Equivariant Harmonic Maps between Compact Riemannian Manifolds of Cohomogeneity 1

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Dedicated to Professor Tadashi Nagano on his sixtieth birthday

Introduction

The aim of this paper is to develop the earlier works of Smith [12; 13] (see also [5; 7]) on symmetric harmonic maps between spheres and also his reduction theorem on equivariant harmonic maps. As applications, we construct explicitly (1) new harmonic maps from 2-flat tori into spheres, complex projective spaces, quaternion projective spaces or complex quadrics, (2) new harmonic maps from complex projective spaces into spheres, and (3) new nonholomorphic harmonic maps from complex projective spaces into other complex projective spaces.

At the present time, the only known method to construct harmonic maps from a higher-dimensional Riemannian manifold to another compact Riemannian manifold of positive curvature involves investigation of minimal submanifolds, holomorphic maps, or equivariant maps whose symmetry of large isometry group action reduces the Euler–Lagrange equation to an ordinary differential equation. We are concerned with the last one, and assume that both domain and target Riemannian manifolds are of cohomogeneity 1; that is, both admit the isometry group actions having orbits of codimension 1. Then we derive the ordinary differential equations of the equivariant harmonic maps between them (cf. §2), and solving these equations, we construct the above harmonic maps (cf. §§3–6).

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1. Preliminaries

In this section, we prepare the materials which will be needed in the following arguments.

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1.1. Cohomogeneity 1 actions

Let (M, g) be a compact Riemannian manifold and K a compact Lie group. It is said that the group K acts *cohomogeneity 1* on (M, g) (cf. [2; 8; 17]) if K acts isometrically and effectively on (M, g) which has orbits of codimension 1; that is, there exists a point x in M such that $\dim(Kx) = \dim M - 1$. The orbit space $K \backslash M$ of K on M is the closed interval $[0, l]$ or the circle. In the former case we give the fine structure of such K and (M, g), and show the examples following [2], [15], and [17].

Let $c(t)$, $0 \leq t \leq l$, be the geodesic of (M, g) representing the orbit space $K \backslash M$. Let J_t be the isotropy subgroup of K at $c(t)$. Then, for $0 < t < l$, the subgroups J_t are the same group J. The Lie algebra $\mathfrak{k}$ of K can be decomposed orthogonally with respect to the $\mathrm{Ad}(K)$-invariant inner product $\langle\, ,\rangle$ on $\mathfrak{k}$ as

$$\mathfrak{k} = \mathfrak{j} \oplus \mathfrak{m},$$

where $\mathfrak{j}$ is the Lie algebra of J and $\mathfrak{m}$ is an $\mathrm{Ad}(J)$-invariant subspace of $\mathfrak{k}$.

The mapping $K/J \times [0, l] \ni (kJ, t) \mapsto kc(t) \in M$ is an onto mapping, and the restriction to $K/J \times (0, l)$ is smooth and its image of $K/J \times (0, l)$, denoted by $\mathring{M}$, is open and dense in M. The metric g on M can be expressed on $\mathring{M}$ *as*

$$g = dt^2 + g_t. \tag{1.1}$$

Here g_t is the K-invariant metric on the orbit $Kc(t)$, $0 < t < l$, given by

$$g_t(X_{c(t)}, Y_{c(t)}) = \alpha_t(X, Y), \quad X, Y \in \mathfrak{m}, \tag{1.2}$$

where, for $X \in \mathfrak{m}$, we define a vector field on M, denoted by the same letter X, by

$$X_p = \left.\frac{d}{dt}\right|_{t=0} \exp tX \cdot p \quad \text{for } p \in M.$$

We assume the inner product α_t on $\mathfrak{m}$ is given as

$$\alpha_t(X_i, X_j) = f_i(t)^2 \delta_{ij}, \quad 1 \leq i, j \leq m-1, \tag{1.3}$$

where $m = \dim M$. Here $\{X_i\}_{i=1}^{m-1}$ is an orthonormal basis of $(\mathfrak{m}, \langle\, ,\rangle)$.

We also give an orthonormal frame field $\{e_i\}_{i=1}^{m-1}$ on a neighborhood W of $c(t)$, $0 < t < l$, as follows:

$$W := \{kc(s);\, k \in U \subset \exp(\mathfrak{m}),\, |s - t| < \epsilon\}; \tag{1.4}$$

$$\begin{cases} (e_i)_{kc(s)} := f_i(s)^{-1} \tau_{k*} X_{ic(s)}, & 1 \leq i \leq m-1, \\ (e_m)_{kc(s)} := \tau_{k*} \dot{c}(s), \end{cases} \tag{1.5}$$

where τ_k, $k \in K$, is the action of K on M, $\dot{c}(s)$ is the tangent vector of $c(s)$, and U is a small neighborhood of e in $\exp(\mathfrak{m})$.

1.2. Examples

EXAMPLE 1.1 (S^n, can). In this case, Hsiang–Lawson [8], Takagi–Takahashi [14], and Asoh [1] classified cohomogeneity 1 actions on (S^n, can). A typical example is $K = SO(p+1) \times SO(n-p) \subset SO(n+1)$ acting naturally on the unit sphere S^n, $0 \le p \le n$. The representing geodesic $c(t)$ is

$$c(t) = \cos t\, \xi_1 + \sin t\, \xi_{p+2} \in S^n, \tag{1.6}$$

where $0 \le t \le \pi/2$ when $1 \le p \le n-2$ and $0 \le t \le \pi$ when $p = 0$ or $n-1$, and where $\{\xi_j\}_{j=1}^{n+1}$ is the standard basis of $\mathbf{R}^{n+1}$, that is, $\xi_j = (0, \ldots, 0, 1, 0, \ldots, 0)$. We only treat the case $1 \le p \le n-2$. In this case, $J_t = J = SO(p) \times SO(n-p-1)$, $J_0 = SO(p) \times SO(n-p)$, and $J_{\pi/2} = SO(p+1) \times SO(n-p-1)$. We can take an orthonormal basis $\{X_i\}_{i=1}^{n-1}$ of $\mathfrak{m}$ as

$$X_i := \begin{pmatrix} E_i & 0 \\ 0 & 0 \end{pmatrix}, \ 1 \le i \le p; \quad X_i := \begin{pmatrix} 0 & 0 \\ 0 & E_{i-p} \end{pmatrix}, \ p+1 \le i \le n-1,$$

where E_i is the $(p+1) \times (p+1)$ matrix whose $(i+1, 1)$ entry is 1, $(1, i+1)$ entry is -1, and all other entries are zero, and where E_{i-p} is the similar $(n-p) \times (n-p)$ matrix. Then the functions $f_i(t)$, $0 < t < \pi/2$, are

$$f_1(t) = \cdots = f_p(t) = \cos t, \qquad f_{p+1}(t) = \cdots = f_{n-1}(t) = \sin t. \tag{1.7}$$

EXAMPLE 1.2 ($\mathbf{C}P^n$, can). Classification of cohomogeneity 1 actions on ($\mathbf{C}P^n$, can) was done by Uchida [15]. A typical example of cohomogeneity 1 action on ($\mathbf{C}P^n$, can) is $K = SU(p+1) \times SU(n-p) \subset SU(n+1)$, $0 \le p \le n$, acting in the natural way on $\mathbf{C}P^n$. The representing geodesic is given by

$$c(t) = [\cos t\, \xi_1 + \sin t\, \xi_{p+2}] \in \mathbf{C}P^n, \quad 0 \le t \le \pi/2. \tag{1.8}$$

The isotropy subgroups J_t of K at $c(t)$ are

$$J_t = \left\{ \left(\begin{array}{cc|cc} \xi & 0 & & \\ 0 & x & \multicolumn{2}{c}{0} \\ \hline & & \xi & 0 \\ \multicolumn{2}{c|}{0} & 0 & y \end{array} \right) ; \begin{array}{l} \xi \in U(1), x \in U(p), y \in U(n-p-1), \\ \det x = \det y = \xi^{-1} \end{array} \right\}, \quad 0 < t < \frac{\pi}{2},$$

$$J_0 = \left\{ \left(\begin{array}{cc|c} \xi & 0 & \\ 0 & x & 0 \\ \hline \multicolumn{2}{c|}{0} & y \end{array} \right) ; \begin{array}{l} \xi \in U(1), x \in U(p), y \in SU(n-p), \\ \det x = \xi^{-1} \end{array} \right\}, \quad \text{and}$$

$$J_{\pi/2} = \left\{ \left(\begin{array}{c|cc} x & \multicolumn{2}{c}{0} \\ \hline & \xi & 0 \\ 0 & 0 & y \end{array} \right) ; \begin{array}{l} x \in SU(p+1), \xi \in U(1), y \in U(n-p-1), \\ \det y = \xi^{-1} \end{array} \right\}.$$

In this paper, we always fix our Riemannian metric g on $\mathbf{C}P^n$ as one-fourth of the Fubini–Study metric. Then the inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ is given by

$$\langle X, Y\rangle = -\tfrac{1}{2}\operatorname{Tr}(XY), \quad X, Y \in \mathfrak{su}(n+1).$$

We can take as an orthonormal basis of $(\mathfrak{m}, \langle\ ,\ \rangle)$

$$X_j = \begin{pmatrix} E_j & 0 \\ 0 & 0 \end{pmatrix},\ 1 \le j \le p; \quad X_j = \begin{pmatrix} F_{j-p} & 0 \\ 0 & 0 \end{pmatrix},\ p+1 \le j \le 2p;$$

$$X_j = \begin{pmatrix} 0 & 0 \\ 0 & E_{j-2p} \end{pmatrix},\ 2p+1 \le j \le a; \quad X_j = \begin{pmatrix} 0 & 0 \\ 0 & F_{j-b} \end{pmatrix},\ a+1 \le j \le 2n-2,$$

where $b := n+p-1$ and X_{2n-1} is the diagonal $(n+1)\times(n+1)$ matrix

$$[ip\eta_1, -i\eta_1, \dots, -i\eta_1, -i(n-p-1)\eta_2, i\eta_2, \dots, i\eta_2],$$

and where $\eta_1 := (n-p)\eta$, $\eta_2 = (p+1)\eta$, and

$$\eta^2 := (n-p)^{-1}(p+1)^{-1}((n-p)p + (n-p-1)(p+1))^{-1}.$$

E_j (resp. F_j) are $(p+1)\times(p+1)$ or $(n-p)\times(n-p)$ matrices whose $(j+1, 1)$ entry is 1 (resp. i), $(1, j+1)$ entry is -1 (resp. i), and all other entries are zero. The functions $f_i(t)$, $0 < t < \pi/2$, are

$$f_j(t) = \begin{cases} \cos t, & 1 \le j \le 2p, \\ \sin t, & 2p+1 \le j \le 2n-2; \end{cases}$$

$$f_j(t) = \frac{1}{\sqrt{2}}\left(\frac{p}{p+1} + \frac{n-p-1}{n-p}\right)^{1/2} \sin 2t, \quad j = 2n-1.$$

