{d-1}$ go to infinity. For such f the Julia set $J(f)$ represents an example of a Cantor repeller. But now we would like to note that the result of Brolin together with Manning's formula [Man] gives the following estimate of the dimension of harmonic measure on the Cantor-like Julia set of a polynomial f of degree d

$$\dim w = \frac{\log d}{\int_{J(f)} \log|f'|\,dw} < 1. \tag{1.1}$$

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In fact $\int_{J(f)} \log|f'|\,dw = \log d + \sum_{i=1}^{d-1} G(c_i)$, where G is the Green function of $\bar{\mathbf{C}}\setminus J(f)$ and $c_1, \dots, c_{d-1}$ are the critical points of f (as we assumed, all of them lie in $\bar{\mathbf{C}}\setminus J(f)$).

Estimate (1.1) holds for a wide class of conformal dynamical systems. For the harmonic measure of any Cantor repeller, it was proved in [C2]. Moreover, it holds for each compact set K with "many annuli" of fixed modules in the complement $\bar{\mathbf{C}}\setminus K$; that result was proved by Jones and Wolff in [JW2]. For an arbitrary compact set K, Jones and Wolff proved in [JW1] that

$$\dim w \le 1. \tag{1.2}$$

For connected K (not a point) the famous result of Makarov [Ma1] computes

$$\dim w = 1. \tag{1.3}$$

Looking at (1.1)–(1.3), one may conclude that harmonic measure always finds some place to hide. And this makes plausible the conjecture that

$$\dim w < \dim K \tag{1.4}$$

when K is totally disconnected. An unpublished example of Christopher Bishop shows that in general (1.4) does not hold. However for some "self-similar" sets (1.4) turns out to be true [MV; Z]. This supports the conjecture that (1.4) is true for all "dynamically defined" disconnected compacts (disconnected fractals).

We shall prove this conjecture here for a wide class of conformal dynamical systems. Let us introduce this class of systems, the so-called (expanding) Cantor repellers. Let $U, U_1, \dots, U_d$ be $d+1$ topological discs with real analytic boundaries such that the closures $\bar{U}_i$ are contained in U, $i = 1, \dots, d$. Consider a map $f\colon \bigcup_{i=1}^{d} U_i \to U$ which is univalent on $\bar{U}_i$, $i = 1, \dots, d$, and is a conformal isomorphism $f_i\colon U_i \to U$ on each U_i (see Figure 1). By the *(expanding) Cantor repeller* we mean the set

$$J = J(f) = \left\{x : f^n x \in \bigcup_{i=1}^{d} U_i,\ n = 0, 1, \dots\right\}.$$

The aim of this paper is to prove the following theorem.

THEOREM 1.1. *Let w be the harmonic measure on an expanding Cantor repeller $J(f)$, let U be symmetric with respect to* $\mathbf{R}$, *and let*

$$f(\bar{z}) = \overline{f(z)}. \tag{S}$$

Then

$$\dim w < \dim J(f). \tag{1.5}$$

Thus, (1.1) *can be strengthened by*

$$\dim w < \min(1, \dim J).$$

As was mentioned before, (1.5) was proved [MV] for "standard Cantor sets" depictured on Figure 2. These Cantor repellers $(U, U_1, \dots, U_d; f)$ are

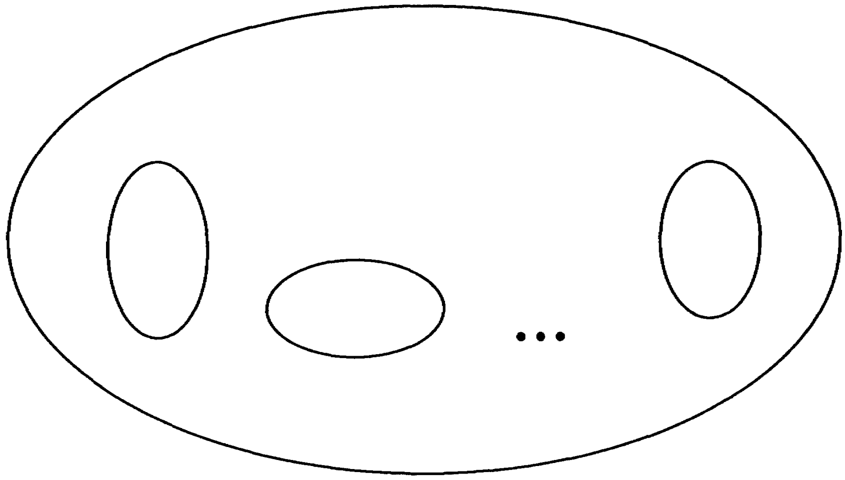

Figure 1

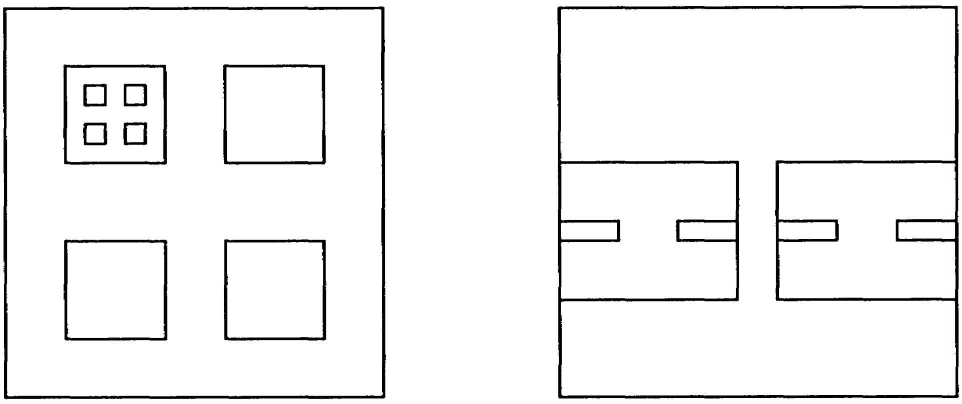

Figure 2

distinguished by the property that all $f_i = f \mid U_i$ are linear. On the other hand, (1.5) was also proved for polynomials f by Zdunik [Z]. Both approaches used the specific nature of J in a very important way.

2. Gibbs Property of the Harmonic Measure of Cantor Repellers

2.1

We refer to [Bo] or [EL] for the exposition of the theory of Gibbs measures, and state here only the facts we need in what follows. On a Cantor repeller $J = J(f)$, the dynamical system $f: J \hookleftarrow$ is naturally topologically conjugate to the one-sided shift $T: \Sigma_d^+ \hookleftarrow$ in d symbols. Providing Σ_d^+ with the natural metric $\rho(\bar{x}, \bar{y}) = 1/2^n$, where n is the first moment for which $x_n \neq y_n$, we see that the conjugacy $h: J \to \Sigma_d^+$ is Hölder continuous. So, the class of Hölder functions is well-defined if we identify J and Σ_d^+ via h.

For each f-quasi-invariant measure ν on J one can consider its Jacobian

$$\mathcal{G}_\nu(x) = \frac{df^*\nu}{d\nu}(x),$$

defined for all x in J except on a set of ν measure zero, which is the derivative of f with respect to ν. The potential of ν is defined to be $\phi_\nu(z) = -\log \mathcal{G}_\nu(x)$. For measures with Hölder potential a proper theory exists—the theory of Gibbs measures. Let us state now the definitions and some properties of Gibbs measures.

