Two Metric Invariants for Riemannian Manifolds

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1. Examples and Main Theorems

In this short note we shall define two metric invariants for a closed metric manifold (i.e., a topological manifold with a metric). Each of them has its own topological significance. The purpose of this note is to associate each invariant with some familiar geometric invariants of Riemannian manifolds, for example, curvatures, diameter and volume. Our first metric invariant concerns the orbits of finite groups acting on metric manifolds. In [N], Newman proved that at least one orbit is not too small if the action of a finite group on a closed connected metric manifold is nice. More precisely, Newman proved the following theorem.

THEOREM (Newman). *If M is a closed topological manifold with a metric d, then there exists a positive number $\eta=\eta(M,d)$ depending only on M and d such that every finite group G acting effectively on M has at least one orbit of diameter at least η.*

For instance, it can be shown that the unit n-sphere S^n with its canonical metric d has $\eta(S^n,d)\geq 1$, and that the flat torus $(T^n,d)=(\mathbf{R}^n,\text{can})/\mathbf{Z}^n$ has $\eta(T^n,d)\geq\frac{1}{4}$.

Cernavskii [C] generalized Newman's theorem to the setting of finite-to-one open mappings on metric manifolds. We shall say that an open finite-to-one proper surjective map f (which is not a homeomorphism) from a metric manifold M to a metric space Y is a *pseudo-submersion,* and that $f^{-1}(f(x))$ is an orbit of f at x and is denoted by $O_f(x)$. In [MR], McAuley and Robinson expanded upon Cernavskii's result and obtained the following theorem.

THEOREM. *If M is a closed topological manifold with a metric d, then there is a positive number $\eta=\eta(M,d)$ such that if Y is a metric space and $f: M\to Y$ is a pseudo-submersion then there is at least one point $y\in Y$ with* $\operatorname{diam} f^{-1}(y)\geq\eta$.

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These results motivate the following definition of the Newman constant $\eta(M,d)$ of a closed metric manifold (M,d). Assume that there exists at least one pseudo-submersion f from M to a metric space. Then we define the Newman constant $\eta(M,d)$ of M to be the supremum of η with the property that, for every pseudo-submersion $f: M \to Y$, there exists an $x \in M$ with diam $O_f(x) \geq \eta$. Otherwise, we define $\eta(M,d)$ to be infinity.

For a closed connected Riemannian manifold (M,g) with induced metric d, it is believed that the Newman constant $\eta(M,d)$ should depend only on some geometric invariants of (M,g). In fact, H. T. Ku, M. C. Ku, and Mann [KKM] obtained an estimate of $\eta(M,d)$ in terms of the injectivity radius $i(M)$ and the sectional curvature $K(M)$ of M. Using our Newman constant, their result can be stated in the following manner.

THEOREM (Ku–Ku–Mann).
(1) *If $K(M) \leq 0$, then $\eta(M,d) \geq i(M)/2$.*
(2) *If $K(M) \leq k^2$, then $\eta(M,d) \geq (2/\pi)\min\{\pi/2k, i(M)/2\}$.*

Note that the number $r_0 = \min\{\pi/2k, i(M)/2\}$ is related to the convexity radius of M. Namely, for each $x \in M$ and $r \leq r_0$, the geodesic ball $B(x,r)$ is strongly convex (cf. [CE]); that is, for each pair of points p and q in $B(p,r)$, there is a unique minimal geodesic from p to q inside the ball $B(x,r)$.

Our first result will show that the Newman constant depends only on some "weaker" geometric invariants. Let $\mathbf{M}(k,D,v,n)$ denote the class of closed connected Riemannian n-manifolds (M,g) with sectional curvature $K(M) \geq k$, diameter $d(M) \leq D$ and volume $\mathrm{vol}(M) \geq v > 0$. It is worth noticing that without an upper bound on the sectional curvature there is no estimate for the convexity radius, the injectivity radius, or even the contractibility radius of M. Our first main result is contained in Theorem 1.

THEOREM 1. *There is a positive number $\eta^* > 0$ depending only on the constants k, D, v, and n such that if $M \in \mathbf{M}(k,D,v,n)$ then $\eta(M,d) \geq \eta^*$.*

The following three examples show that Theorem 1 is optimal.