EXAMPLE 1.3 ($\mathbf{H}P^n$, can). The classification of cohomogeneity 1 actions on ($\mathbf{H}P^n$, can) was done by Iwata [9]. A typical cohomogeneity 1 action is $K = S_p(n)\times S_p(1) \subset S_p(n+1) = \{x \in U(2n+2);\ {}^t x J_{n+1} x = J_{n+1}\}$, where

$$J_{n+1} = \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix}$$

and I_{n+1} is the identity matrix of degree $n+1$. The representing geodesic $c(t) = z(t)\cdot o \in \mathbf{H}P^n$, $0 \le t \le \pi/2$, where o is the origin of $\mathbf{H}P^n = S_p(n+1)/S_p(n)\times S_p(1)$ and

$$z(t) = \exp t\begin{pmatrix} E_1 & 0 \\ 0 & E_1 \end{pmatrix} \in S_p(n+1),$$

where E_1 is the same $(n+1)\times(n+1)$ matrix as in Examples 1.1 and 1.2. The isotropy subgroups J_t, $0 \le t \le \pi/2$, are $J_t = S_p(1)\times S_p(n-1)$, $0 < t < \pi/2$, $J_0 = K$, and $J_{\pi/2} = S_p(1)\times S_p(1)\times S_p(n-1)$. We define the inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ by $\langle X, Y\rangle = -\frac{1}{4}\operatorname{Tr}(XY)$, $X, Y \in \mathfrak{su}(2n+2)$, which corresponds to a constant multiple of the standard $S_p(n+1)$-invariant Riemannian meric on $\mathbf{H}P^n$. We take an orthonormal basis $\{X_j\}_{j=1}^{4n-1}$ of $(\mathfrak{m}, \langle\ ,\ \rangle)$ in the way that the functions $f_j(t)$ are given as

$$f_j(t)=\begin{cases}\sin t, & 1\le j\le 4n-4,\\ \sin 2t, & 4n-3\le j\le 4n-1.\end{cases} \tag{1.10}$$

EXAMPLE 1.4 (Q_n, can). The complex quadric $Q_n = SO(n+2)/SO(2)\times SO(n)$ has a cohomogeneity 1 action of $K=SO(n+1)\subset SO(n+2)$ (cf. [15]). The representing geodesic $c(t)$, $0\le t\le \pi/2$, is $c(t)=z(t)\cdot o$, where o is the origin of $Q_n=SO(n+2)/SO(2)\times SO(n)$ and $z(t)$ is the 1-parameter subgroup of $SO(n+2)$ given by

$$z(t)=\begin{pmatrix}\cos t & 0 & -\sin t & & & \\ 0 & 1 & 0 & & & \\ \sin t & 0 & \cos t & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1\end{pmatrix}.$$

The isotropy subgroups J_t of K at $c(t)$ are $J_t=SO(n-1)$ when $0<t<\pi/2$, $J_0=SO(n)$, and $J_{\pi/2}=SO(2)\times SO(n-1)$. The inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ is given by

$$\langle X,Y\rangle=-\tfrac{1}{2}\mathrm{Tr}(XY),\quad X,Y\in\mathfrak{so}(n+2),$$

which corresponds to the $SO(n+2)$-invariant Riemannian metric on Q_n. An orthonormal basis $\{X_j\}_{j=1}^{2n-1}$ of $(\mathfrak{m},\langle\ ,\ \rangle)$ can be chosen in such a manner that the functions $f_j(t)$ are given by

$$\begin{cases} f_1(t)=\cos t;\\ f_j(t)\equiv 1, \quad 2\le j\le n;\\ f_j(t)=\sin t, \quad n+1\le j\le 2n-2.\end{cases} \tag{1.11}$$

2. Reduction of the Euler–Lagrange Equation

2.1. Setting

Our situation is that both compact Riemannian manifolds (M,g) and (N,h) admit cohomogeneity 1 actions of compact Lie groups K and G, respectively. We may write the orbit spaces $K\backslash M=[0,l]$ and $G\backslash N=[0,\bar{l}]$ and fix the geodesics $c(t)$ $(0\le t\le l)$ and $\bar{c}(r)$ $(0\le r\le \bar{l})$ of (M,g) and (N,h), which represent the orbit spaces $K\backslash M$ and $G\backslash M$, respectively. We denote by J_t and H_r the isotropy subgroups of K and G at $c(t)$ and $\bar{c}(r)$. Then, for $0<t<l$ and $0<r<\bar{l}$, J_t and H_r are the same groups J and H, respectively.

Let $A\colon K\to G$ be a Lie group homomorphism. A mapping $\phi\colon M\to N$ is *A-equivariant* (cf. [5, (4.17)]) if $\phi(kx)=A(k)\phi(x)$, $k\in K$, $x\in M$. Then, for any A-equivariant map $\phi\colon M\to N$, there exist a function $r\colon[0,l]\to[0,\bar{l}]$ and a map $\Psi\colon[0,l]\to G$ such that $\phi(c(t))=\Psi(t)\bar{c}(r(t))$, $t\in[0,l]$. The A-equivariance of ϕ implies that

$$\Psi(t)^{-1}A(J_t)\Psi(t)\subset H_{r(t)},\quad t\in[0,l]. \tag{2.1}$$

Conversely, given a function $r: [0,l] \to [0,\bar{l}]$ and a map $\Psi: [0,l] \to G$ with (2.1), we get an A-equivariant map $\phi: M \to N$ by

$$\phi(kc(t)) = A(k)\Psi(t)\bar{c}(r(t)), \quad k \in K, \ t \in [0,l], \tag{2.2}$$

and every A-equivariant map of M into N can be obtained in this way.

2.2. *Continuity*

PROPOSITION 2.1. *Assume that the functions $r: [0, l] \to [0, \bar{l}]$ and Ψ: $[0,l] \to G$ satisfy* (2.1) *and extend continuously to $(-\epsilon, l+\epsilon)$, $\epsilon > 0$. Then the A-equivariant map $\phi: M \to N$ given by* (2.2) *is continuous.*

Proof. (1) Continuity on $\mathring{M}$. We take a coordinate around any point $k_0c(t)$ in $\mathring{M}$ as $k_0 \exp(\sum_{j=1}^{m-1} x_jX_j)c(t) \mapsto (x_1, \ldots, x_{m-1}, t)$. Since ϕ satisfies

$$\phi\left(k_0 \exp\left(\sum_{j=1}^{m-1} x_jX_j\right)c(t)\right) = A(k_0)\exp\left(\sum_{j=1}^{m-1} x_jA(X_j)\right)\Psi(t)\bar{c}(r(t)),$$

the continuity of ϕ at $k_0c(t)$ follows immediately from those of Ψ and r.

(2) Continuity on $Kc(0)$. Recall that the coordinate around any point in $Kc(0)$ is given as follows (cf. [2; 17]): Let $\mathfrak{j}_0$ be the Lie algebra of J_0 in $\mathfrak{k}$, and let $\mathfrak{l}_0, \mathfrak{m}_0$ be the subspaces of $\mathfrak{j}_0, \mathfrak{k}$ which are orthogonal to $\mathfrak{j}, \mathfrak{j}_0$ with respect to $\langle\,,\rangle$, respectively; that is, $\mathfrak{k} = \mathfrak{j}_0 \oplus \mathfrak{m}_0$, $\mathfrak{j}_0 = \mathfrak{j} \oplus \mathfrak{l}_0$. Defining $\phi(c(-t)) := \Psi(-t)\bar{c}(r(-t))$, $\phi(c(t))$ is continuous in t with $|t| < \epsilon$. Any point p around $k_0c(0)$ with $k_0 \in \exp(\mathfrak{l}_0)$ can be expressed as

$$p = k_0 \exp Y \exp X\, c(t) = k_0 \exp Y \exp(X + \pi X)c(-t),$$

with $Y \in \mathfrak{m}_0$, $|t| < \epsilon$ and normalized $X \in \mathfrak{l}_0$ by $\exp 2\pi X \in J$, and $p \to k_0c(0)$ if and only if $Y \to 0$ and $t \to 0$. Since

$$\begin{aligned}\phi(p) &= A(k_0)\exp A(Y) \exp A(X)\phi(c(t)) \\ &= A(k_0)\exp A(Y) \exp(A(X) + \pi A(X))\phi(c(-t)),\end{aligned}$$

if $p \to k_0c(0)$ then

$$\phi(p) \to A(k_0)\exp A(X)\phi(c(0)) = A(k_0)\exp(A(X) + \pi A(X))\phi(c(0)).$$

But since $\phi(c(0)) = \Psi(0)\bar{c}(r(0))$, $\Psi(0)^{-1}A(J_0)\Psi(0) \subset H_{r(0)}$ (by condition (2.1)), and $X \in \mathfrak{l}_0 \subset \mathfrak{j}_0$, we have

$$\exp A(X)\phi(c(0)) = \exp(A(X) + \pi A(X))\phi(c(0)) = \phi(c(0)).$$

Therefore $\phi(p) \to A(k_0)\phi(c(0)) = \phi(k_0c(0))$ if $p \to k_0c(0)$.

(3) Continuity in $Kc(l)$ follows in a similar way. □

2.3. *The Euler-Lagrange Equation*

We calculate the tension field

$$\tau(\phi) := \sum_{j=1}^{m} (\tilde{\nabla}_{e_j}\phi_* e_j - \phi_*\nabla_{e_j}e_j)$$

for an A-equivariant map $\phi: M \to N$ which is smooth on $\mathring{M}$.

Since $\phi\circ\tau_k=\tau_{A(k)}\circ\phi$, $k\in K$, and K and G act isometrically on (M,g) and (N,h), the tension field satisfies

$$(2.3)\qquad \tau(\phi)(kx)=\tau_{A(k)*}\tau(\phi)(x),\quad k\in K,\ x\in \mathring{M}.$$

Then we only have to calculate $\tau(\phi)$ at $c(t)$, $0<t<l$. In order to do this, let us recall that the metrics g and h on M and N are described as follows (cf. §1.1): Let $\mathfrak{m}$ and $\mathfrak{n}$ be the subspaces of $\mathfrak{k}$ and $\mathfrak{g}$ which are invariant under $\mathrm{Ad}(J)$ and $\mathrm{Ad}(H)$, and orthogonal to $\mathfrak{j}$ and $\mathfrak{h}$ with respect to the inner product $\langle\ ,\ \rangle$ on $\mathfrak{k}$ and $\mathfrak{g}$, respectively. Then

$$g=dt^2+g_t,\qquad h=dr^2+h_r,$$

and the inner products α_t and β_r on $\mathfrak{m}$ and $\mathfrak{n}$ are induced from g_t and h_r. Choose orthonormal bases $\{X_j\}_{j=1}^{m-1}$ and $\{Y_a\}_{a=1}^{n-1}$ of $(\mathfrak{m},\langle\ ,\ \rangle)$ and $(\mathfrak{n},\langle\ ,\ \rangle)$ in such a way that

$$\alpha_t(X_i,X_j)=f_i(t)^2\delta_{ij}\quad\text{and}\quad \beta_r(Y_a,Y_b)=h_a(r)^2\delta_{ab}.$$

As in formula (1.5), define orthonormal frame fields $\{e_j\}_{j=1}^{m-1}$ and $\{\bar{e}_a\}_{a=1}^{n-1}$ on neighborhoods W and $\bar{W}$ of $c(t)$ and $\bar{c}(r)$, respectively. Then we obtain the following theorem.