(1) By a Gibbs measure on $J(f)$ we mean an f-invariant measure ν with Hölder potential $\phi_\nu = -\log \mathcal{G}_\nu$.

(2) If η is an f-quasi-invariant measure with Hölder potential, then there exists a unique f-invariant measure ν that is absolutely continuous with respect to η. This measure is ergodic and is a Gibbs measure, and $\log(d\nu/d\eta)$ is Hölder continuous.

(3) If η is an f-quasi-invariant measure with Hölder potential ϕ_η, and ν is the corresponding Gibbs measure, then its potential ϕ_ν satisfies the so-called homologeous equation

$$\phi_\nu = \phi_\eta + \gamma \circ f - \gamma, \tag{2.1}$$

where γ is Hölder continuous. Actually, $\gamma = \log(d\nu/d\eta)$.

2.2 Estimates for Harmonic Measure

For a cylinder set $X = x_1 \cdots x_n \subset \Sigma_d^+$, let

$$Q_X = f_{x_1}^{-1} f_{x_2}^{-1} \cdots f_{x_n}^{-1} U = U_{x_1} \cap f^{-1} U_{x_2} \cap \cdots \cap f^{-(n-1)} U_{x_n},$$

and let $\nu(X) = \nu(Q_X)$ for each measure ν on $J = J(f)$. For two cylinder sets $X = x_1 \cdots x_n$ and $Y = y_1 \cdots y_m$ the symbol XY will denote the cylinder set $x_1 \cdots x_n y_1 \cdots y_m$. For $X = x_1 \cdots x_n$, let $|X| = n$. In [C2] and [MV] it was proved that for any Cantor repeller there exist constants C and $q \in (0,1)$ such that, for all cylinders X, Y, Z,

$$\left| \log \left[\frac{w(XYZ)}{w(XY)} : \frac{w(YZ)}{w(Y)} \right] \right| \le C q^{|Y|}. \tag{2.2}$$

Let us remark that (2.2) can be also easily derived from the so-called "boundary Harnack principle", proved in [JK].

This inequality may be interpreted as follows: On the probability space (J, w) the sequence of random variables $\{X_n\}_{n \ge 1}$, with $X_n(x) := x_n$, is almost independent. The past and the future are exponentially independent. Now, taking $X = x_1$, $Y = x_2 \cdots x_n$, and $Z = x_{n+1}$, we get

$$\left| \log \frac{w(x_1 \cdots x_n)}{w(x_2 \cdots x_n)} - \log \frac{w(x_1 \cdots x_{n+1})}{w(x_2 \cdots x_{n+1})} \right| \le C q^n.$$

Consequently, the potential ϕ_w exists for all $x \in J$ and is Hölder continuous. We are now in position to apply the Gibbs theory. Summing up, we get the following proposition.

PROPOSITION 2.1. *The function* $\log \mathcal{G}_w(x)$ *is Hölder continuous of* $J(f)$. *There exists a unique* f*-invariant measure* μ *that is absolutely continuous with respect to* w; *its potential* $\phi_\mu = -\log \mathcal{G}_\mu$ *is Hölder continuous. This measure is ergodic and* $\log(d\mu/dw)$ *is Hölder continuous.*

2.3 Estimates of the Green Function

Let G be the Green function of $\mathbf{C}\backslash J$ with pole at infinity. Let us consider the function $\mathcal{G}$ defined by

$$\mathcal{G}(z) = \frac{G(fz)}{G(z)} \quad \text{for } z \in \bigcup_{i=1}^{d} U_i \backslash J.$$

The next two propositions show that $\mathcal{G}$ is an extension of the Jacobian J_w with certain nice properties.

PROPOSITION 2.2.

(1) *For each cylinder set* X *and each* $z \in Q_X$,

$$\left|\log\left[\frac{G(fz)}{G(z)} : \frac{w(fX)}{w(X)}\right]\right| \leq Cq^{|X|}; \tag{2.3}$$

and

(2) *for each* $z \in Q_X$ *and each* $x \in Q_X \cap J$,

$$\left|\log\left[\frac{G(fz)}{G(z)} : J_w(x)\right]\right| \leq Cq^{|X|}. \tag{2.4}$$

PROPOSITION 2.3. $-\log(G(fz)/G(z))$ *is a Hölder continuous extension of* $\phi_w = -\log J_w$ *into* $\bigcup_{i=1}^{d} U_i$.

Each of these results follows easily from (2.2). In what follows we will also need the next simple lemma.

LEMMA 2.4. *Let* U_1, U_2 *be two topological discs with* $\bar{U}_1 \subset U_2$. *Let* g *map* U_1 *onto* U_2 *univalently. Let* ϕ *be a Hölder function in* U_1. *Then there exists a unique (up to an additive constant) Hölder solution of the homologeous equation*

$$\gamma(g(z)) - \gamma(z) = \phi(z), \quad z \in U_1. \tag{2.5}$$

Proof. Let z_0 be a fixed point of g. Then the function

$$\gamma(z) = \sum_{n\geq 1} [\phi(g^{-n}z) - \phi(z_0)]$$

satisfies (2.5) provided that the series converges. This series converges because the Hölder property of ϕ ensures that $\phi(g^{-n}z) - \phi(z_0)$ decreases exponentially. Now it is clear that γ is also Hölder continuous.

To prove uniqueness, let us consider the homogeneous equation

$$\gamma(gz) - \gamma(z) = 0.$$

It follows that γ is constant along the orbits of g^{-1}. These orbits accumulate to z_0 and so the continuity of γ implies $\gamma(z) \equiv \gamma(z_0)$. □

2.4. *Invariant Hausdorff Measure and Variational Principle*

Let $J = J(f)$ be a Cantor repeller and $\delta = \dim J$ be its Hausdorff dimension. If H_δ denotes the Hausdorff measure of dimension δ on J, then its Jacobian $J_\delta = J_{H_\delta} = |f'|^\delta$. In particular, J_δ is Hölder continuous. By h_δ we will denote the f-invariant measure equivalent to H_δ, which exists as was explained in Section 2.1. The potential of h_δ is equal to $-\delta \log|f'|$.

PROPOSITION 2.5. *Let ν be a Gibbs measure on $J(f)$ and* $\dim \nu = \delta$. *Then* $\nu = h_\delta$.

Proof. Let h_μ denote the entropy of an f-invariant measure μ [Bi]. We will use the following variational principle [Bo] for Gibbs measures. Let ρ be a Gibbs measure and let ϕ_ρ be its potential. Consider the functional defined on probability f-invariant measures

$$V(\mu) = h_\mu + \int \phi_\rho \, d\mu.$$

Then V attains its maximum at a unique point, and this point is the measure ρ. $V(\rho)$ is said to be the *pressure* of ϕ_ρ. Apply this principle to h_δ and $-\delta \log|f'|$. Suppose that $\nu \neq h_\delta$. Then

$$h_\nu - \delta \int_J \log|f'| \, d\nu < h_{h_\delta} - \delta \int_J \log|f'| \, dh_\delta. \tag{2.6}$$

Now we use the Manning formula [Man], which computes the dimension of an f-invariant measure μ, as follows:

$$\dim \mu = \frac{h_\mu}{\int_J \log|f'| \, d\mu}. \tag{2.7}$$

So, applying (2.7) to $\mu = h_\delta$, we see that the right part in (2.6) vanishes. Now the combination of (2.6) and (2.7) applied to $\mu = \nu$ implies that $\dim \nu < \delta$. □