EXAMPLE 1. Let $S(r)$ denote the circle in $\mathbf{R}^2$ with radius r. Consider for $n \geq 1$ the 2-tori $T_n = S(1/n) \times S(1)$ with the product metric d. We have $K(T_n) = 0$, $d(T_n) \leq 4\pi$, and $\mathrm{vol}(T_n) = 4\pi^2/n \to 0$ as $n \to \infty$. Let the group $\mathbf{Z}_2 = \{e, g\}$ act effectively on T_n by

$$g\left(\frac{1}{n}e^{i\theta}, e^{i\phi}\right) = \left(\frac{1}{n}e^{i(\theta+\pi)}, e^{i\phi}\right)$$

for all $\theta, \phi \in [0, 2\pi]$. Thus for every point $x \in T_n$, the orbit $\mathbf{Z}_2 x$ of x has diameter $= \pi/n$. This implies that $\eta(T_n, d) \leq \pi/n \to 0$ as $n \to \infty$. This shows that the lower volume bound is needed.

EXAMPLE 2. To see that one needs the diameter assumption, consider for $n \geq 1$ the 2-tori $T_n = S(1/n) \times S(n)$ with the product metric d. We have

$K(T_n)=0$, $\mathrm{vol}(T_n)=4\pi^2$, and $d(T_n)\geq n\pi\to\infty$ as $n\to\infty$. Again consider the group $\mathbf{Z}_2$ acting effectively on T_n by

$$g\left(\frac{1}{n}e^{i\theta}, ne^{i\phi}\right)=\left(\frac{1}{n}e^{i(\theta+\pi)}, ne^{i\phi}\right)$$

for all $\theta,\phi\in[0,2\pi]$. Then $d(x,gx)=\pi/n$ for all $x\in T_n$. This implies that $\eta(T_n,d)\leq\pi/n\to 0$ as $n\to\infty$.

EXAMPLE 3. Consider a smooth function $f_n\colon[-1,1]\to\mathbf{R}$ with $f_n(t)=\sqrt{1-t^2}$ when $\sqrt{3}/2\leq|t|\leq 1$, $f_n(t)\in[\frac{1}{4},1]$ when $|t|\leq\sqrt{3}/2$, and the curve $(t,f_n(t))\,|_{t\in[-\sqrt{3}/2,\sqrt{3}/2]}$ in $\mathbf{R}^2$ has length in $[n,2n]$. Such a function does exist and is easy to construct. Let Σ_n be the surface of revolution of the curve $(t,(1/n)f_n(t))$, $t\in[-1,1]$, in $\mathbf{R}^3$ around the x-axis; that is,

$$\Sigma_n=\left\{\left(t,\frac{1}{n}f_n(t)\sin\theta,\frac{1}{n}f_n(t)\cos\theta\right),(t,\theta)\in[-1,1]\times[0,2\pi]\right\}.$$

We have $\pi/2\leq\mathrm{vol}(\Sigma_n)\leq\pi$, while $\min K(\Sigma_n)\to-\infty$ and $d(\Sigma_n)\geq n\to\infty$ as $n\to\infty$. To obtain surfaces with uniform diameter upper bound, we modify the surfaces Σ_n as follows.

Consider a smooth function $f_n(t,\theta)\colon[-1,1]\times[0,2\pi]\to\mathbf{R}$ with the properties: $f_n(t,\theta)=f_n(t)$ when $(t,\theta)\in[-\sqrt{3}/2,\sqrt{3}/2]\times[\pi/2,2\pi]$ or $\sqrt{3}/2\leq|t|\leq 1$; $f_n(t,\theta)=\sqrt{1-t^2}$ when $(t,\theta)\in[-\sqrt{3}/2,\sqrt{3}/2]\times[\pi/8,\pi/4]$, and $f_n(t,\theta)\in[\frac{1}{8},2]$, otherwise. It is also easy to check that such a function does exist and can be constructed via the Urysohn lemma. Let $\tilde{\Sigma}_n$ be the surface with induced metric in $\mathbf{R}^3$ given by

$$\tilde{\Sigma}_n=\left\{\left(t,\frac{1}{n}f_n(t,\theta)\sin\theta,\frac{1}{n}f_n(t,\theta)\cos\theta\right),(t,\theta)\in[-1,1]\times[0,2\pi]\right\}.$$