THEOREM 2.2. (i) *Assume that the function $r(t)\colon[0,l]\to[0,\bar{l}]$ satisfies $r(0)=0$, $r(l)=\bar{l}$, and $0<r(t)<\bar{l}$ for $0<t<l$. Then the tension field of an A-equivariant map $\phi\colon(M,g)\to(N,h)$, which is smooth on $\mathring{M}$, can be described as*

$$\begin{aligned}(2.4)\quad \tau(\phi)(c(t))=&({}^N\nabla_{\phi_*\dot c}\phi_*\dot c)(\phi(c(t)))+\left(\sum_{j=1}^{m-1}f_j(t)^{-1}\frac{df_j}{dt}\right)\phi_*\dot c\\ &-\sum_{j=1}^{m-1}\sum_{i=1}^{n-1}f_j(t)^{-2}h_i(r)^{-3}\frac{dh_i}{dr}\beta_r(Y_i,U_j)\tau_{\Psi_*}\dot{\bar c}(r)\\ &+\sum_{j=1}^{m-1}f_j(t)^{-2}\tau_{\Psi_*}\{[V_i,U_i]+V_r(U_i,U_i)\\ &\qquad-(\mathrm{Ad}(\Psi^{-1})A(U_t(X_i,X_i))_{\mathfrak{n}}\}_{\bar c(r)}\end{aligned}$$

for $r=r(t)$, $\Psi=\Psi(t)$, $0<t<l$. Here we put $A(X_j)=\mathrm{Ad}(\Psi)(U_j+V_j)$, $U_j\in\mathfrak{n}$, $V_j\in\mathfrak{h}$, and $X_{\mathfrak{n}}$ is the $\mathfrak{n}$-component of $X\in\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{n}$. Also $U_t(X,Y)\in\mathfrak{m}$ and $V_r(X',Y')\in\mathfrak{n}$ are defined by

$$2\alpha_t(U_t(X,Y),Z)=\alpha_t(X,[Z,Y]_{\mathfrak{m}})+\alpha_t([Z,X]_{\mathfrak{m}},Y),\quad X,Y,Z\in\mathfrak{m};$$

$$2\beta_r(V_r(X',Y'),Z')=\beta_r(X',[Z',Y']_{\mathfrak{n}})+\beta_r([Z',X']_{\mathfrak{n}},Y'),\quad X',Y',Z'\in\mathfrak{n}.$$

(ii) *In particular, if the orbits $(Kc(t),g_t)$, $(G\bar{c}(r),h_r)$ satisfy*

$$(2.5)\qquad U_t(X,X)=0\quad\textit{and}\quad V_r(X',X')=0\quad\textit{for } X\in\mathfrak{m},X'\in\mathfrak{n},$$

and $\phi(c(t))=\bar{c}(r(t))$—that is, $\Psi(t)=e$, $0<t<l$—then

$$\text{(2.6)}\quad \tau(\phi)(c(t))=\Bigg\{\ddot r+\left(\sum_{j=1}^{m-1} f_j(t)^{-1}\frac{df_j}{dt}\right)\dot r - \sum_{j=1}^{m-1}\sum_{i=1}^{n-1} f_j(t)^{-2}h_i(r)^{-3}\frac{dh_i}{dr}\beta_r(Y_i, A(X_j))^2\Bigg\}\dot{\bar c}(r(t)) + \sum_{j=1}^{m-1} f_j(t)^{-2}[V_j, U_j]_{\bar c(r(t))}$$

for $r=r(t)$, $0<t<l$. *In this case,* $\tau(\phi)(c(t))\equiv 0$ *if and only if*

$$\text{(2.7)}\quad \ddot r+\left(\sum_{j=1}^{m-1} f_j(t)^{-1}\frac{df_j}{dt}\right)\dot r-\sum_{j=1}^{m-1}\sum_{i=1}^{n-1} f_j(t)^{-2}h_i(r)^{-3}\frac{dh_i}{dr}\beta_r(Y_i, A(X_j))^2=0$$

and

$$\text{(2.8)}\qquad C:=\sum_{j=1}^{m-1} f_j(t)^{-2}[V_j, U_j]=0 \ \ in\ \mathfrak{g}.$$

REMARK 2.3. The condition (2.5) is satisfied for all the Examples 1.1–1.5.

Proof. By the assumption $0<r(t)<\bar l$ for $0<t<l$, we may take the orthonormal frame fields $\{\bar e_a\}_{a=1}^{n-1}$ around $\bar c(r(t))$. We first calculate $h(\bar e_a, \phi_* e_j)$ around $c(t)$, $0<t<l$, in the equation

$$\text{(2.9)}\qquad \tilde\nabla_{e_j}\phi_* e_j=\sum_{a=1}^{n}\{e_j(h(\bar e_a, \phi_* e_j))\bar e_a + h(\bar e_a, \phi_* e_j)\tilde\nabla_{e_j}\bar e_a\}.$$

For $k\in U\subset \exp(\mathfrak{m})$, we write

$$A(k)=\Psi(t)n(k)h(k)\Psi(t)^{-1}$$

with $n(k)\in\exp(\mathfrak{n})$, $h(k)\in H$. Then we use the following lemmas.

LEMMA 2.4. *For* $\Psi=\Psi(t)$, $r=r(t)$:

(i)
$$\phi_* e_{j\,kc(t)}=\begin{cases} f_j(t)^{-1}\tau_{\Psi_*}\tau_{n(k)_*}\mathrm{Ad}(h(k))U_{j\bar c(r)}, & 1\le j\le m-1,\\ \dot r(t)\tau_{\Psi_*}\tau_{n(k)_*}\dot{\bar c}(r), & j=m.\end{cases}$$

(ii)
$$\bar e_{a\,\phi(kc(t))}=\begin{cases} h_a(r)^{-1}\tau_{\Psi_*}\tau_{n(k)_*}Y_{a\bar c(r)}, & 1\le a\le n-1,\\ \tau_{\Psi_*}\tau_{n(k)_*}\dot{\bar c}(r), & a=n.\end{cases}$$

(iii) $h(\bar e_a, \phi_* e_j)(kc(t))$
$$=\begin{cases} h_a(r)^{-1}f_j(t)^{-1}\beta_r(Y_a, \mathrm{Ad}(h(k))U_j), & 1\le j\le m-1,\ 1\le a\le n-1,\\ 0, & j=m,\ 1\le a\le n-1 \ \text{or}\ 1\le j\le m-1,\ a=n,\\ \dot r(t), & j=m,\ a=n.\end{cases}$$

LEMMA 2.5. (i) *For* $1\le j\le m-1$,

$$e_j(h(\bar e_a, \phi_* e_j))(c(t))=\begin{cases} f_j(t)^{-2}h_a(r)^{-1}\beta_r(Y_a, [V_j, U_j]), & 1\le a\le n-1,\\ 0, & a=n.\end{cases}$$

(ii) *For* $j=m$,

$$e_j(h(\bar{e}_a,\phi_* e_m))(c(t))=\begin{cases}0 & 1\leq a\leq n-1,\\ \ddot{r}(t), & a=n.\end{cases}$$

Therefore, at $c(t)$,

$$(2.10)\quad \sum_{a=1}^{n} e_j(h(\bar{e}_a,\phi_* e_j))\bar{e}_a = \begin{cases}\sum_{a=1}^{n-1} f_j(t)^{-2}h_a(r)^{-1}\beta_r(Y_a,[V_j,U_j])\bar{e}_a, & 1\leq j\leq m-1,\\ \ddot{r}(t)\bar{e}_n, & j=m.\end{cases}$$

On the other hand, $\tilde{\nabla}_{e_j}\bar{e}_a(c(t))=\sum_{b=1}^{n}h(\bar{e}_b,\phi_* e_j)({}^N\nabla_{\bar{e}_b}\bar{e}_a)_{\phi(c(t))}$.

LEMMA 2.6. (i) *For* $1\leq a,b\leq n-1$, *and at a point* $\phi(c(t))=\Psi(t)\bar{c}(r(t))$,

$${}^N\nabla_{\bar{e}_b}\bar{e}_a=h_a(r)^{-1}h_b(r)^{-1}\tau_{\Psi_*}\left\{-h_b(r)\frac{dh_b}{dr}\delta_{ab}\dot{\bar{c}}(r)+\frac{1}{2}[Y_b,Y_a]+V_r(Y_b,Y_a)\right\}.$$

(ii) *For* $a=b=n$,

$${}^N\nabla_{\bar{e}_n}\bar{e}_n=\|\phi_*\dot{c}\|^{-1}\left(\frac{d}{dt}\|\phi_*\dot{c}\|^{-1}\right)\phi_*\dot{c}+\|\phi_*\dot{c}\|^{-2}\,{}^N\nabla_{\phi_*\dot{c}}\phi_*\dot{c}.$$

In (2.9), since

$$\sum_{a=1}^{n}h(\bar{e}_a,\phi_* e_j)\tilde{\nabla}_{e_j}\bar{e}_a=\sum_{a,b=1}^{n}h(\bar{e}_a,\phi_* e_j)h(\bar{e}_a,\phi_* e_j)\,{}^N\nabla_{\bar{e}_b}\bar{e}_a,$$

we may calculate $\tilde{\nabla}_{e_j}\phi_* e_j$ making use of the above lemmas. On the other hand, since

$$(\nabla_{X_j}X_j)_{c(t)}=-f_j\frac{df_j}{dt}\dot{c}(t)+U_t(X_j,X_j)_{c(t)},$$

$\nabla_{\dot{c}}\dot{c}=0$, and $\phi_* X_{c(t)}=\tau_{\Psi_*}(\mathrm{Ad}(\Psi^{-1})A(X))_{\mathfrak{n}\bar{c}(r)}$, for $X\in\mathfrak{m}$ we may compute $\phi_*\nabla_{e_j}e_j$, and thus we obtain Theorem 2.2. □

From this point on, we will consider an A-equivariant map ϕ satisfying the condition

$$(*)\qquad \phi(c(t))=\bar{c}(r(t)),\quad \text{that is,}\quad \Psi\equiv 1,$$

and classify all A-equivariant harmonic maps satisfying $(*)$ between cohomogeneity 1 Riemannian manifolds in Section 1.