Thus, the proof of (1.5) will follow from the fact that w is singular with respect to the measure H_δ. So, now we shall prove that on a Cantor repeller with condition (S), harmonic measure is singular with respect to Hausdorff measure; that is,

$$w \perp H_\sigma \text{ on } J(f). \tag{2.8}$$

3. Homologeous Equation

The negation of (2.8) is supposed from now on, and we will arrive at a contradiction. First we conclude that invariant harmonic and Hausdorff measures on $J(f)$ coincide, that is, $\mu = h_\delta$. Section 2.1.3 then implies that the potential of w and H_δ are connected by the homologeous equation

$$\phi_w(x) = -\delta \log|f'(x)| + \gamma(fx) - \gamma(x), \quad x \in J(f). \tag{3.1}$$

Fix an index i with $1 \le i \le d$, let p_i be a fixed point of $f_i = f\,|\,U_i$, and let $\lambda_i = |f'(p_i)|$ be its multiplier. Consider a point $z \in U$ and its backward orbit $\{z_{-n}\}_{n=1}^{\infty}$ converging to p_i where $z_{-n} = f_i^{-n}(z)$. We will use the following two simple lemmas. Recall that $\mathcal{G}(z) = G(fz)/G(z)$.

LEMMA 3.1. *Let U be a compact topological disc in U. Then there exist constants $C = C(\tilde{U})$ and $\epsilon > 0$ such that, for all z' and $z'' \in \tilde{U}$,*

$$|\mathcal{G}(f^{-n}z') - \mathcal{G}(f^{-n}z'')| \le C|z'-z''|^{\epsilon}.$$

Proof. By the Koebe distortion theorem,

$$|f^{-n}z' - f^{-n}z''| \le C|z'-z''|,$$

where C depends on $\tilde{U}$ but does not depend on n. An application of Proposition 2.3 finishes the proof. □

LEMMA 3.2. *Let $\sum \epsilon_n(x)$ be a series of Hölder functions with uniformly bounded Hölder norms. Let $\|\epsilon_n\|_\infty \le Cq^n$ for $q \in (0,1)$. Then the sum of the series is also Hölder.*

The proof is trivial and so it is omitted.

Now we are going to derive some consequences of the homologeous equation (3.1). First, it follows from Proposition 2.2 and from (3.1) that

$$\begin{aligned}\log \frac{G(z_{-(n-1)})}{G(z_{-n})} &= \log \mathcal{G}_w(p_i) + O(q^n)\\ &= \delta \log|f'(p_i)| + O(q^n) = \log \lambda_i^{\delta} + O(q^n).\end{aligned} \tag{3.2}$$

Hence, the series

$$\sum_{n=1}^{\infty} \log \frac{G(z_{-(n-1)})}{\lambda_i^{\delta} G(z_{-n})} = \lim_{n\to\infty} \log \frac{G(z)}{\lambda_i^{n\delta} G(z_{-n})} \tag{3.3}$$

is convergent on $\tilde{U}$ and its sum is Hölder on $\tilde{U}$ by Lemmas 3.1 and 3.2.

Second, analogously we have

$$\log \frac{G(z_{-(n-1)})}{G(z_{-n})} = \delta \log|f'(z_{-n})| + O(q^n). \tag{3.4}$$

Hence the series

$$\sum_{n=1}^{\infty} \log \frac{G(z_{-(n-1)})}{|f'(z_{-n})|^{\delta} G(z_{-n})} = \lim_{n\to\infty} \log \frac{G(z)}{|(f^n)'(z_{-n})|^{\sigma} G(z_{-n})} \tag{3.5}$$

is convergent on $\tilde{U}$ and its sum is Hölder on $\tilde{U}$. Using (3.3), we may consider the following functions:

$$\tau_i(z) = \lim_{n\to\infty} \lambda_i^{n\delta} G(f_i^{-n}z), \quad z \in U, \ i = 1, \dots, d,$$

which are positive and harmonic in U. These functions are the main objects of our study; we would like to prove that they are proportional. This proof is accomplished in Sections 4 and 5. In Section 6, the proportionality of the functions τ_i is used to prove that the dynamics $f=(f_1,\dots,f_d)$ is conformally equivalent to a linear dynamics $F=(F_1,\dots,F_d)$—that is, to the situation where $|F_i'|\equiv\lambda_i$ is constant. For linear dynamics (on the line) w and H_δ on $J(F)$ are singular; this was proved in [MV]. This reference shows that the proportionality of τ_i really leads to a contradiction. However, to make the present article self-contained, we would like to avoid reliance upon [MV] here.

So we assume that $F=(F_1,\dots,F_d)$ is a linear dynamics; that $|F_i'(z)|=\lambda_i$ is a constant for $z\in U_i$; and that the τ_i are proportional. We will now show that this leads to a contradiction.

Multiplying τ_i by suitable positive constants, we obtain a function τ which is strictly positive in $U\setminus J$, vanishes on J, and satisfies the functional equation

$$\tau(F_i z)=\lambda_i^\delta\tau(z),\quad z\in U_i,\ i=1,\dots,d.$$

Now assume that

$$\exists i:|F_i'|=\lambda_i>1. \tag{E}$$

The functional equation and (E) allow us to extend τ onto the whole complex plane. Let us denote by G the group of linear transformations generated by $F_1,\dots,F_d$. For a given set K, let GK denote $\bigcup_{g\in G} gK$. Now obviously the set $Z(\tau)$ of zeros of τ is the closure of GJ. That τ is positive in $U\setminus J$ implies

$$GJ\cap(U\setminus J)=\emptyset.$$

The following lemma was pointed out to me by A. Eremenko and is published with his permission. It shows that the previous assertion is always false, which means that the proportionality of τ will lead us to a contradiction.

LEMMA 3.3. *Let G be a group generated by linear mappings $F_1,\dots,F_d$ satisfying* (E). *Then, for an arbitrary compact K and an arbitrary neighborhood N of K, the following assertion holds:*

$$GK\cap(N\setminus K)\neq\emptyset.$$

Proof. We assume that $\lambda_1>1$. It is sufficient to consider the case $d=2$. Conjugating G with a linear map, we can always assume that

$$F_1(z)=\lambda_1 z;\qquad F_2(z)=\lambda_2 z+\delta_2.$$

Then

$$F_0(z)=F_1^{-1}F_2^{-1}F_1F_2(z)=z+\delta_2\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1\lambda_2}\right)=z+\delta_0.$$

So a shift belongs to G. Considering $F_1^{-k}F_0F_1^k(z)=z+\delta_0/\lambda_1^k$, we see that arbitrary small shifts are in G (we use (E) here). Now, let K and N be fixed and let $\epsilon=\operatorname{dist}(K,\partial N)$. We choose $g=z+h\in G$, where $|h|<\epsilon/2$. Let $h=|h|e^{i\theta_0}$ and let z_0 be a point on K such that

$$\operatorname{Re} z_0e^{-\theta_0}=\max_{z\in K}\operatorname{Re} ze^{-i\theta_0}.$$

Then $gz_0 = z_0 + h$ does not belong to K but does belong to N, and Lemma 3.3 is proved. □

Returning to our functions τ_i, let us recall that the proof of their proportionality will take almost the entire remainder of the paper.