Now one has $\mathrm{vol}(\tilde{\Sigma}_n)\geq\frac{3}{4}\mathrm{vol}(\Sigma_n)\geq 3\pi/8$, $d(\tilde{\Sigma}_n)\leq 4\pi$, and $\min K(\tilde{\Sigma}_n)\to-\infty$ as $n\to\infty$. Let the group $\mathbf{Z}_2$ act effectively on $\tilde{\Sigma}_n$ by

$$g\left(t,\frac{1}{n}f_n(t,\theta)\sin\theta,\frac{1}{n}f_n(t,\theta)\cos\theta\right)$$
$$=\left(t,\frac{1}{n}f_n(t,\theta+\pi)\sin(\theta+\pi),\frac{1}{n}f_n(t,\theta+\pi)\cos(\theta+\pi)\right)$$

for all $(t,\theta)\in[-1,1]\times[0,2\pi]$. Thus, for all x in $\tilde{\Sigma}_n$, $d(x,gx)\leq\pi/n$. This implies that $\eta(\tilde{\Sigma}_n,d)\leq\pi/n\to 0$ as $n\to\infty$. Hence, the lower curvature bound is necessary.

In contrast to the Newman constant, our second metric invariant comes from a consideration of continuous maps all of whose orbits (or preimages) are small. Let (Y,d) be a metric space. A continuous map $f\colon Y\to X$ is called a *δ-map* if $\mathrm{diam}\, f^{-1}(x)\leq\delta$ for each $x\in X$. We define the homotopy constant $\delta(M,d)$ of a closed metric manifold (M,d) to be the supremum of $\delta\geq 0$ with the property that for every δ-map $f\colon M\to N$ where N is a closed n-manifold, f is a homotopy equivalence.

In the next section we shall see that the homotopy constant $\delta(M,d)$ is always a positive number. It will not be surprising that the homotopy constant of a closed Riemannian manifold (M,g) can also be estimated by the convexity radius of M. In fact, we shall show that it depends only on some weaker geometric invariants.

THEOREM 2. *There is a $\delta^*>0$ depending only on the constants k, D, v, and n such that if $M\in \mathbf{M}(k,D,v,n)$ then $\delta(M,d)\geq \delta^*$.*

Following Examples 1–3, we construct three examples to show that Theorem 2 is also optimal.

EXAMPLE 4. Let the 2-tori $T_n=S(1/n)\times S(1)$ be as in Example 1. We have $K(T_n)=0$, $d(T_n)\leq 4\pi$, and $\operatorname{vol}(T_n)=4\pi^2/n\to 0$ as $n\to\infty$. Define a proper map $f_n\colon T_n\to S(1)\times S(1)$ by

$$f_n\left(\frac{1}{n}e^{i\theta}, e^{i\phi}\right)=(e^{i\pi}, e^{i\phi})$$

for all $\theta,\phi\in[0,2\pi]$. Hence f_n is a (π/n)-map. Since $f_n(T_n)=\{e^{i\pi}\}\times S(1)$, the homology groups of T_n and $S(1)$ show that f_n can not be a homotopy equivalence. This means that the lower volume bound in Theorem 2 is necessary.

EXAMPLE 5. Let the 2-tori $T_n=S(1/n)\times S(1)$ be as in Example 2. We have $K(T_n)=0$, $\operatorname{vol}(T_n)=4\pi^2$, and $d(T_n)\geq n\pi\to\infty$ as $n\to\infty$. Define a proper map $f_n\colon T_n\to S(1)\times S(1)$ by

$$f_n\left(\frac{1}{n}e^{i\theta}, ne^{i\phi}\right)=(e^{i\pi}, e^{i\phi})$$

for all $\theta,\phi\in[0,2\pi]$. Again, $f_n(T_n)=\{e^{i\pi}\}\times S(1)$ and thus f_n cannot be a homotopy equivalence. On the other hand, f_n is also a (π/n)-map. This implies that one needs the diameter assumption in Theorem 2.