3. Equivariant Harmonic Maps from Tori

3.1

In this section, we classify all A-equivariant harmonic maps satisfying the condition $(*)$ from a flat torus into a Riemannian manifold (N,h) in the examples in Section 1 which admit a cohomogeneity 1 action of a compact Lie group G.

Let $(M, g) = (T^2, g)$ be a flat torus where

$$K = SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \theta \in \mathbf{R} \right\}$$

acts cohomogeneity 1. Then $T^2 = \mathbf{R}/2\pi\mathbf{Z} \times \mathbf{R}/T\mathbf{Z}$ with some $T > 0$. The constant $T > 0$ will be determined as the period of the periodic solution of some ODE. In this case, the isotropy subgroup J_t of K consists only of the identity

$$\mathfrak{m} = \mathbf{R}X_2 \quad \text{with } X_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } f_1(t) \equiv 1.$$

Let A be a homomorphism of $K = SO(2)$ with the compact Lie group G. Then any A-equivariant map ϕ with $(*)$ of (T^2, g) into (N, h) is of the form

$$\phi(t, \theta) := \exp\theta A(X_1)\bar{c}(r(t)), \tag{3.1}$$

where $\bar{c}(r)$, $0 \le r \le \bar{l}$, is the representing geodesic of (N, h), and A satisfies always the condition (2.1). The conditions for ϕ to be harmonic are, due to Theorem 2.2, as follows:

(1) If we put $A(X_1) = U_1 + V_1$, $U_1 \in \mathfrak{n}$, and $V_1 \in \mathfrak{h}$, then

$$[V_1, U_1] = 0; \tag{3.2}$$

(2) the function $r(t)$ satisfies

$$\ddot{r} - \sum_{i=1}^{n-1} h_i(r)\frac{dh_i}{dr}\beta_r(Y_i, U_1)^2 = 0. \tag{3.3}$$

Then, since $\exp\theta A(X_1) = \exp\theta U_1 \exp\theta V_1$ due to (3.2) and $\exp\theta V_1\bar{c}(r(t)) = \bar{c}(r(t))$, we get

$$\phi(t, \theta) = \exp\theta U_1\bar{c}(r(t)). \tag{3.4}$$

Note that (3.3) and (3.4) depend only on $U_1 \in \mathfrak{n}$.

Summing up, we obtain the following proposition.

PROPOSITION 3.1. *Let A be a homomorphism $K = SO(2) \to G$ satisfying $A(X_1) = U_1 + V_1$, $U_1 \in \mathfrak{n}$, $V_1 \in \mathfrak{h}$, and $[U_1, V_1] = 0$. Then all the A-equivariant harmonic maps $\phi: (T^2, g) \to (N, h)$ with condition $(*)$ which admit cohomogeneity 1 action of G are exhausted by all solutions $r(t)$ of* (3.3) *such that $\bar{c}(r(t))$ is periodic in t with period T, and the corresponding harmonic maps are given by* (3.4).

In the rest of this section, we will give explicitly A-equivariant harmonic maps of flat tori into several kinds of cohomogeneity 1 Riemannian manifolds.

3.2. Spheres [I]

We first consider the action of $G = SO(p+1) \times SO(n-p) \subset SO(p+1)$ on S^n as in Example 1.1. In this case, the map ϕ in (3.4) is of the form

$$\phi(\theta,t)=\cos r(t)\exp\theta\begin{pmatrix}0 & -{}^tX\\ X & 0\end{pmatrix}\xi_1 + \sin r(t)\exp\theta\begin{pmatrix}0 & -{}^tY\\ Y & 0\end{pmatrix}\xi_{p+2}\in S^n, \tag{3.5}$$

where $\{\xi_j\}_{j=1}^{n+1}$ is the standard basis of $\mathbf{R}^{n+1}$, and $X\in\mathbf{R}^p$ and $Y\in\mathbf{R}^{n-p-1}$ are arbitrary vectors such that both matrices

$$\exp\theta\begin{pmatrix}0 & -{}^tX\\ X & 0\end{pmatrix} \quad\text{and}\quad \exp\theta\begin{pmatrix}0 & -{}^tY\\ Y & 0\end{pmatrix},$$

are periodic in θ with period 2π. The function $r(t)$ is a solution of

$$\ddot r+(\|X\|^2-\|Y\|^2)\sin r\cos r=0 \tag{3.6}$$

such that $(\cos r(t),\sin r(t))$ is periodic in t with period T. Then these exhaust all the A-equivariant harmonic maps with the condition $(*)$ of (T^2,g) into (S^n,can) with the $(SO(p+1)\times SO(n-p))$-action. In particular, in the case of $n=2$ or 3, all such harmonic maps are given as follows.

Case $n=2$: For each $a\in\mathbf{Z}$, all solutions of $\ddot r-a\sin r\cos r=0$ with periodic $(\cos r(t),\sin r(t))$ in t with period T yield harmonic maps

$$T^2\ni(\theta,t)\mapsto(\cos r(t),\sin r(t)\cos a\theta,\sin r(t)\sin a\theta)\in S^2. \tag{3.7}$$

Case $n=3$: For all $a,b\in\mathbf{Z}$, the similar solutions $r(t)$ of

$$\ddot r+(a^2-b^2)\sin r\cos r=0$$

yield harmonic maps

$$T^2\ni(\theta,t)\mapsto(\cos r(t)\cos a\theta,\cos r(t)\sin a\theta,\sin r(t)\cos b\theta,\sin r(t)\sin b\theta)\in S^3. \tag{3.8}$$

Here we remark that our ODE is $\ddot r-k\sin r\cos r=0$, where $k\in\mathbf{R}$. This is just the pendulum equation $\ddot\phi-k\sin\phi=0$, by putting $\phi=2r$. For each solution ϕ, the function $E=\frac12\dot\phi^2+k\cos\phi$ is constant in t. Therefore it is known that there exist many solutions $r(t)$ such that the position of the pendulum $(\cos r(t),\sin r(t))$ is periodic in time t.

3.3. Spheres [II]

The odd-dimensional sphere S^{2n-1}, $n\ge 3$, admits another cohomogeneity 1 action of $G=SO(2)\times SO(n)$ by $M(n,2,\mathbf{R})\ni X\mapsto lXk^{-1}\in M(n,2,\mathbf{R})$ for $(k,l)\in G$, where $S^{2n-1}=\{X\in M(n,2,\mathbf{R});\ \|X\|=1\}$. In this case, the representing geodesic $\bar c(r)$ is

$$\bar c(r)={}^t\begin{pmatrix}0 & \sin r & 0 & \dots & 0\\ \cos r & 0 & 0 & \dots & 0\end{pmatrix},\quad 0\le r\le\pi/4,$$

and the orthogonal complement $\mathfrak{n}$ of the isotropy Lie algebra of $\mathfrak{g}$ at $\bar c(r)$, $0<r<\pi/4$, is given by

$$\mathfrak{n}=\left\{\left(X,\begin{pmatrix} Y & -{}^tZ \\ Z & 0 \end{pmatrix}\right); X, Y\in\mathfrak{so}(2),\ Z\in M(n-2,2,\mathbf{R})\right\}.$$

We can choose an orthonormal basis $\{Y_i\}_{i=1}^{2n-2}$ of $(\mathfrak{n},\langle\ ,\ \rangle)$ such that the functions $h_j(r)$ are $h_1(r)=(1/\sqrt{2})(\cos r-\sin r)$, $h_2(r)=(1/\sqrt{2})(\cos r+\sin r)$, $h_i(r)=\cos r$ $(3\le i\le n)$, and $h_i(r)=\sin r$ $(n+1\le i\le 2n-2)$.

Then for all $a\in\mathbf{Z}$, $b\in\mathbf{R}$, and $Z, W\in\mathbf{R}^{n-2}$ such that

$$B(\theta):=\exp\theta\begin{pmatrix} 0 & -b & -{}^tZ \\ b & 0 & -{}^tW \\ Z & W & 0 \end{pmatrix}$$

is periodic in θ with period 2π, the mappings

$$(\theta,t)\mapsto B(\theta)\bar{c}(r(t))\begin{pmatrix} \cos a\theta & \sin a\theta \\ -\sin a\theta & \cos a\theta \end{pmatrix}\in S^{2n-1} \tag{3.9}$$

give harmonic maps of (T^2,g) into (S^{2n-1},can) if $r(t)$ is a solution of

$$\ddot{r}+\frac{a^2-b^2}{2}\cos 2r+(\|Z\|^2-\|W\|^2)\sin r\cos r=0 \tag{3.10}$$

such that $(\cos r(t),\sin r(t))$ is periodic in t with period T. All A-equivariant harmonic maps with $(*)$ of (T^2,g) into (S^{2n-1},can) with the action of $SO(2)\times SO(n)$ are obtained in this way.

3.4. Complex Projective Spaces

In the case of Example 1.2, with $G=SU(p+1)\times SU(n-p)\subset SU(n+1)$ acting on $N=\mathbf{C}P^n$, we take arbitrary $a\in\mathbf{R}$, $Z\in\mathbf{C}^p$, and $W\in\mathbf{C}^{n-p-1}$ satisfying that both $p\eta_1 a$ and $(n-p-1)\eta_2 a$ are integers, and that both matrices

$$Z(\theta):=\exp\theta\left(ap\xi_1X_0'+\begin{pmatrix} 0 & -{}^t\bar{Z} \\ Z & 0 \end{pmatrix}\right)\quad\text{and}$$

$$W(\theta):=\exp\theta\left(-a(n-p-1)\xi_2X_0''+\begin{pmatrix} 0 & -{}^t\bar{W} \\ W & 0 \end{pmatrix}\right)$$

are periodic in θ with period 2π, where

$$X_0'=\left[i,-\frac{i}{p},\dots,-\frac{i}{p}\right]\quad\text{and}\quad X_0''=\left[i,-\frac{i}{n-p-1},\dots,-\frac{i}{n-p-1}\right]$$

are diagonal matrices. Then, for any solution of

$$\ddot{r}+(\|W\|^2-\|Z\|^2)\sin r\cos r-2a^2\sin 2r\cos 2r=0 \tag{3.11}$$

such that $(\cos r(t),\sin r(t))$ is periodic in t with period T, the maps

$$(\theta,t)\mapsto[\cos r(t)Z(\theta)\xi_1+\sin r(t)W(\theta)\xi_{p+2}]\in\mathbf{C}P^n$$

exhaust all A-equivariant harmonic maps with $(*)$ of (T^2,g) into $(\mathbf{C}P^n,\mathrm{can})$ with $G=(SU(p+1)\times SU(n-p))$-action. Here η_1,η_2 are the numbers in Example 1.2 and $\{\xi_j\}_{j=1}^{n+1}$ is the standard basis of $\mathbf{R}^{n+1}$.