The first step is to introduce the functions

$$v_i(z) = \lim_{n\to\infty} |(f^n)'(f_i^{-n}z)|^\delta G(f_i^{-n}z), \quad z \in U_i, \ i = 1, \dots, d. \tag{3.6}$$

These functions are not harmonic, but they have the following advantage. Let us divide G by v_i in U so that

$$G(z) = v_i(z) e^{\gamma_i(z)}, \quad i = 1, \dots, d, \tag{3.7}$$

where γ_i is given by the series (3.5). In particular, all γ_i are Hölder. On U_i this function satisfies the homologeous equation

$$\gamma_i(fz) - \gamma_i(z) = \log \mathcal{G}(z) - \delta \log|f'(z)|.$$

Restricting this to x in $J \cap U_i$ and comparing the result with (3.1), we conclude that $\gamma_i(x) = \gamma(x) + C_i$ for $x \in J$, where the C_i are constants. Using the uniqueness part of Lemma 2.4 we can renormalize γ_i and v_i, preserving (3.7) in such a way that

$$\gamma_i(x) \equiv \gamma(x), \quad x \in J, \ i = 1, \dots, d. \tag{3.8}$$

It is useful to note that the functions τ_i and v_i satisfy the following functional equations:

$$\tau_i(fz) = \lambda_i^\delta \tau_i(z), \quad z \in U_i; \tag{3.9}$$

$$v_i(fz) = |f'(z)|^\delta v_i(z), \quad z \in U_i. \tag{3.10}$$

We will also use the following notation.

Again for $z \in U$ and fixed i with $1 \le i \le d$, let $\{z_{-n}\}$ be the backward orbit of z; that is, $z_{-n} = f_i^{-n} z$. Define $a_i(z)$ by

$$a_i(z) = \frac{\delta}{2} \sum_{n=1}^{\infty} \log \frac{f'(z_{-n})}{f'(p_i)}. \tag{3.11}$$

Each term here is a well-defined holomorphic in U function, and by the Koebe distortion theorem the series converges absolutely and uniformly in each $\tilde{U}$ for $\bar{\tilde{U}} \subset U$. It is clear that

$$v_i(z) = |e^{a_i(z)}|^2 \tau_i(z), \quad z \in U, \ i = 1, \dots, d. \tag{3.12}$$

4. Flatness of $v_i - v_j$ on J

We begin with the following simple lemma.

LEMMA 4.1. *Given a Cantor repeller $J = J(f)$, there exist constants c_1, c_2 such that*

(1) $c_1 \operatorname{dist}(\partial Q_X, J) \le \operatorname{length}(\partial Q_X) \le c_2 \operatorname{dist}(\partial Q_X, J)$, *and*
(2) $c_1|(f^n)'(z)| \le \operatorname{dist}(z, J) \le c_2|(f^n)'(z)|$ *if* $z \in \partial Q_X$ *and* $n = |X|$.

This is a direct consequence of the Koebe theorem. Now, for a given z let Q_X be smallest cylinder containing z and let $|X| = n$. Fix an $x \in J \cap Q_X$ and let $x^0 = x, x^1 = fx, \dots, x^n = f^n x$. Using (3.1) with $x^0, \dots, x^{n-1}$ we get

$$\begin{aligned} -\sum_{k=0}^{n-1} \log \mathcal{G}_w(x^k) &= -\delta \sum_{k=0}^{n-1} \log|f'(x^k)| + \gamma(x^n) - \gamma(x^0) \\ &= -\delta \log|(f^n)'(x)| + \gamma(x^n) - \gamma(x). \end{aligned} \tag{4.1}$$

The Hölder property of $\log \mathcal{G}_w$ implies that

$$\forall y \in Q_Y \cap J, \quad \left|\log\left[\frac{w(fY)}{w(Y)} : \mathcal{G}_w(y)\right]\right| \le Cq^{|Y|}. \tag{4.2}$$

Put $Y_0 = X$ and $Y_k = f^k X$, and use (4.2) in (4.1) to arrive at

$$\log w(X) = \log \frac{1}{|(f^n)'(x)|^\delta} + \gamma(x^n) - \gamma(x).$$

Hence, there are c_1 and c_2 such that

$$c_1|(f^n)'(x)|^{-\delta} \le w(X) \le c_2|(f^n)'(x)|^{-\delta} \quad \forall x \in Q_X \cap J. \tag{4.3}$$

LEMMA 4.2. *Given a Cantor repeller $J(f)$, there exist constants c_1 and c_2 such that*

$$c_1 w(X) \le G(z) \le c_4 w(X) \quad \forall z \in \partial Q_X. \tag{4.4}$$

Combining Lemmas 4.1, 4.2, and 4.3, we obtain the following proposition.

PROPOSITION 4.3. *For a Cantor repeller J, either $w \perp H_\delta$ or*

$$c_1 \operatorname{dist}(z, J)^\delta \le G(z) \le c_2 \operatorname{dist}(z, J)^\delta. \tag{4.5}$$

The estimate of $v_i - v_j$ will be obtained in two steps. The first step follows immediately from (4.5), (3.8), and the Hölder continuity of the functions γ_i for $i = 1, \dots, d$. Suppose their Hölder exponent is $\epsilon > 0$. Then

$$\begin{aligned} |v_i(z) - v_j(z)| &= G(z)|e^{-\gamma_i(z)} - e^{-\gamma_j(z)}| \\ &\le c \operatorname{dist}(z, J)^\delta |e^{-\gamma_i} - e^{-\gamma_j}| \le c \operatorname{dist}(z, J)^{\delta+\epsilon}. \end{aligned} \tag{4.6}$$

Next we derive from (4.6) a much better estimate:

$$\exists \eta > 0 : |v_i(z) - v_j(z)| \le c \operatorname{dist}(z, J)^{1+\eta}. \tag{4.7}$$

This requires some work, which will be finished at (4.18). To start, let us estimate ∇v_i in $\tilde{U} \setminus J$ when $\bar{\tilde{U}} \subset U$. Let $V_0 = \bar{\tilde{U}} \setminus \bigcup_{i=1}^d f_i^{-1} \tilde{U}$. Functions τ_i are harmonic and hence real analytic in V_0; by (3.12) the v_i are also real analytic in V_0. For $X = x_1 \cdots x_n$ let $V_X = f_{x_1}^{-1} \cdots f_{x_n}^{-1} V_0$. Now (3.12) and (3.7) imply that

$$\tau_i(z) = e^{-\gamma_i(z)}|e^{-a_i(z)}|^2 G(z), \quad i = 1, \dots, d, \; z \in U.$$

Combining this with (4.5), we see that

$$c_1 \operatorname{dist}(z,J)^{\delta} \le |\tau_i(z)| \le c_2 \operatorname{dist}(z,J)^{\delta}, \quad i=1,\dots,d. \tag{4.8}$$

Using the fact that all τ_i are harmonic in $U\setminus J$, we derive from (4.8) that

$$|\nabla \tau_i(z)| \le c \operatorname{dist}(z,J)^{\delta-1}. \tag{4.9}$$

By (3.12), $\nabla v_i = |e^{a_i}|^2 \nabla \tau_i + \tau_i \nabla |e^{a_i}|^2$. As each $|e^{a_i}|^2$ is a real analytic function in U, we conclude that

$$|\nabla v_i(z)| \le c \operatorname{dist}(z,J)^{\delta-1}. \tag{4.10}$$

Let dA denote the area measure on $\mathbf{C}$.