EXAMPLE 6. Let $\tilde{\Sigma}_n$ be the surface $\mathbf{R}^3$ as in Example 3. We have $d(\tilde{\Sigma}_n)\leq 4\pi$, $\operatorname{vol}(\tilde{\Sigma}_n)\geq 3\pi/8$, and $\min K(\tilde{\Sigma}_n)\to-\infty$ as $n\to\infty$. Define a proper map $h_n\colon \tilde{\Sigma}_n\to S(1)\times S(1)$ by

$$h_n\left(t, \frac{1}{n}f_n(t,\theta)\sin\theta, \frac{1}{n}f_n(t,\theta)\cos\theta\right)=(e^{i\pi}, e^{it})$$

for all $(t,\theta)\in[-1,1]\times[0,2\pi]$. Hence, h_n is a (π/n)-map. Since $\tilde{\Sigma}_n$ and $S(1)\times S(1)$ have different homotopy types, h_n cannot be a homotopy equivalence. Thus, the lower curvature bound in Theorem 2 is also needed.

2. The Newman Constant and the Homotopy Constant

In this section we shall prove Theorems 1 and 2. Our method is based on a combination of a recent result of Grove, Petersen, and the author and a theorem of Chapman and Ferry. Let us first recall these two results.

THEOREM 3 [GPW]. *Given a sequence of closed Riemannian n-manifolds M_i in $\mathbf{M}=\mathbf{M}(k,D,v,n)$, $n\geq 2$, there exists a subsequence M_j with the following properties:*

(1) *M_j converges to a compact metric space X in the Gromov–Hausdorff topology;*

(2) *the product space $X\times N$ is a topological manifold for any closed topological manifold N with dimension ≥ 2; and*

(3) *there exist ϵ_j-homotopy equivalences: $f_j: M_j\to X$ and $g_j: X\to M_j$ with*

$$|d(f_j(x),f_j(y))-d(x,y)|<\epsilon_j \quad \text{and} \quad |d(g_j(p),g_j(q))-d(p,q)|<\epsilon_j$$

for all $x,y\in M_j$ and $p,q\in X$, with $\epsilon_j\to 0$ as $j\to\infty$.

THEOREM 4 [CF; F]. *Let M be a topological manifold of dimension $n\geq 5$ with a metric d. Given any $\epsilon>0$, there is a positive number δ depending on (M,d) and ϵ such that if N is another closed n-manifold and $f: M\to N$ is a δ-map, then f is homotopic through ϵ-maps to a homeomorphism.*

Now we start to prove Theorem 1. In view of Theorem 3(2) and Theorem 4, we need a property of the Newman constant for product spaces in order to lift the manifolds in $\mathbf{M}(k,D,v,n)$ to higher dimensions.

PROPOSITION 1. *Let (M_1,d_1) and (M_2,d_2) be two closed metric manifolds. Then*

$$\eta(M_1\times M_2, d_1\times d_2)\leq \min\{\eta(M_1,d_1),\eta(M_2,d_2)\}.$$

Proof. If there is no pseudo-submersion from M_1 to a metric space, then $\eta(M_1,d_1)=\infty\geq\eta(M_1\times M_2,d_1\times d_2)$. Otherwise, for any pseudo-submersion $f: M_1\to N$, we can define $F: M_1\times M_2\to N\times M_2$ by $F(x,y)=(f(x),y)$ for all $x\in M_1$ and $y\in M_2$. It is easy to see that F is also a pseudo-submersion. By the definition of the Newman constant, there is a point $(x,y)\in M_1\times M_2$ with $\operatorname{diam} F^{-1}(F(x,y))\geq\eta=\eta(M_1\times M_2,d_1\times d_2)$. Since $F^{-1}(F(x,y))=F^{-1}(f(x),y)=(f^{-1}(f(x)),y)$, we have $\operatorname{diam} f^{-1}(f(x))\geq\eta$. Hence $\eta\leq\eta(M_1,d_1)$. Similarly, $\eta\leq\eta(M_2,d_2)$ and the proposition holds. □