3.5. Quaternion Projective Spaces

In the case of Example 1.3, for any $X=\sum_{i=1}^{4n-1} a_i X_i$ ($a_i \in \mathbf{R}$) such that $\exp\theta X$ is periodic in θ with period 2π, and for any solution of

$$\ddot{r}-\left(\sum_{i=1}^{4n-4} a_i^2\right)\sin r\cos r-2(a_{4n-3}^2+a_{4n-2}^2+a_{4n-1}^2)\sin 2r\cos 2r=0 \tag{3.12}$$

such that $(\cos r(t), \sin r(t))$ is periodic in t with period T, the maps

$$(\theta, t)\mapsto \exp\theta X\bar{c}(r(t))\in \mathbf{H}P^n$$

give all A-equivariant harmonic maps with $(*)$ of (T^2, g) into $(\mathbf{H}P^n, \mathrm{can})$ with the action of $S_r(n)\times S_r(1)\subset S_r(n+1)$.

3.6. Complex Quadrics

In the case of Example 1.4, for $a\in\mathbf{R}$ and $X, Y\in\mathbf{R}^{n-1}$ such that

$$Z(\theta):=\exp\theta\begin{pmatrix}0 & -a & -{}^tX\\ a & 0 & -{}^tY\\ X & Y & 0\end{pmatrix}$$

is periodic in θ with period 2π, and for any solution $r(t)$ of

$$\ddot{r}+(\|Y\|^2-a^2)\sin r\cos r=0 \tag{3.13}$$

such that $(\cos r(t), \sin r(t))$ is periodic in t with period T, the maps

$$(\theta, t)\mapsto\begin{pmatrix}1 & 0\\ 0 & Z(\theta)\end{pmatrix}z(r(t))\cdot o\in Q_n$$

yield all A-equivariant harmonic maps with $(*)$ of (T^2, g) into (Q_n, can) with the action of $G=SO(n+1)$.

4. Smith's Examples Revisited

Let $(M, g)=(S^m, \mathrm{can})$ and $(N, h)=(S^n, \mathrm{can})$, the standard spheres. Let $K=SO(p+1)\times SO(m-p)\subset SO(m+1)$, $G=SO(q+1)\times SO(n-q)\subset SO(n+1)$ act on S^m and S^n as in Example 1.1, respectively. The representing geodesics $c(t)$, $0\le t\le\pi/2$ and $\bar{c}(r)$, $0\le r\le\pi/2$, are as in (1.6). Let J_t, H_r be the isotropy subgroups of K, G at $c(t), \bar{c}(r)$, respectively. Then the homomorphisms $A\colon K\to G$ satisfying $A(J_t)\subset H_{r(t)}$ under the conditions $0\le r(t)\le\pi/2$ $(0<t<\pi/2)$, $r(t)=0$, $r(\pi/2)=\pi/2$, are described as

$$A\begin{pmatrix}x & 0\\ 0 & y\end{pmatrix}=\begin{pmatrix}\pi'(x) & 0\\ 0 & \pi''(y)\end{pmatrix}. \tag{4.1}$$

Here $(\pi', V_{\pi'})$ and $(\pi'', V_{\pi''})$ are real $(q+1)$- and $(n-q)$-dimensional representations of $SO(p+1)$ and $SO(m-p)$ which are spherical with respect to $SO(p)$ and $SO(m-p-1)$, respectively. Namely, let $\{v_j\}_{j=1}^{q+1}, \{w_i\}_{i=1}^{n-q}$ be the orthonormal bases of $V_{\pi'}, V_{\pi''}$ such that v_1, w_1 are fixed under the actions of

$\pi'(SO(p))$, $\pi''(SO(m-p-1))$, respectively. Represent π' and π'' by matrices with respect to the bases $\{v_j\}$ and $\{w_l\}$ as

$$\pi'(x)v_j=\sum_i \pi'_{ij}(x)v_i \quad \text{and} \quad \pi''(y)w_l=\sum_k \pi''_{kl}(y)w_k,$$

with $x\in SO(p+1)$ and $y\in SO(m-p)$. Then the homomorphism A given by

$$A\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}=\begin{pmatrix} \pi'_{ij}(x) & 0 \\ 0 & \pi''_{kl}(y) \end{pmatrix}$$

satisfies the conditions $A(J_t)\subset H_{r(t)}$, $0\le t\le \pi/2$, and all such homomorphisms are given by this way. The corresponding A-equivariant map with $(*)$ $\phi: S^m\to S^n$ is given by

$$\text{(4.2)}\quad \phi: S^m\ni \cos t\, x\cdot\xi_1+\sin t\, y\cdot \xi_{p+2}$$
$$\mapsto \cos r(t)\sum_{i=1}^{q+1}\pi'_{i1}(x)\xi_i+\sin r(t)\sum_{i=1}^{n-q}\pi''_{i1}(y)\xi_{q+1+i}\in S^n$$

with

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\in K,$$

where the $\{\xi_i\}$s are the standard bases of the Euclidean spaces. The equations for ϕ to be harmonic are

$$\text{(4.3)}\quad \ddot r+\{-p\tan t+(m-p-1)\cot t\}\dot r+\left\{\frac{\lambda'}{\cos^2 t}-\frac{\lambda''}{\sin^2 t}\right\}\sin r\cos r=0$$

and

$$\text{(4.4)}\quad C=(\cos t)^{-2}\sum_{j=1}^{p}[V_j,U_j]+(\sin t)^{-2}\sum_{j=p+1}^{m-1}[V_j,U_j]=0,$$

where $\lambda'=k(k+p-1)$ and $\lambda''=l(l+m-p-2)$, $k,l=0,1,2,\ldots$, are the eigenvalues of the Laplacian on (S^p,can) and (S^{m-p-1},can) corresponding to the spherical representations $(\pi',V_{\pi'})$ and $(\pi'',V_{\pi''})$, respectively, and $A(X_j)=U_j+V_j$; $U_j\in\mathfrak{n}$, $V_j\in\mathfrak{h}$. Then it can be proved by the similar way as Proposition 6.4 that (4.4) holds automatically. Furthermore, for the solution $r(t)$ of (4.3) with the conditions

$$\text{(4.5)}\quad 0\le r(t)\le\pi/2\ \ (0<t<\pi/2),\quad r(0)=0,\quad \text{and}\quad r(\pi/2)=\pi/2,$$

the A-equivariant map ϕ (4.2) is a continuous weakly harmonic map of (S^m,can) into (S^n,can) of Sobolev class L_1^2. By the regularity theorem (cf. [7, p. 397; 3; 11]), this ϕ is a smooth harmonic map. On the other hand, Ding [4] gave the necessary and sufficient conditions for the existence of solutions $r(t)$ of (4.3) with (4.4), which are as follows. Under the assumption $\lambda''(p-1)\ge\lambda'(m-p-2)$, either

(i) $(p-1)^2<4\lambda'$, or
(ii) $(p-1)^2\ge 4\lambda'$ and $\sqrt{(m-p-2)^2+4\lambda''}+\sqrt{(p-1)^2-4\lambda'}<m-3$.

Now let us consider the following particular case:

$$(4.6) \qquad \begin{cases} p=m-2, \ m\geq 3, \\ \pi': SO(m-1)\ni x\mapsto x\in SO(m-1), \\ \pi'': SO(2)\ni y\mapsto y^l\in SO(2), \end{cases}$$

for $l\in\mathbf{Z}$. Then our A-equivariant map ϕ_l is of the form

$$(4.7) \quad \phi_l: S^m\ni \cos t\, x\cdot\xi_1+\sin t(\cos\theta\xi_m+\sin\theta\xi_{m+1}) \\ \mapsto \cos r(t)x\cdot\xi_1+\sin r(t)(\cos a\theta\xi_m+\sin a\theta\xi_{m+1})\in S^m$$

for $x\in SO(m-1)$, where $\{\xi_j\}$ is the standard basis of $\mathbf{R}^{m+1}$. Ding's conditions are reduced to the following by applying $\lambda'=m-2$, $\lambda''=l^2$ ad $p=m-2$ $(m\geq 3)$:

(i) $\Leftrightarrow 3\leq m\leq 7$;

(ii) $\Leftrightarrow m\geq 8$ and $2|l|+\sqrt{(m-3)^2-4(m-2)}<m-3\Leftrightarrow m\geq 8$ and $l=0,\pm 1$.

Making use of the formula $\deg(\phi)=\int_M \varphi^* v_h/\int_M v_g$ for a smooth map ϕ: $(M^m,g)\to(S^m,h)$, we get $\deg(\phi_l)=l$. We therefore obtain the following.

PROPOSITION 4.1 (Smith [12]). *For $3\leq m\leq 7$, the map ϕ_l is a harmonic map of (S^m,can) into itself of degree l for all $l\in\mathbf{Z}$.*

5. Harmonic Maps of $\mathbf{C}P^m$ into S^n

In this section, we give new harmonic maps of $\mathbf{C}P^m$ into S^n. We assume that $(M,g)=(\mathbf{C}P^m,g)$ and $(N,h)=(S^n,\mathrm{can})$, where $g=\frac{1}{4}\times$the Fubini–Study metric. Consider the group actions $K=SU(p+1)\times SU(m-p)\subset SU(m+1)$ on $\mathbf{C}P^m$ as in Example 1.2, and $G=SO(q+1)\times SO(n-q)\subset SO(n+1)$ on S^n as in Example 1.1.