LEMMA 4.4. *For a Cantor repeller J, either $w \perp H_\delta$ or the function $\rho(z) = \operatorname{dist}(z,J)^{\delta-1}$, $z\in U$, belongs to $L^{2+\delta}(dA)$.*

Proof. We fix an integer k and let $X_1,\dots,X_{m_k}$ be that family of maximal cylinders having the property $\operatorname{diam} V_{X_i} \le 2^{-k}$. The Koebe distortion theorem and maximality of X_i imply that $\operatorname{diam} V_{X_i} \ge c\cdot 2^{-k}$, where c does not depend on k. To estimate the number m_k we note that, according to (4.3) and Lemma 4.1,

$$w(X_i) \ge c\cdot 2^{-k\delta}, \quad i=1,\dots,m_k. \tag{4.11}$$

Cylinders $X_1,\dots,X_{m_k}$ being disjoint, we have

$$m_k \le c\cdot 2^{k\delta}.$$

Note that the area of V_X, is less than $c2^{-2k}$. Let $V^k = \bigcup_{i=1}^{m_k} V_{X_i}$. Then $U = \bigcup_{k=1}^{\infty} V^k$ and

$$\int_{\bar U} \rho^{2+\delta}(z)\, dA(z) = \sum_{k=1}^{\infty} \int_{V^k} \cdots \le c \sum_{k=1}^{\infty} 2^{-2k}\cdot 2^{k\delta} 2^{-k(\delta-1)(2+\delta)} < \infty.$$

REMARK. Clearly $\rho \in L^{2+\eta}(dA)$ for any $\eta < \delta/(1-\delta)$.

We will use also the following standard version of the Sobolev embedding theorem.

PROPOSITION 4.5. *Let f be a function from L^p, $p>2$, with compact support. Then the function*

$$\hat f(z) = \int \frac{f(\xi)}{\xi - z}\, dA(\xi)$$

belongs to the Hölder class with exponent $1-2/p$.

Let

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \bar\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

and $\Phi_{ij} = \partial(v_i - v_j)$. Function Φ_{ij} is not holomorphic, but still it has a good $\bar\partial$-estimate:

$$\bar\partial \Phi_{ij} = \bar\partial\partial(v_i - v_j) = \bar\partial\partial(|e^{a_i}|^2\tau_i - |e^{a_j}|^2\tau_j).$$

We continue the estimate, using the notation $\alpha_i = \operatorname{Re} a_i$, to obtain

$$\begin{aligned}\bar\partial \Phi_{ij} = \bar\partial\partial(e^{2\alpha_i}\tau_i - e^{2\alpha_j}\tau_j) = e^{2\alpha_i}\bar\partial\partial\tau_i - e^{2\alpha_j}\bar\partial\partial\tau_j \\ + \tau_i\bar\partial\partial e^{2\alpha_i} - \tau_j\bar\partial\partial e^{2\alpha_j} + \bar\partial e^{2\alpha_i}\partial\tau_i + \partial e^{2\alpha_i}\bar\partial\tau_i \\ - \bar\partial e^{2\alpha_j}\partial\tau_j - \bar\partial\tau_j\partial e^{2\alpha_j}.\end{aligned}$$

The first two terms vanish as $\bar\partial\partial = \frac{1}{4}\Delta$ and τ_i and τ_j are harmonic. The next two terms are bounded since α_i and α_j are real analytic. Each of the last four terms has an estimate $c\operatorname{dist}(z, J)^{\delta-1}$ according to (4.9). Let us denote

$$w_{ij} = \frac{1}{\pi}\int_U \frac{\bar\partial\Phi_{ij}(\xi)}{\xi - z}\, dA(\xi).$$

Combining (4.9), Lemma 4.4, and Proposition 4.5, we see that

$$w_{ij} \in \Lambda^{\delta/(2+\delta)}. \tag{4.12}$$

Let us fix an integer k, and let $X_1, \ldots, X_{m_k}$ be the same maximal cylinders as in the proof of Lemma 4.4. Recall that $V^k = \bigcup_{i=1}^{m_k} V_{X_i}$. We are now in a position to prove (4.7). To do this, let us use the Green formula for $\Phi = \Phi_{ij}$ in $\tilde U \setminus V^k$:

$$\begin{aligned}\Phi(z) = \frac{1}{2\pi i}\int_{\partial U}\frac{\Phi(\xi)}{\xi - z}\,d\xi + \sum_{j=1}^{m_k}\frac{1}{2\pi i}\int_{\partial V_{X_i}}\frac{\Phi(\xi)}{z-\xi}\,d\xi \\ + \frac{1}{\pi}\int_{U\setminus V^k}\frac{\bar\partial\Phi(\xi)}{z-\xi}\,dA(\xi) = I + \Sigma + A.\end{aligned}$$

The next estimate

$$|\nabla(v_i - v_j)(z)| \le c\operatorname{dist}(z, J)^{\delta+\epsilon-1} \tag{4.13}$$

will be proven later. Taking it for granted at this point, let us estimate Σ. According to (4.13), $|\Phi| \le c2^{-k(\delta+\epsilon-1)}$ on ∂V_{X_i} for $i = 1, \ldots, m_k$. So

$$\begin{aligned}|\Sigma| &\le C\sum_{j=1}^{m_k} 2^{-k(\delta+\epsilon-1)}\int_{\partial V_{x_i}} |dz| \\ &\le Cm_k 2^{-k(\delta+\epsilon-1)}\cdot 2^{-k} \le C2^{k\delta - k(\delta+\epsilon-1)-k} = C2^{-k\epsilon}.\end{aligned}$$

Letting k tend to infinity, we obtain

$$\Phi_{ij}(z) = \frac{1}{2\pi i}\int_{\partial U}\frac{\phi_{ij}(\xi)}{\xi - z} - w_{ij}(z), \quad z \in U.$$

Hence, $\partial(v_i - v_j)$ is a restriction to $U\setminus J$ of a function from $\Lambda^{\delta/(2+\delta)}$ (see (4.12)). As $v_i - v_j$ is real valued, the same is true for $\bar\partial(v_i - v_j)$ and for $\nabla(v_i - v_j)$. As J is nowhere dense and $v_i - v_j$ is at least continuous on U, we conclude that

$$v_i - v_j \in C^{1+\delta/(2+\delta)}(\tilde U). \tag{4.14}$$

The last assertion means that $v_i - v_j$ has continuous partial derivatives and, moreover, these partial derivatives belong to $\Lambda^{\delta/(2+\delta)}$ on $\tilde U$. Let us now use the symmetry condition (S) given in the statement of Theorem 1.1. In this case it is clear that

$$\tau_i(\bar{z}) = \tau_i(z), \qquad v_i(\bar{z}) = v_i(z), \tag{4.15}$$

and

$$a_i(\bar{z}) = \overline{a_i(z)} \quad \text{for } i = 1, \ldots, d. \tag{4.16}$$

From (4.14) it follows that

$$\frac{\partial}{\partial x}(v_i - v_j)(x) = 0 \quad \text{for } x \in J.$$

Using (4.14) once again, we conclude that

$$|(v_i - v_j)(x)| \le c \operatorname{dist}(x, J)^{1+\eta} \quad \text{for } x \in \mathbf{R},\ \eta = \delta/(2+\delta). \tag{4.17}$$

The symmetry condition (4.15) implies that

$$\frac{\partial}{\partial y} v_i(x) = \frac{\partial}{\partial y} v_j(x) = 0 \quad \text{for all } x \in \mathbf{R}.$$

Combining this with (4.14), we get

$$|(v_i - v_j)(z)| \le c \operatorname{dist}(z, J)^{1+\delta/(2+\delta)}, \quad z \in U. \tag{4.18}$$

To justify (4.18) we need only prove (4.13). This is done in the following lemma.