Proof of Theorem 1. Suppose that Theorem 1 is not true. Then one could find a sequence of Riemannian n-manifolds M_i in $\mathbf{M}(k,D,v,n)$ with $\eta_i=\eta(M_i,d_i)\to 0$ as $i\to\infty$. According to Theorem 3, there is a subsequence M_j with the properties (1)–(3) of Theorem 3. Let S^5 be the unit 5-sphere in $\mathbf{R}^6$ with its canonical metric, and consider the product manifold $M_j\times S^5$ with the product metric. Hence, $M_j\times S^5$ converges to the metric manifold $X\times S^5$ in the Gromov–Hausdorff topology.

Define the map $G_j: X\times S^5\to M_j\times S^5$ by $G_j(p,q)=(g_j(p),q)$ for all $(p,q)\in X\times S^5$, where g_j is as in Theorem 3(3). Note that G_j is also an ϵ_j-map with $\epsilon_j\to 0$ as $j\to\infty$. Thus the Chapman–Ferry theorem implies that there are homeomorphisms $\tilde{G}_j: X\times S^5\to M_j\times S^5$ such that, for all $x,y\in X\times S^5$,

$$(+)\qquad |d(\tilde{G}_j(x),\tilde{G}_j(y))-d(x,y)|<\epsilon_j+\eta_0/4$$

for large j, where $\eta_0 = \eta(X \times S^5, d)$. Since $X \times S^5$ is homeomorphic to $M_j \times S^5$ for large j, there are pseudo-submersions from $X \times S^5$ to metric spaces. Hence, the theorem of McAuley and Robinson implies that η_0 is a finite positive number.

By Proposition 1, one has $\eta_j = \eta(M_j \times S^5, d) \le \eta(M_j, d) \to 0$ as $j \to \infty$. Hence, we can choose a j_0 such that if $j \ge j_0$ then $\eta_j \le \eta_0/4$ and $\epsilon_j \le \eta_0/4$. Suppose that $F: M_j \times S^5 \to Y$ is a pseudo-submersion; then $F \circ \tilde{G}_j: X \times S^5 \to Y$ is also a pseudo-submersion. Hence, there is at least a point $y \in Y$ with $\operatorname{diam}(F \circ \tilde{G}_j)^{-1}(y) \ge \eta_0$.

Pick two points p, q in $(F \circ \tilde{G}_j)^{-1}(y)$ with $d(p, q) \ge \eta_0$. Then (+) gives

$$d(\tilde{G}_j(p), \tilde{G}_j(q)) \ge d(p, q) - (\epsilon_j + \eta_0/4) \ge \eta_0 - \eta_0/2 = \eta_0/2$$

for all $j \ge j_0$. Thus $\eta_j \ge \eta_0/2$ for all $j \ge j_0$. This contradicts the fact that $\eta_j \le \eta_0/4$ for all $j \ge j_0$, and the proof of Theorem 1 is completed. □

For the homotopy constant, we also have a similar property about the product spaces.

PROPOSITION 2. *Let (M_1, d_1) and (M_2, d_2) be two closed metric manifolds. Then we have*

$$\delta(M_1 \times M_2, d_1 \times d_2) \le \min\{\delta(M_1, d_1), \delta(M_2, d_2)\}.$$

Proof. The proof is quite similar to that of Proposition 2. Let

$$\delta_0 = \delta(M_1 \times M_2, d_1 \times d_2).$$

Suppose that N is a closed manifold with the same dimension as M_1, and that $f: M_1 \to N$ is a (δ_0)-map. The map $F: M_1 \times M_2 \to N \times M_2$ is defined by $F(x, y) = (f(x), y)$ for all $x \in M_1$ and $y \in M_2$. Then F is also a δ_0-map. By the definition of the homotopy constant δ_0, F is a homotopy equivalence; that is, there exists a map $G: N \times M_2 \to M_1 \times M_2$ and two homotopies H_1 and H_2 such that

$$F \circ G \overset{H_1}{\simeq} \mathrm{Id}_{N \times M_2} \quad \text{and} \quad G \circ F \overset{H_2}{\simeq} \mathrm{Id}_{M_1 \times M_2}.$$