For a function $r(t)$ with $0\leq r(t)\leq\pi/2$ for $0<t<\pi/2$, $r(0)=0$, and $r(\pi/2)=\pi/2$, each homomorphism A of K into G satisfying $A(J_t)\subset H_{r(t)}$ for $0\leq t\leq\pi/2$ is also described as

$$(5.1) \quad A\begin{pmatrix} x & 0 \\ 0 & y\end{pmatrix}=\begin{pmatrix}\pi'(x) & 0 \\ 0 & \pi''(y)\end{pmatrix}, \quad x\in SU(p+1),\ y\in SU(m-p),$$

where $(\pi',V_{\pi'})$ and $(\pi'',V_{\pi''})$ are respectively the real $(q+1)$- and $(n-q)$-dimensional representations of $SU(p+1)$ and $SU(m-p)$ which are spherical with respect to $SU(p)$ and $SU(m-p-1)$—that is, the orthogonal representations of $SU(p+1)$ and $SU(m-p)$ whose representation spaces are the spaces of real-valued eigenfunctions of the Laplacian of $\mathbf{C}P^p$ and $\mathbf{C}P^{m-p-1}$ with the eigenvalues $\lambda'=4k(k+p-1)$ and $\lambda''=4l(l+m-p-2)$, $k,l=0,1,2,\dots$, respectively. Note that

$$q+1=p(p+2k)\left(\frac{(p+k-1)!}{p!\,k!}\right)^2;$$

$$n-q=(m-p-1)(m-p-1+2l)\left(\frac{(m-p+l-2)!}{(m-p-1)!\,l!}\right)^2.$$

Our A-equivariant map $\phi:\mathbf{C}P^m\to S^n$ is

$$\text{(5.2)} \quad \phi: \mathbf{C}P^m \ni [\cos t\, x\cdot\xi_1 + \sin t\, y\cdot \xi_{p+2}]$$
$$\mapsto \cos r(t) \sum_{i=1}^{q+1} \pi'_{i1}(x)\xi_i + \sin r(t) \sum_{i=1}^{n-q} \pi''_{i1}(y)\xi_{q+1+i} \in S^n,$$

with

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in K,$$

where the $\{\xi_i\}$s are the standard bases of the Euclidean spaces. Then the equations for ϕ to be harmonic are

$$\text{(5.3)} \quad \ddot{r} + 2\{-p\tan t + (m-p-1)\cot t + \cot 2t\}\dot{r} + \left(\frac{\lambda'}{\cos^2 t} - \frac{\lambda''}{\sin^2 t}\right)\sin r \cos r = 0$$

and

$$\text{(5.4)} \quad C = (\cos t)^{-2} \sum_{j=1}^{2p} [V_j, U_j] + (\sin t)^{-2} \sum_{j=2p+1}^{2n-2} [V_j, U_j] = 0,$$

where $A(X_j) = U_j + V_j$, $U_j \in \mathfrak{n}$, $V_j \in \mathfrak{h}$, $1 \leq j \leq 2m-2$. But (5.3) is just Smith's equation

$$\text{(5.3}'\text{)} \quad \ddot{r} + \{-(2p+1)\tan t + (2m-2p-1)\cot t\}\dot{r} + \left(\frac{\lambda'}{\cos^2 t} - \frac{\lambda''}{\sin^2 t}\right)\sin r \cos r = 0,$$

since $(\cos 2t)/(\sin 2t) = \frac{1}{2}(\cot t - \tan t)$. It can be also proved in the same manner as in Proposition 6.4 that (5.4) holds.

Ding gave necessary and sufficient conditions of existence for solutions of (5.3′) with $0 \leq r(t) \leq \pi/2$ for $0 < t < \pi/2$, $r(0) = 0$, and $r(\pi/2) = \pi/2$. For such solution $r(t)$, the corresponding A-equivariant map (5.2) is a continuous weakly harmonic map of $(\mathbf{C}P^m, g)$ into (S^n, can) of Sobolev class L_1^2. By the regularity theorem, it is smooth and harmonic. Therefore we obtain the next theorem.

THEOREM 5.2. (1) (cf. Ding [4]) *Assume that* $\lambda'' p \geq \lambda'(m-p-1)$. *Then equation* (5.3) *has a solution* $r(t)$ *with the condition* $0 \leq r(t) \leq \pi/2$ *for* $0 < t < \pi/2$, $r(0) = 0$, *and* $r(\pi/2) = \pi/2$ *if and only if either*

(i) $p^2 < \lambda'$ *or*

(ii) $p^2 \geq \lambda'$ *and* $\sqrt{(m-p+1)^2 + \lambda''} + \sqrt{p^2 - \lambda'} < m-1$.

(2) *In particular, let* $p \geq m-p+1$. *If either*

(i) $l \geq k \geq ((\sqrt{2}-1)/2)p$ *or*

(ii) $k = l = 1$,

then for $\lambda' = 4k(k+p-1)$ *and* $\lambda'' = 4l(l+m-p-2)$ *the A-equivariant map* (5.2) *of* $(\mathbf{C}P^m, g)$ *into* (S^n, can) *is harmonic.*

REMARK 5.2. (i) For $p = m-p-1 = s$ and $k = l = 1$, (5.2) gives an A-equivariant harmonic map of $(\mathbf{C}P^{2s+1}, g)$ into $(S^{2s(s+2)-1}, \text{can})$. If $s = 1$, then it is a harmonic map of $(\mathbf{C}P^3, g)$ into (S^5, can). However, it is not horizontal with respect to the Hopf fibering $S^5 \to \mathbf{C}P^2$, whose composition of (5.2) is not a harmonic map of $\mathbf{C}P^3$ into $\mathbf{C}P^2$.

(ii) For $p = m-p = s$ and $k = l = 1$, (5.2) gives an A-equivariant harmonic map of $\mathbf{C}P^{2s}$ with $S^{2s(s+1)-2}$.

6. Harmonic Maps from $\mathbf{C}P^m$ into $\mathbf{C}P^n$

In this section, we give new nonholomorphic harmonic maps from $\mathbf{C}P^m$ into $\mathbf{C}P^n$. We assume $(M, g) = (\mathbf{C}P^m, g)$ and $(N, h) = (\mathbf{C}P^n, h)$, where g, h are $\frac{1}{4}$ times the Fubini–Study metrics. Consider the group actions $K = SU(p+1) \times SU(m-p) \subset SU(m+1)$ on $\mathbf{C}P^m$ and $G = SU(q+1) \times SU(n-q) \subset SU(n+1)$ on $\mathbf{C}P^n$, as in Example 1.2.

6.1

For a function $r(t)$ satisfying the conditions $0 \leq r(t) \leq \pi/2$ for $0 < t < \pi/2$, $r(0) = 0$, and $r(\pi/2) = \pi/2$, each homomorphism A of K into G satisfying $A(J_t) \subset H_{r(t)}$ for $0 \leq t \leq \pi/2$ is given by

$$A\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} \pi'(x) & 0 \\ 0 & \pi''(y) \end{pmatrix}, \quad x \in SU(p+1), \ y \in SU(m-p), \tag{6.1}$$

where $(\pi', V_{\pi'})$ and $(\pi'', V_{\pi''})$ are the $(q+1)$- and $(n-q)$-dimensional unitary representations of $SU(p+1)$ and $SU(m-p)$, satisfying the restrictions that $\pi'|_{S(U(1)\times U(p))}$ and $\pi''|_{S(U(1)\times U(m-p-1))}$ have 1-dimensional invariant subspaces $[v_1]_{\mathbf{C}}$ and $[w_1]_{\mathbf{C}}$ of $V_{\pi'}$ and $V_{\pi''}$, respectively, and satisfy

$$\langle \pi'(X_0')v_1, v_1 \rangle = \langle \pi''(X_0'')w_1, w_1 \rangle. \tag{6.2}$$

Here $X_0' \in \mathfrak{s}(\mathfrak{u}(1) \times \mathfrak{u}(p))$ and $X_0'' \in \mathfrak{s}(\mathfrak{u}(1) \times \mathfrak{u}(m-p-1))$ are the diagonal matrices whose diagonal entries are

$$\left[i, -\frac{i}{p}, \ldots, -\frac{i}{p}\right] \quad \text{and} \quad \left[i, -\frac{i}{m-p-1}, \ldots, -\frac{i}{m-p-1}\right],$$

respectively. Such representations $(\pi', V_{\pi'})$ and $(\pi'', V_{\pi''})$ are determined in terms of their highest weights in the follwing manner.

Let us recall the unitary representation theory of the special unitary group $SU(n+1)$. For a Cartan subalgebra $\mathfrak{t} = \{[ix_1, \ldots, ix_{n+1}]; \sum_{i=1}^{n+1} x_i = 0, x_i \in \mathbf{R}\}$ of a Lie algebra $\mathfrak{su}(n+1) = \{X \in M_{n+1}(\mathbf{C}); {}^t\bar{X} + X = 0\}$, let us define an element λ_j in the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ by $[ix_1, \ldots, ix_{n+1}] \mapsto x_j$, $1 \leq j \leq n+1$, and introduce a lexicographic order $>$ on $\mathfrak{t}^*$ in the way $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0 > \lambda_{n+1}$. Put

$$D(SU(n+1)) = \left\{\Lambda = \sum_{i=1}^{n} m_i \lambda_i;\ m_i \in \mathbf{Z},\ m_1 \geq \cdots \geq m_n \geq 0\right\}.$$

Then there exists a bijection between a complete set of non-equivalent irreducible unitary representations of $SU(n+1)$ and $D(SU(n+1))$, assigning $\Lambda \in D(SU(n+1))$ to an irreducible unitary representation (π_Λ, V_Λ) of $SU(n+1)$ with the highest weight Λ. Then we get the next proposition.

PROPOSITION 6.1. *Each homomorphism A of K into G satisfying $A(J_t)\subset H_{r(t)}$ for $0\le t\le \pi/2$ is of the form* (6.1). *The unitary representations $(\pi', V_{\pi'})$ and $(\pi'', V_{\pi''})$ of $SU(p+1)$ and $SU(m-p)$ must have the following highest weights, respectively:*

$$\Lambda'=(l+2k')\lambda_1+k'\lambda_2+\cdots+k'\lambda_p,$$
$$\Lambda''=(l+2k'')\lambda_1+k''\lambda_2+\cdots+k''\lambda_{m-p-1},$$

for $l, k', k''\in\mathbf{Z}$ with nonnegative $l+2k'$, $l+2k''$, k', k''. Then (6.2) *can be written as*

$$\langle\pi'(X_0')v_1, v_1\rangle=\langle\pi''(X_0'')w_1, w_1\rangle=il, \tag{6.2$'$}$$

where v_1, w_1 are fixed under $\pi'(SU(p))$ and $\pi''(SU(m-p-1))$, respectively.

The proof follows by a direct computation, making use the following branching theorem.

LEMMA 6.2 (cf. [16, Thm. 4.1]). *Let $V=V_\Lambda$ be an irreducible unitary representation of $SU(n+1)$ with highest weight $\Lambda=\sum_{i=1}^n m_i\lambda_i$. Then V_Λ decomposes as $S(U(1)\times U(n))$-modules into irreducible representations as*

$$V_\Lambda=\sum V_{k_1\lambda_1+\cdots+k_n\lambda_n},$$

where the summation runs over all the integers $k_1,\dots,k_n$ for which there exists a nonnegative integer k such that

$$m_1\ge k_2+k\ge m_2\ge k_3+k\ge m_3\ge\cdots\ge m_{n-1}\ge k_n+k\ge m_n\ge k$$

and $\sum_{i=1}^n m_i=\sum_{i=1}^n k_i+(n+1)k$.