LEMMA 4.6. *Let τ_1 and τ_2 be harmonic in $U \setminus J$; let α_1 and α_2 be real analytic in U, $\gamma \le 1+\sigma$, and*

$$|\tau_1| + |\tau_2| \le c \operatorname{dist}(z, J)^{\sigma}; \qquad |(e^{\alpha_1}\tau_1 - e^{\alpha_2}\tau_2)(z)| \le c \operatorname{dist}(z, J)^{\gamma}. \tag{4.19}$$

Then

$$|\nabla(e^{\alpha_1}\tau_1 - e^{\alpha_2}\tau_2)(z)| \le c \operatorname{dist}(z, J)^{\gamma-1}. \tag{4.20}$$

Proof. Fix $z_0 \in U \setminus J$ and $\mathfrak{D} = \mathfrak{D}_{z_0, r_0}$, where $r_0 = \frac{1}{2}\operatorname{dist}(z_0, J)$. Let $\alpha_1^0 = \alpha_1(z_0)$, $\alpha_2^0 = \alpha_2(z_0)$, $\Delta_1(z) = e^{\alpha_1(z)} - e^{\alpha_1(z_0)}$, and $\Delta_2(z) = e^{\alpha_2(z)} - e^{\alpha_2(z_0)}$. Then

$$\begin{aligned}
\nabla(e^{\alpha_1}\tau_1 - e^{\alpha_2}\tau_2) &= e^{\alpha_1}\nabla\tau_1 - e^{\alpha_2}\nabla\tau_2 + \tau_1\nabla e^{\alpha_1} - \tau_2\nabla e^{\alpha_2} \\
&= (e^{\alpha_1^0}\nabla\tau_1 - e^{\alpha_2^0}\nabla\tau_2) + (\tau_1\nabla e^{\alpha_1} - \tau_2\nabla e^{\alpha_2}) + (\Delta_1\nabla\tau_1 - \Delta_2\nabla\tau_2) \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Now $I_1 = \nabla(e^{\alpha_1^0}\cdot\tau_1 - e^{\alpha_2^0}\cdot\tau_2)$. The function under the ∇ sign is harmonic and bounded by $c\operatorname{dist}(z, J)^{1+\sigma} + c\operatorname{dist}(z, J)^{\gamma}$ in the disc D, so the estimate (4.20) holds for $|I_1|$. Obviously, $|I_2| \le c\operatorname{dist}(z, J)^{\sigma} \le c\operatorname{dist}(z, J)^{\gamma-1}$ and

$$|I_3| \le |\Delta_1||\nabla\tau_1| + |\Delta_2||\tau_2| \le c\operatorname{dist}(z, J)^{1+\sigma-1} \le \operatorname{dist}(z, J)^{\gamma-1}. \qquad \square$$

We can apply this lemma with $\tau_1 = \tau_i$, $\tau_2 = \tau_j$, $\sigma = \delta$, and $\gamma = \delta + \epsilon$ because (4.8) and (4.6) imply (4.19). Thus the proof of (4.18) is finished. Applying Lemma 4.6 for the second time (now with $\sigma = \delta$ and $\gamma = 1 + \delta/(2+\delta)$), we get for all pairs i, j, $1 \le i, j \le d$,

$$|\partial(e^{2\alpha_i}\tau_i - e^{2\alpha_j}\tau_j)(z)| \le c\,\mathrm{dist}(z,J)^{\delta/(2+\delta)}, \tag{4.21}$$

where $\alpha_i = \operatorname{Re} a_i$ for $i=1,\dots,d$. Using (4.8) again we have

$$|(e^{2\alpha_i}\partial\tau_i - e^{2\alpha_j}\partial\tau_j)(z)| \le c(\mathrm{dist}(z,J)^{\delta/(2+\delta)} + \mathrm{dist}(z,J)^{\delta}).$$

Let us rewrite this inequality for $z = x \in \mathbf{R}$ and $\eta = \delta/(2+\delta)$ to obtain

$$|e^{a_i(x)}e^{\overline{a_i(x)}}(\partial\tau_i)(x) - e^{a_j(x)}e^{\overline{a_j(x)}}(\partial\tau_j)(x)| \le c\,\mathrm{dist}(x,J)^{\eta}. \tag{4.22}$$

5. The Lemma of S. V. Hruščëv Removes Singularities

Let us consider the auxiliary function

$$\Psi_{ij}(z) = e^{a_i(z)}e^{\overline{a_i(\bar z)}}(\partial\tau_i)(z) - e^{a_j(z)}e^{\overline{a_j(\bar z)}}(\partial\tau_j)(z).$$

Each term is holomorphic in $U\setminus J$. On the real axis,

$$\begin{aligned}\Psi_{ij}(x) &= e^{2\alpha_i(x)}(\partial\tau_i)(x) - e^{2\alpha_j(x)}\partial\tau_j(x)\\ &= e^{2\alpha_i(x)}\frac{\partial\tau_i}{\partial x}(x) - e^{2\alpha_j(x)}\frac{\partial\tau_j}{\partial x}(x)\end{aligned} \tag{5.1}$$

because $\partial\tau_i/\partial y = \partial\tau_j/\partial y = 0$ on $\mathbf{R}$ according to (4.15). Using the lemma due to Hruščëv [H], we will prove that Ψ_{ij} is holomorphic on U; that is, we remove the singularity of Ψ_{ij} on J. By (4.22), $\Psi_{ij}(x)=0$ for $x\in J$; hence $\Psi_{ij}(z)\equiv 0$. In particular, $\Psi_{ij}(x)\equiv 0$ for $x\in\mathbf{R}$. Using (5.1) we see that (where prime means x-derivative)

$$\tau_i' - e^{2(\alpha_j-\alpha_i)}\tau_j' \equiv 0 \text{ on } \mathbf{R}. \tag{5.2}$$

The function $\tau_i - e^{2(\alpha_i-\alpha_j)}\tau_j$ is continuous on $\mathbf{R}$ and continuously differentiable on each complementary interval of J. By (4.8) this function vanishes at the ends of each interval. Thus, for any complementary interval l, there exists a point x_l such that

$$(\tau_i - e^{2(\alpha_i-\alpha_j)}\tau_j)'(x_l) = 0.$$

Combining with (5.2), we get $2e^{2(\alpha_i-\alpha_j)}\tau_j(\alpha_i-\alpha_j)'(x_l)=0$, and on the sequence $\{x_l\}$

$$(\alpha_i-\alpha_j)'(x_l)=0. \tag{5.3}$$

The functions $\alpha_i-\alpha_j$ are real analytic, and thus (5.3) implies $\alpha_i-\alpha_j\equiv c_{ij}$, where c_{ij} is a constant. These constants c_{ij} are not arbitrary. The condition $c_{ij}+c_{jk}+c_{ki}=0$ holds for each i,j,k. Hence, we can find the constants $\{c_i\}_{i=1}^d$ such that $c_{ij}=c_i-c_j$. Returning to (5.2), we conclude that

$$(e^{2c_i}\tau_i)' - (e^{2c_j}\tau_j)' = 0 \text{ on } \mathbf{R}\cap U.$$

So $e^{2c_i}\tau_i - e^{2c_j}\tau_j$ is constant on $\mathbf{R}$, and this constant must be zero by (4.8). For later use we will record here that we have proved the following assertion:

$$\alpha_i - c_i \equiv \alpha_j - c_j. \tag{5.4}$$

Renormalizing τ_i for $\tau_i := e^{2c_i}\tau_i$, we get that $\tau_i \equiv \tau_j$ on $\mathbf{R}$. Combining this with (4.15) we have $\partial\tau_i \equiv \partial\tau_j$ on $\mathbf{R}$ and hence $\partial\tau_i \equiv \partial\tau_j$ on U. The same holds for $\bar{\partial}$ derivatives. Finally,

$$\tau_i(z) \equiv \tau_j(z), \quad z \in U, \; i, j = 1, \ldots, d,$$

and this implies that there exists a positive harmonic function τ in $U \setminus J$ such that

$$\tau(fz) = \lambda_i^{\delta}\tau(z), \quad z \in U_i, \; i = 1, \ldots, d. \tag{5.5}$$