Fix a point $y_0 \in M_2$. Then the map $p \mapsto G(p, y_0)$ can be expressed by $G(p, y_0) = (g_1(p), g_2(p))$, where $g_1: N \to M_1$ and $g_2: N \to M_2$ are continuous maps determined by G and y_0. Let $\pi_i: M_1 \times M_2 \to M_i$, $i = 1, 2$, be the projections, and let $\bar{H}_i(p, t) = \pi_i \circ H_i(p, y_0, t)$, $i = 1, 2$. Then we have

$$f \circ g_1 \overset{\bar{H}_1}{\simeq} \mathrm{Id}_N \quad \text{and} \quad g_1 \circ f \overset{\bar{H}_2}{\simeq} \mathrm{Id}_{M_1}.$$

That is, f is a homotopy equivalence. Thus $\delta(M_1, d_1) \ge \delta_0$. Similarly, $\delta(M_2, d_2) \ge \delta_0$, and the proposition follows. □

Taking $(M_2, d_2) = (S^5, \text{can})$ in Proposition 2 and $\epsilon = 1$ in Theorem 4, Proposition 2 along with Theorem 4 then yields the following proposition.

PROPOSITION 3. *For each closed topological n-manifold M with a metric d, the homotopy constant $\delta(M, d)$ is positive.*

Proof of Theorem 2. We will prove Theorem 2 by contradiction. Suppose that Theorem 2 is not true. Then one could find a sequence of manifolds M_i in $\mathbf{M}(k, D, v, n)$ such that the homotopy constant of M_i goes to zero as i approaches infinity. According to Theorem 3 and the argument in the proof of Theorem 1, we have a subsequence M_j that converges to a compact metric space X and has the following properties:

(1) $X \times S^5$ is a metric $(n+5)$-manifold;

(2) there are homeomorphisms $\tilde{G}_j: X \times S^5 \to M_j \times S^5$ with

$$|d(\tilde{G}_j(x), \tilde{G}_j(y)) - d(x, y)| < \epsilon_j$$

for all $x, y \in X \times S^5$, where $\epsilon_j \to 0$ as $j \to \infty$; and

(3) by Proposition 2, $\delta_j = \delta_j(M_j \times S^5, d) \leq \delta(M_j, d) \to 0$ as $j \to \infty$.

Proposition 3 implies that $\delta_0 = \delta(X \times S^5, d) > 0$. Hence one can choose a j_0 such that if $j \geq j_0$ then $\epsilon_j \leq \delta_0/4$. Suppose that N is a closed $(n+5)$-manifold and that $F: M_j \times S^5 \to N$ is a $(\delta_0/2)$-map. Then $F \circ \tilde{G}_j: X \times S^5 \to N$ is a (δ_0)-map for all $j \geq j_0$. By the definition of the homotopy constant δ_0, we know that $F \circ \tilde{G}_j$ must be a homotopy equivalence for all $j \geq j_0$. Hence, $F = (F \circ \tilde{G}_j) \circ \tilde{G}_j^{-1}$ is also a homotopy equivalence for all $j \geq j_0$. Thus, one has $\delta_j \geq \delta_0/2$. This contradicts property (3), and the proof of Theorem 2 is completed. □

REMARK. Our present proofs do not yield estimates on the constants $\eta(M, d)$ and $\delta(M, d)$. It will be interesting to obtain explicit estimates of these two constants in terms of the numbers k, D, v, and n. Our proofs of Theorems 1 and 2 also show that the Newman constant and the homotopy constant have uniform positive lower bound for a certain class $\mathbf{M}$ of closed Riemannian n-manifolds, provided that $\mathbf{M}$ is C^0-precompact in the Gromov–Hausdorff topology; see also [W]. However, these two constants $\eta(M, d)$ and $\delta(M, d)$, viewed as functions of (M, d), are not continuous with respect to the Gromov–Hausdorff topology.

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