6.2

Our A-equivariant map ϕ of $\mathbf{C}P^m$ into $\mathbf{C}P^n$ is of the form

$$\begin{aligned}\phi:\mathbf{C}P^m\ni{}&[\cos t\, x\cdot\xi_1+\sin t\, y\cdot\xi_{p+2}]\\ &\mapsto\left[\cos r(t)\sum_{i=1}^{q+1}\pi_{i1}'(x)\xi_1+\sin r(t)\sum_{i=1}^{n-q}\pi_{i1}''(y)\xi_{q+1+i}\right]\in\mathbf{C}P^n,\end{aligned}\tag{6.3}$$

with

$$\begin{pmatrix}x&0\\0&y\end{pmatrix}\in K,$$

where the $\{\xi_i\}$s are the standard bases of the Euclidean spaces.

PROPOSITION 6.3. *The equations for ϕ to be harmonic are*

$$\begin{aligned}&\ddot r+\{-(2p+1)\tan t+(2m-2p-1)\cot t\}\dot r\\ &\quad+\left(\frac{\mu'}{\cos^2 t}-\frac{\mu''}{\sin^2 t}\right)\sin r\cos r-2l^2\frac{\sin 2r\cos 2r}{\sin^2 2t}=0\end{aligned}\tag{6.4}$$

and

$$C=\sum_{j=1}^{m-1} f_j(t)^{-2}[V_j, U_j]=0.\tag{6.5}$$

Here $\mu' := 2l(2k'+p)+4k'(k'+p)$, $\mu'' := 2l(2k''+a)+4k''(k''+a)$, *and* $a := m-p-1$.

The proof follows from Theorem 2.2 and the following considerations. Taking the basis $\{X_j\}_{j=1}^{2m-1}$ of $\mathfrak{m}$ in Example 1.2, $A(X_j)$ are of the form

$$
(6.6)\qquad \begin{cases} A(X_j) = \sum_{i=1}^{2q} a_{ij} Y_i + V_j, & 1 \leq j \leq 2p, \\ A(X_j) = \sum_{i=2q+1}^{2n-2} a_{ij} Y_i + V_j, & 2p+1 \leq j \leq 2m-2, \\ A(X_{2m-1}) = a_{2m-1} Y_{2n-1} + V_{2m-1}, \end{cases}
$$

where $V_j \in \mathfrak{h}$ and the coefficients a_{ij}, a_{2m-1} satisfy

$$
(6.7)\qquad \begin{cases} \sum_{j=1}^{2p} \sum_{i=1}^{2q} a_{ij}^2 = \langle \Lambda' + 2\delta', \Lambda' \rangle - l^2 \dfrac{2p}{p+1}, \\ \sum_{j=2p+1}^{2m-2} \sum_{i=2q+1}^{2n-2} a_{ij}^2 = \langle \Lambda'' + 2\delta'', \Lambda'' \rangle - l^2 \dfrac{2a}{a+1}, \\ a_{2m-1}^2 = l^2 \left(\dfrac{q}{q+1} + \dfrac{n-q-1}{n-q} \right)^{-1} \left(\dfrac{p}{p+1} + \dfrac{a}{a+1} \right). \end{cases}
$$

The numbers $\langle \Lambda' + 2\delta', \Lambda' \rangle$ and $\langle \Lambda'' + 2\delta'', \Lambda'' \rangle$ are

$$
(6.8)\qquad \begin{cases} \langle \Lambda' + 2\delta', \Lambda' \rangle = l^2 \dfrac{2p}{p+1} + 2l(2k'+p) + 4k'(k'+p), \\ \langle \Lambda'' + 2\delta'', \Lambda'' \rangle = l^2 \dfrac{2a}{a+1} + 2l(2k''+a) + 4k''(k''+a). \end{cases}
$$

6.3

Here we show that the vertical equation (6.5) holds automatically.

PROPOSITION 6.4. *Under the above conditions* (6.5), $C = 0$.

Proof. Let us recall Example 1.2 and Section 2. Our situations are as follows. For the homomorphism A of $\mathfrak{k} = \mathfrak{j} \oplus \mathfrak{m}$ into $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$, we put $A(X_j) = U_j + V_j$ with $U_j \in \mathfrak{n}$ and $V_j \in \mathfrak{h}$ for $1 \leq j \leq 2m-1$. Then decompose C as $C = C_1 + C_2 + C_3$, where

$$
C_1 := (\cos t)^{-2} \sum_{j=1}^{2p} [V_j, U_j], \qquad C_2 := (\sin t)^{-2} \sum_{j=2p+1}^{2m-2} [V_j, U_j],
$$

$$
\text{and} \quad C_3 := 2 \left\{ \frac{p}{p+1} + \frac{a}{a+1} \right\}^{-1} (\sin 2t)^{-2} [V_{2m-1}, U_{2m-1}].
$$

Putting the subspaces $\mathfrak{n}_1 = \{Y_1, \ldots, Y_{2q}\}_{\mathbf{R}}$, $\mathfrak{n}_2 = \{Y_{2q+1}, \ldots, Y_{2n-2}\}_{\mathbf{R}}$, and noting that $[Y_{2n-1}, \mathfrak{h}] = 0$, we get

$$
(6.9)\qquad C_1 \in \mathfrak{n}_1, \quad C_2 \in \mathfrak{n}_2, \quad \text{and} \quad C_3 = 0.
$$

On the other hand, by definition the homomorphism A satisfies

$$A(\mathrm{Ad}(j)X)=\mathrm{Ad}(A(j))A(X), \quad x\in\mathfrak{m},\ j\in J; \tag{6.10}$$

$$\mathrm{Ad}(A(j))C=C, \quad j\in J. \tag{6.11}$$

Indeed (6.11) can be proved together with (6.10) and the fact that C_1 and C_2 do not depend on the choice of orthonormal bases of $\mathfrak{m}_1=\{X_1,\dots,X_{2p}\}_{\mathbf{R}}$ and $\mathfrak{m}_2=\{X_{2p+1},\dots,X_{2m-2}\}_{\mathbf{R}}$, respectively.

Then, by (6.11),

$$A\begin{pmatrix} j' & 0\\ 0 & 1\end{pmatrix}(Cv_1)=Cv_1 \quad \text{for } j'\in SU(p+1),$$

$$A\begin{pmatrix} 1 & 0\\ 0 & j''\end{pmatrix}(Cw_1)=Cw_1 \quad \text{for } j''\in SU(m-p),$$

and

$$A\begin{pmatrix} \exp\theta X_0' & 0\\ 0 & 1\end{pmatrix}(Cv_1)=e^{il\theta}Cv_1, \qquad A\begin{pmatrix} 1 & 0\\ 0 & \exp\theta X_0''\end{pmatrix}(Cw_1)=e^{il\theta}Cw_1$$

for all $\theta\in\mathbf{R}$. Therefore Cv_1 and Cw_1 belong to $V_{l\lambda_1}\subset V_{\Lambda'}$ and $V_{l\lambda_1}\subset V_{\Lambda''}$ as $S(U(1)\times U(p))$ and $S(U(1)\times U(m-p-1))$ submodules, respectively. Using the branching theorem (cf. Lemma 6.2), $Cv_1=\lambda v_1$ and $Cw_1=\mu w_1$ for some $\lambda,\mu\in\mathbf{C}$. Therefore $C\in\mathbf{R}Y_{2n-1}\oplus\mathfrak{h}$, which implies by (6.9) that $C=0$. □

6.4. The ODE

The next theorem concerns the existence of solutions of ODE (6.4) with boundary conditions.

THEOREM 6.5. *Suppose that $a\mu'\le p\mu''$, $a=m-p-1$. The ordinary differential equation* (6.4) *has a solution with the conditions*

$$0\le r(t)\le\pi/2\ (0<t<\pi/2),\ r(0)=0,\ \text{and}\ r(\pi/2)=\pi/2, \tag{6.12}$$

provided that either

(i) $p^2\mu'-l^2$ *and* $0<\mu''-l^2$, *or*
(ii) $p^2\ge\mu'-l^2$, $0<\mu''-l^2$, $0<\mu'-l^2$, *and* $\sqrt{a^2+\mu''+l^2}+\sqrt{p^2-\mu'+l^2}<m-1$.

The proof is accomplished in a way similar to that of Ding in [4].

Our ODE (6.4) is the Euler–Lagrange equation of the energy

$$J(r):=\int_0^{\pi/2}\left\{\dot r^2+\mu''\frac{\sin^2 r}{\sin^2 t}+\mu'\frac{\cos^2 r}{\cos t}+l^2\frac{\sin^2 2r}{\sin^2 2t}\right\}f(t)\,dt,$$

where $f(t):=\frac{1}{2}\cos^{2p}t\sin^{2a}t\sin 2t=\cos^{2p+1}t\sin^{2m-2p-1}t$. The energy J is a functional on the Hilbert space X defined by

$$X=\left\{r\in H^1_{\mathrm{loc}}(0,\pi/2);\ \|r\|^2=\int_0^{\pi/2}(\dot r^2+r^2)f(t)\,dt<\infty\right\}.$$

In the case $p>1$ or $a=m-p-1>1$, J is well-defined on X, but in the case $p=1$ and $a=1$, we still have J as a functional on X by allowing it to assume $+\infty$.

Note that $J(0)=a\mu' A$ and $J(\pi/2)=p\mu'' A$, where $A=(a-1)!\,(p-1)!/2(m-1)!$. Then

$$J(0)\le J(\pi/2)\Leftrightarrow a\mu'\le p\mu''. \tag{6.13}$$

We also define the closed convex subset

$$X_0=\{r\in X;\ 0\le r(t)\le \pi/2 \text{ when } 0<t<\pi/2\}.$$

Then, as in [4], we obtain the following lemma.

LEMMA 6.6. *There exists an r_0 which minimizes J over X_0; that is, $J(r_0)=c_0=\inf\{J(r);\ r\in X_0\}$. This r_0 is a solution of* (6.4) *with $0\le r_0(t)\le\pi/2$ for $0<t<\pi/2$.*

In [4], Ding claimed that if $r_0\not\equiv 0,\pi/2$ then r_0 satisfies $r_0(0)=0$ and also $r_0(\pi/2)=\pi/2$. But, in our case, we need the additional assumptions (see Lemma 6.10).

COROLLARY 6.7. *If $c_0<\min\{J(0),J(\pi/2)\}$, there exists a solution $r_0(t)$ of* (6.4) *with the conditions $0\le r_0(t)\le\pi/2$ $(0<t<\pi/2)$ and $r_0(t)\not\equiv 0,\pi/2$.*

In the following, we assume (6.13). We will get the condition for $c_0<J(0)$ as in [4]. We calculate the second variation of J at the critical point 0. Defining

$$2I(v):=\left.\frac{d^2}{ds^2}\right|_{s=0} J(sv) \quad \text{for } v\in X,$$

we get

$$I(v)=\int_0^{\pi/2}\{\dot v^2+Q(t)v^2\}f(t)\,dt, \tag{6.14}$$

where

$$Q(t):=\frac{\mu''+l^2}{\sin^2 t}-\frac{\mu'-l^2}{\cos^2 t}.$$

LEMMA 6.8 (cf. [4, Lemma 2.3]). *If there exists $v\in X$ such that $v\ge 0$ and $I(v)<0$, then $c_0<J(0)$.*

Using Lemma 6.8, we get the following lemmas in a manner similar to that of Lemmas 2.4 and 2.5 in [4].