An immediate consequence of (5.5) is the fact that

$$\sum_{i=1}^{d} \lambda_i^{-\delta} = 1. \tag{5.6}$$

In Section 6 we will show that (5.4) or (5.5) allows us to replace our dynamics f with a linear dynamics for which (5.5) still holds. In the case when all f_i are linear, there is no τ satisfying (5.5)—this was shown in Section 3. This contradiction will finish the proof of Theorem 1.1.

Section 6 is devoted to the reduction of (5.4) to the linear case. We will devote the rest of this section to the proof of the removability of J for functions Ψ_{ij} that are a priori holomorphic in $U \setminus J$. Obviously it is sufficient to prove that Ψ_{ij} is bounded in a neighborhood of J. Then we will finish the proof using the fact that the set J on $\mathbf{R}$ is removable for bounded and analytic functions on $U \setminus J$ if $H_1(J) = 0$.

Let J be a subset of $[-1, 1]$, Q be the square $[-2, 2] \times [-2, 2]$, and $Q_{\pm}$ be its upper and lower halves. By considering a portion of J and stretching, we may suppose that $Q \subset \tilde{U}$. The following result plays a key role here.

Lemma [H, pp. 184–185]. *Let f be a function holomorphic in the unit disc $\mathbf{D}$ and having there the estimate* $(0 < \sigma < 1)$

$$|f(z)| \le C_1 \exp\left(\frac{C_2}{(1 - |z|)^{\sigma}}\right).$$

Also, outside a closed set E on $\partial\mathbf{D}$, the function f has boundary values; that is, $f^*(\xi) = \lim_{z \to \xi} f(z)$ for $\xi \in \partial\mathbf{D} \setminus E$, and

$$|f^*(\xi)| \le 1 \quad \forall \xi \in \partial\mathbf{D} \setminus E.$$

If $H_{1-\sigma}(E) = 0$, then

$$|f(z)| \le 1, \quad z \in \mathbf{D}.$$

For the convenience of the reader we will give a simple proof which repeats the one found in [H].

Proof. It is sufficient to prove that $|f(0)| \le 1$. In fact, everything is invariant under Möbius transformation of $\mathbf{D}$. To prove that $|f(0)| \le 1$ let us consider a finite family of disjoint intervals $\{l_i^{\epsilon}\}_{i=1}^{N}$ such that

$$E \subset \bigcup_{i=1}^{N} l_i^\epsilon \quad \text{and} \quad \sum_{i=1}^{N} |l_i^\epsilon|^{1-\sigma} \leq \epsilon. \tag{5.7}$$

Let $Q_i = \{z \in \mathbf{R} : z/|z| \in l_i^\epsilon, 1-|z| \leq |l_i^\epsilon|\}$. Denote by G the domain $\mathbf{D} \setminus \bigcup_{i=1}^N \bar{Q}_i$ and apply the Jensen formula to the function $\log|f|$, which is subharmonic in G and continuous on $\bar{G}$ (w_G is the harmonic measure of G). Then

$$\log|f(0)| \leq \int_{\partial G} \log|f(\xi)|\, dw_G(\xi) \leq \sum_{i=1}^{n} \int_{\partial Q_i} \log|f(\xi)|\, dw_G(\xi).$$

It is easy to see that

$$\int_{\partial Q_i} \log|f(\xi)|\, dw_G(\xi) \leq C_3 \int_0^{|l_i^\epsilon|} \frac{dx}{x^\sigma} \leq C_4 |l_i^\epsilon|^{1-\sigma}.$$

Taking (5.7) into account completes the proof. □

Now we wish to apply this lemma to Ψ_{ij} in Q_+ and Q_- separately. The "singular" set now is J and $H_{\delta+\epsilon}(J) = 0$. So it is sufficient to have a global estimate

$$|\Psi_{ij}(z)| \leq C_1 \exp\left(\frac{C_2}{|\operatorname{Im} z|^{(1-\delta-\epsilon)}}\right) \quad \text{for } z \in \tilde{U}.$$

But a much better estimate

$$\Psi_{ij}(z)| \leq c \operatorname{dist}(z, J)^{\delta-1} \tag{5.8}$$

follows from (4.9). The second condition of Hruščëv's lemma—the fact that Ψ_{ij} is bounded outside the "singular" set J—follows from (4.22).

REMARK. We cannot remove the singularities of Ψ_{ij} by a standard "Cauchy integral procedure," because (5.8) is not sufficient for that. An estimate by $\operatorname{dist}(z, J)^{\delta+\epsilon-1}$ would be sufficient. From (4.13) one deduces easily that

$$|e^{2\alpha_i}\partial\tau_i - e^{2\alpha_j}\partial\tau_j| \leq c \operatorname{dist}(z, J)^{\delta+\epsilon-1}.$$

But $e^{2\alpha_i}\partial\tau_i - e^{2\alpha_j}\partial\tau_j$ coincides with Ψ_{ij} only on the real line.

We would like to note, however, that by using just a slightly more tedious consideration one can avoid the application of Hruščëv's lemma. Its application is justified for our goals by its elegance and straightforwardness.

6. Reducing (5.4) to the Linear Case

Here we use an idea of M. Lyubich. We call our dynamics $f: \bigcup_{i=1}^d U_i \to U$ *linear* if $f_i' \equiv \lambda_i$ for constants λ_i and $i = 1, \ldots, d$.

Our goal is to show that f is conformally equivalent to a linear dynamics. Let $z \in U$ and $\{z_{-n}\}$ be the backward orbit $z_{-n} = f_i^{-n} z$ for $i = 1, \ldots, d$. We put

$$\beta_i(z) = \sum_{n=1}^{\infty} (\log|f'(z_{-n})| - \log|\lambda_i|), \tag{6.1}$$

where λ_i is $f'(p_i)$ and p_i is a fixed point of f_i. We restate (5.4) as follows: There exist constants $b_1, \ldots, b_d$ such that $\beta_i - b_i \equiv \beta_j - b_j$. Denoting these differences by β we see that β satisfies the homologeous equation

$$\beta(fz)-\beta(z)=\log|f'(z)|-\log|\lambda_i|, \quad z\in U_i. \tag{6.2}$$

Function β is harmonic in U; denoting by $\tilde{\beta}$ its harmonic conjugate, we introduce the following function that is holomorphic in U:

$$\log h'(z)=-(\beta(z)+i\tilde{\beta}(z)). \tag{6.3}$$

In a small neighborhood V of J this function h is univalent. Let $W=h(V)$, $V_i=f_i^{-1}(V)$, $W_i=h(V_i)$, and $\lambda(z)=h\circ f\circ h^{-1}(z)$ on W. By this definition of λ,

$$\frac{h'(fz)}{h'(z)}=\frac{\lambda'(h(z))}{f'(z)}, \quad z\in V_i,\ i=1,\dots,d. \tag{6.4}$$

At the same time, combining (6.2) and (6.3) yields

$$\frac{|h'(fz)|}{|h'(z)|}=\frac{|\lambda_i|}{|f'(z)|}, \quad z\in V_i,\ i=1,\dots,d. \tag{6.5}$$

Comparing (6.4) and (6.5), we conclude that

$$|\lambda'(w)|\equiv\lambda_i, \quad w\in W_i,\ i=1,\dots,d.$$

But λ' is holomorphic on W_i and so coincides with a constant.

The result is that (5.4) implies that the dynamics f is conformally equivalent to linear dynamics. But for linear dynamics, assertion (5.5) was disproved in Section 3. Finally, we conclude that for any Cantor repeller J with the property (S), $w\perp H_\delta$ holds and

$$\dim w<\dim J.$$

This completes the proof of Theorem 1.1.

7. Polynomial-like Mappings

In this section we would like to give another proof that (5.5) leads to a contradiction. This proof will work only for polynomial-like mappings, but it is short and worthy of inclusion.

In [DH], Douady and Hubbard introduced an important class of holomorphic dynamical systems—so-called polynomial-like (p.-l.) mappings. A p.-l. mapping of degree d is a triple (f,U,W), where U and W are conformal discs with analytic boundaries with $\bar{U}\subset W$, and $f:U\to W$ is a branched covering of degree d. One may consider sets

$$K_f=\bigcap_{n\ge1} f^{-n}U$$

and $J(f)=\partial K_f$. The following proposition reveals the topological structure of K_f (see [DH, p. 296]).

PROPOSITION 7.1. *The set K_f is connected if and only if all the critical points of f belong to K_f. If none of them belongs to K_f, then $K_f=J(f)$ and is a Cantor set.*

We will be interested in the last case, in which we deal with a certain subfamily of the family of all Cantor repellers.

PROPOSITION 7.2. *Let f be a p.-l. mapping of degree d and $J=J(f)$ be a Cantor repeller. Let $\delta=\dim J$. Let $f^{-1}(U)=\bigcup_{i=1}^{d} U_i$, where the U_i are the components of $f^{-1}U$. Let p_i be a fixed point of $f_i=f\,|\,U_i$ and $\lambda_i=|f'(p_i)|$. Then each nonnegative function τ that is harmonic in $U\setminus J$ and satisfies*

$$\tau(fz)=\lambda_i^{\delta}\tau(z) \quad \textit{for } z\in U_i \textit{ and } i=1,\dots,d \tag{7.1}$$

must be zero.

Proof. As before, we conclude that (7.1) holds and so w is absolutely continuous with respect to H_δ (see Section 2.4).

Now let us extend τ to a harmonic function in W:

$$\hat{\tau}(z)=\lambda_i^{\delta}\tau(f_i^{-1}z).$$

Locally $\hat{\tau}$ is a well-defined harmonic function on W. Furthermore, continuation about the critical points of f shows $\hat{\tau}$ to be single-valued because of (7.1). Thus

$$\hat{\tau}(fz)=\lambda_i^{\delta}\tau(z)=\lambda_j^{\delta}\tau(z), \quad z\in U,$$

and so $\lambda_i=\lambda_j$ for all $i,j=1,\dots,d$. Taking (5.6) into account we see that $\lambda_i^{\delta}=d$, and we have shown that

$$\tau(fz)=d\tau(z), \quad z\in\bigcup_{i=1}^{d} U_i, \tag{7.2}$$

where τ is nonnegative and harmonic in $U\setminus J$. Suppose now that τ does not vanish. Assertion (7.2) with positive harmonic function τ in $U\setminus J$ implies that the harmonic measure is absolutely continuous with respect to the measure m of maximal entropy of $f: J\hookleftarrow$.

The mutual absolute continuity of w and m implies that f is conformally equivalent to a polynomial. This is proved in [LV]. Using this conformal change of variable we conclude that, for a polynomial p with Cantor-like $J(p)$, the harmonic measure is absolutely continuous with respect to the Hausdorff measure H_δ and $\delta=\dim J(p)$. But this is impossible by a result of Zdunik [Z]. □

8. Dynamical Dichotomy

Let $\delta_0=\dim w$. We have proved that $\delta_0<\delta=\dim J(f)$. But even more is proved. Namely, let us show also that $w\perp H_{\delta_0}$. Let us determine the order of magnitude of $w(Q_X)$ when $x\in X$, $X\to x$ for w a.e. x. Recall that μ denotes the invariant harmonic measure. By the Manning formula ([Man]; see also (2.7)),

$$\delta_0=\frac{\int_J \phi_\mu\,d\mu}{\int_J \log|f'|\,d\mu},$$

where $\phi_\mu = -\log \mathcal{G}_\mu$ is the potential of μ and $\mathcal{G}_\mu$ is the Jacobian of μ as in Section 2.

We introduce the function ψ defined on J by $\psi = \phi_\mu - \delta_0 \log|f'|$, and consider the sequence

$$\psi_n = \psi \circ f^n \quad \text{for } n = 0, 1, 2, \ldots$$

of functions as a stationary process on the probability space (J, μ). In Section 2.2 it was shown that the sequence of random variables $X_0(x) = x_0$, $X_n = X_0 \circ f^n$ on (J, μ), consists of exponentially independent variables. Thus,

$$\left| \int_J \psi_k \psi_{n+k} \, d\mu \right| \le C q^n$$

for certain C and $q \in (0,1)$. Such random variables are said to be *asymptotically independent*. Hence there exists a finite number

$$\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \int (\psi_1 + \cdots + \psi_n)^2 \, d\mu$$

called the *variance* of the process $\{\psi_n\}_{n \ge 1}$. The condition $\sigma \ne 0$ plays a crucial part in the theory of asymptotically independent random variables (see [IL; PS]). In [IL] it is proved that only the following two possibilities can occur:

(1) If $\sigma = 0$, then there exists $\gamma \in L^2(\mu)$ such that

$$\psi = \gamma \circ f - \gamma. \tag{8.1}$$

(2) If $\sigma \ne 0$, then

$$\mu\{S_n > \sigma \sqrt{n \log\log n} \text{ indefinitely often}\} = 1. \tag{8.2}$$

If (8.1) holds, then ϕ_μ and $\delta_0 \log|f'|$ are $L^2(\mu)$-homological. As both functions are Hölder, it is easy to derive that they are Hölder homological and

$$\phi_\mu - \delta_0 \log|f'| = \gamma \circ f - \gamma, \quad \gamma \in \Lambda^\epsilon(J).$$

In particular,

$$c_i \le \sum_{|X|=n} (\operatorname{diam} Q_X)^{\delta_0} \le c_2,$$

so $\dim w = \delta_0 = \delta = \dim J$, contradicting our main result. In other words, (8.1) is impossible and (8.2) implies that for w a.e. x for infinitely many cylinders X^i ($|X^i| = n_i$), we have

$$w(X^i) \ge (\operatorname{diam} Q_{X^i})^{\delta_0} \exp(\sigma \sqrt{n_i \log\log n_i}),$$

which means that $w \perp H_{\delta_0}$.

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