LEMMA 6.9.

(i) *If $p^2<\mu'-l^2$, then $c_0<J(0)$;*
(ii) *if $p^2\ge\mu'-l^2$ and $\sqrt{a^2+\mu''+\lambda^2}+\sqrt{p^2-\mu'+l^2}<m-1$, then $c_0<J(0)$.*

LEMMA 6.10. *Under the assumptions $l^2<\mu'$ and $l^2<\mu''$, all solutions $r(t)$ of* (6.4), *with $0\le r(t)\le\pi/2$ for $0<t<\pi/2$ and $r(t)\not\equiv 0,\pi/2$, must satisfy $r(0)=0$ and $r(\pi/2)=\pi/2$.*

Proof. This can be proved as in Lemma 5.5 of [12]. We first change the variable $e^s=\tan t$, $-\infty<s<\infty$; then (6.4) is transformed into

$$(6.15)\qquad \frac{d^2r}{ds^2}+(e^s+e^{-s})^{-1}\{2ae^{-s}-2pe^s\}\frac{dr}{ds} \\ +(e^s+e^{-s})^{-1}\{\mu' e^s-\mu'' e^{-s}\}\sin r\cos r-\frac{l^2}{2}\sin 2r\cos 2r=0.$$

We may prove that $r(-\infty)=0$ and $r(\infty)=\pi/2$ for a solution $r(s)$ of (6.15) with $0\le r(s)\le \pi/2$ ($\forall s\in(-\infty,\infty)$) and $r(s)\not\equiv 0,\pi/2$. We may write (6.15) as

$$(6.16)\qquad \frac{d^2r}{ds^2}=h(s)\frac{dr}{ds}-(g(s)-l^2\cos 2r)\sin r\cos r,$$

where

$$h(s)=-(e^s+e^{-s})^{-1}\{2ae^{-s}-2pe^s\},\qquad g(s)=(e^s+e^{-s})^{-1}\{\mu' e^s-\mu''\epsilon^{-s}\}.$$

Step 1: If $|s|$ is large, $dr/ds>0$, and then $r(\infty)$ and $r(-\infty)$ exist. Indeed, for large s, $h(s)\sim 2p$ and $g(s)\sim \mu'$. Then if $0<\mu'-l^2$, $g(s)-l^2\cos 2r(s)>0$ for large s. Then $dr/ds>0$. In fact, if we assume $dr/ds\le 0$ then the right-hand side of (6.16) is negative, and $d^2r/ds^2<0$. Therefore $r(s)$ is decreasing and becomes negative for large s, which is a contradiction. If $s\to-\infty$, then $h(s)\sim -2a$ and $g(s)\sim -\mu''$. The right-hand side of (6.16) behaves like

$$-2a\frac{d\mu}{ds}+(\mu''+l^2\cos 2r)\sin r\cos r.$$

If $0<\mu''-l^2$, then this second term is positive if $s\to-\infty$. Then $dr/ds>0$, because if $dr/ds\le 0$ then $d^2r/ds^2>0$ for $s\to-\infty$. Then $r(s)$ increases and becomes greater than $\pi/2$ if $s\to-\infty$, which is a contradiction.

Step 2: Assuming $r(\infty)\le\pi/4$, we derive a contradiction. The equation (6.16) is written as

$$(6.17)\qquad \frac{d^2r}{ds^2}=h(s)\frac{dr}{ds}-\frac{f(s)}{2},$$

where $f(s)=(g(s)-l^2\cos 2r(s))\sin 2r(s)$. Differentiation yields

$$\frac{df}{ds}=\left(\frac{dg}{ds}+2l^2\,\dot r\sin 2r\right)\sin 2r+(g(s)-l^2\cos 2r)2\dot r\cos 2r.$$

Since $dg/ds=2(\mu'+\mu'')(e^s+e^{-s})^{-2}$, the first term of df/ds is positive for large s, because $dr/ds>0$. By $\mu'-l^2>0$, $g(s)-l^2\cos 2r>0$ for large s. Since $r(s)$ increases monotonically for large s and $r(\infty)\le\pi/4$, $\cos 2r(s)>0$ for large s, and then the second term of df/ds is positive. Therefore $df/ds>0$ for all $s\ge s_0$ for some $s_0>0$. Then $d^2r/ds^2\le 2p(dr/ds)-\frac{1}{2}f(s_0)$ for all $s\ge s_0$, and $f(s_0)>0$. Therefore $dr/ds\ge(1/4p)f(s_0)$ for $s\ge s_0$, which is a contradiction.

Step 3: Assuming $\pi/4<r(\infty)<\pi/2$, we derive a contradiction. There exists $\epsilon>0$ such that $\pi/4+\epsilon<r(s)<\pi/2-\epsilon$ for all $s\ge s_0$ for some $s_0>0$. Then there exists a positive constant C_0 such that $\sin r\cos r\ge C_0$, and then $-g(s)\sin r\cos r\le -\mu' C_0<0$ for $s\ge s_0$. Also there exists a constant $C_1>0$ such that $l^2\cos 2r\sin r\cos r\le -C_1<0$ for $s\ge s_0$, since $\pi+4\epsilon<4r(s)<2\pi-4\epsilon$. Thus there exists a $C>0$ such that $-f(s)/2\le -C<0$ for $s\ge s_0$.

Then $d^2r/ds^2 \le 2p(dr/ds) - C$ for $s \ge s_0$. Therefore $dr/ds \ge C/2p$ for $s \ge s_0$, which is also a contradiction. We get also $r(-\infty) = 0$ by arguments similar to Steps 2 and 3. □

Summing up, we have proved Theorem 6.5 and the following theorem.

THEOREM 6.11. *Let $(\pi', V_{\pi'})$ and $(\pi'', V_{\pi''})$ be the irreducible unitary representations of $SU(p+1)$ and $SU(m-p)$ with the highest weights*

$$\Lambda' = (l+2k')\lambda_1 + k'\lambda_2 + \cdots + k'\lambda_p \quad \text{and}$$

$$\Lambda'' = (l+2k'')\lambda_1 + k''\lambda_2 + \cdots + k''\lambda_{m-p-1},$$

respectively, where $l, k', k'' \in \mathbf{Z}$ and $l+2k'$, $l+2k''$, k', and k'' are nonnegative. Assume that $a\mu' \le p\mu''$ where $a = m-p-1$,

$$\mu' = 2l(2k'+p) + 4k'(k'+p), \quad \text{and} \quad \mu'' = 2l(2k''+a) + 4k''(k''+a).$$

If either

(i) $p^2 < \mu' - l^2$ *and* $0 < \mu'' - l^2$, *or*
(ii) $p^2 \ge \mu' - l^2$, $0 < \mu'' - l^2$, $0 < \mu' - l^2$ *and* $\sqrt{a^2 + \mu'' + l^2} + \sqrt{p^2 - \mu' + l^2} < m-1$,

then the A-equivariant map (6.3) $\phi: \mathbf{C}P^m \to \mathbf{C}P^n$ *is harmonic.*

Now we ask which representations satisfy the conditions. Assume that $p \ge a = m-p-1$.

(I) If $k'' \ge k' \ge ((\sqrt{2}-1)/2)(p+l)$, then the conditions $a\mu' \le p\mu''$ and (i) of Theorem 6.11 are satisfied.
(II) If $p = a \ge 1$ or $p-1 = a \ge 1$, then $k' = k'' = l = 1$ satisfy the conditions $a\mu' \le p\mu''$ and (i) and (ii) of Theorem 6.11.

For the case $k' = k'' = l = 1$, the dimensions of $V_{\pi'} = V_{\Lambda'}$ and $V_{\pi''} = V_{\Lambda''}$ can be given by Weyl's dimension formula:

$$q+1 = \dim V_{\pi'} = 2^{-1}p(p+1)(p+3);$$

$$n-q = \dim V_{\pi''} = 2^{-1}a(a+1)(a+3).$$

We then obtain this corollary.

COROLLARY 6.12. *For the case $k' = k'' = l = 1$, the A-equivariant maps* (6.3) *give harmonic maps from $\mathbf{C}P^{2p+1}$ into $\mathbf{C}P^{p(p+1)(p+3)-1}$ and from $\mathbf{C}P^{2p}$ into $\mathbf{C}P^{(p/2)(2p^2+5p+1)-1}$.*

REMARK 6.13. The A-equivariant harmonic maps $\phi: \mathbf{C}P^m \to \mathbf{C}P^n$ in Theorem 6.11 are neither holomorphic nor antiholomorphic except for the identity map of $\mathbf{C}P^m$ into itself. Indeed, let J be the complex structures of $\mathbf{C}P^m$ or $\mathbf{C}P^n$. Then $J\dot{c}(t) = X_{c(t)}$, where

$$X = \begin{pmatrix} -\tan t\, X_0' & 0 \\ 0 & \cot t\, X_0'' \end{pmatrix} \in \mathfrak{k}.$$

Then

$$\phi_*(J\dot{c}(t)) = A(X)\bar{c}(r(t)) = \begin{pmatrix} -\tan t \cos r(t) \pi'(X_0')v_1 \\ \cos t \sin r(t) \pi''(X_0'')w_1 \end{pmatrix}$$

and

$$J\phi_* \dot{c}(t) = \dot{r}(t) J\dot{\bar{c}}(r(t)) = \begin{pmatrix} -i\dot{r}(t) \sin r(t) v_1 \\ i\dot{r}(t) \cos r(t) w_1 \end{pmatrix}.$$

By (6.2′), we get

$$\dot{r}(t) \tan r(t) = \pm l \tan t \quad \text{and} \quad \dot{r}(t) \cot r(t) = \pm l \cot t,$$

if we assume ϕ is (anti)holomorphic. Then $r(t) = t$ and $l = 1$. Since $r(t)$ is a solution to (6.4), we conclude that $k' = k'' = 0$. Therefore $\mathbf{C}P^n = \mathbf{C}P^m$ and ϕ is the identity map.

REMARK 6.14. In the case $l = 0$, the A-equivariant map $\phi: \mathbf{C}P^m \to \mathbf{C}P^n$ is the composition of the harmonic map of $\mathbf{C}P^m \to S^n$ in Section 5 and the map $S^n \to \mathbf{R}P^n \hookrightarrow \mathbf{C}P^n$.

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