A Simple Proof of a Theorem of Jean Bourgain

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In this note, we give a very simple proof (compared to earlier known proofs) of Bourgain's version of Grothendieck's theorem for the disk algebra: Every operator on the disk algebra with values in L_1 or L_2 is 2-absolutely summing and hence extends to an operator defined on the whole of C. As far as we know, the currently known proofs are essentially the original one in [B1], the simpler one in [BD], and several new proofs given recently by Kisliakov in [K1; K2]. This implies Bourgain's result that L_1/H^1 is of cotype 2. We also prove more generally that L_r/H^r is of cotype 2 for $0<r<1$.

We first recall the definition of a q-absolutely summing (in short, q-summing) operator for $1 \le q < \infty$. Let $u: X \to Y$ be an operator between two Banach spaces. We say that u is *q-summing* if there is a constant C such that, for all finite sequences $x_1, x_2, \dots, x_n$ in X, we have

$$\left(\sum \|u(x_i)\|^q\right)^{1/q} \le C \sup\{\left(\sum |x^*(x_i)|^q\right)^{1/q} \mid x^* \in X^*, \ \|x^*\| \le 1\}.$$

We denote by $\pi_q(u)$ the smallest possible constant C. Let us denote by A the disc algebra. Then, if $u: A \to Y$ is q-summing, by Pietsch's factorisation theorem there is a probability measure λ on the unit circle such that

$$\forall f \in A, \quad \|u(f)\| \le \pi_q(u)\left(\int |f|^q \, d\lambda\right)^{1/q}.$$

We refer for example to [P1] for more information on this notion.

We will prove the following theorem due to Bourgain.

BOURGAIN'S THEOREM. *There is a constant K such that any bounded operator $u: A \to l_2$ is 2-summing and satisfies*

$$\pi_2(u) \le K\|u\|.$$

Also, u extends to a bounded operator $\hat{u}: C(\mathbf{T}) \to l_2$ such that

$$\|\hat{u}\| \le K\|u\|.$$

Moreover, the same result holds for all operators $u: A \to Y$ if $Y = l_1$ or, more generally, whenever Y is a Banach space of cotype 2.

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Let us recall here the definitions of the K_t and J_t functionals that are fundamental in the real interpolation method. Let A_0, A_1 be a compatible couple of Banach (or quasi-Banach) spaces. For all $x \in A_0+A_1$ and for all $t>0$, we let

$$K_t(x, A_0, A_1) = \inf(\|x_0\|_{A_0} + t\|x_1\|_{A_1} \mid x = x_0+x_1,\ x_0 \in A_0,\ x_1 \in A_1).$$

For all $x \in A_0 \cap A_1$ and for all $t>0$, we let

$$J_t(x, A_0, A_1) = \max(\|x_0\|_{A_0}, t\|x_1\|_{A_1}).$$

Recall that the (real interpolation) space $(A_0, A_1)_{\theta,p}$ is defined as the space of all x in A_0+A_1 such that $\|x\|_{\theta,p} < \infty$, where

$$\|x\|_{\theta,p} = \left(\int (t^{-\theta} K_t(x, A_0, A_1))^p \frac{dt}{t}\right)^{1/p}.$$

We refer to [BL] for more details.

Let $\mathbf{T}$ be the circle group equipped with its normalized Haar measure m. Let $1 \leq p \leq \infty$. When B is a complex Banach space we denote by $L_p(B)$ the usual space of Bochner-p-integrable B-valued functions on $(\mathbf{T}, m)$, so that $L_p \otimes B$ is dense in $L_p(B)$ when $p<\infty$. We denote by $H^p(B)$ the subspace of $L_p(B)$ formed by all the functions f such that their Fourier transform vanishes on the negative integers. When B is 1-dimensional, we write H^p instead of $H^p(B)$. When $0<p<1$, we define H^p as the closure in L_p of the linear span of the functions $\{e^{int} \mid n \geq 0\}$. We refer to [G; GR] for basic information on H^p-spaces.

The next proposition, although very simple, is the key new ingredient in our proof. We refer to [P2] for more applications of the same idea to the interpolation spaces between H^p spaces.

PROPOSITION 1. *Let $1 \leq p \leq q \leq \infty$. Consider a compatible couple of Banach spaces (A_0, A_1); the following are equivalent:*

(i) *There is a constant C such that*

$$\forall f \in H^p(A_0)+H^q(A_1),\ \forall t>0,$$
$$K_t(f, H^p(A_0), H^q(A_1)) \leq C K_t(f, L^p(A_0), L^q(A_1)).$$

(ii) *There is a constant C such that*

$$\forall f \in [L^p(A_0)/H^p(A_0)] \cap [L^q(A_1)/H^q(A_1)],\ \forall t>0, \quad \exists \hat{f} \in L^p(A_0) \cap L^q(A_1)$$

representing the equivalence class of f and satisfying

$$J_t(\hat{f}, L^p(A_0), L^q(A_1)) \leq C J_t(f, L^p(A_0)/H^p(A_0), L^q(A_1)/H^q(A_1)).$$

(iii) *There is a constant C such that*

$$\forall f \in [L^p(A_0)/H^p(A_0)] \cap [L^q(A_1)/H^q(A_1)], \quad \exists \hat{f} \in L^p(A_0) \cap L^q(A_1)$$

representing the equivalence class of f and satisfying

$$\|\hat{f}\|_{L^p(A_0)} \leq C\|f\|_{L^p(A_0)/H^p(A_0)} \quad \text{and} \quad \|\hat{f}\|_{L^q(A_1)} \leq C\|f\|_{L^q(A_1)/H^q(A_1)}.$$

In the above statement we regard the spaces

$$L^p(A_0)/H^p(A_0) \quad \text{and} \quad L^q(A_1)/H^q(A_1)$$

as included via the Fourier transform $f \to (\hat{f}(-1), \hat{f}(-2), \hat{f}(-3), \ldots)$ in the space of all sequences in $A_0 + A_1$. In this way, we may view these quotient spaces as forming a compatible couple for interpolation. (For the subspaces $H^p(A_0), H^q(A_1)$, there is no problem; we may clearly consider them as a compatible couple in the obvious way.)

Proof. For brevity, we will write simply $L^p/H^p(A_0)$ instead of $L^p(A_0)/H^p(A_0)$, and we will also write $L^p, H^p, \ldots$ instead of $L^p(A_0), H^p(A_0), \ldots$; no confusion should arise. The proof is routine. We only indicate the argument for (i)$\Rightarrow$(ii)$\Rightarrow$(iii), which is the one we use below.

Assume (i). Let f be as above such that $J_t(f, L^p/H^p(A_0), L^q/H^q(A_1)) < 1$. Then let $g_p \in L^p(A_0)$ and $g_q \in L^q(A_1)$ be such that

$$\|g_p\|_{L^p} < 1, \quad \|g_q\|_{L^q} < t^{-1}, \qquad f = g_p + H^p(A_0), \quad f = g_q + H^q(A_1).$$

Therefore, $g_p - g_q$ must be in $H^p + H^q$ and

$$K_t(g_p - g_q, L^p(A_0), L^q(A_1)) \leq \|g_p\|_{L^p} + t\|g_q\|_{L^q} < 2.$$

By (i) we have $K_t(g_p - g_q, H^p, H^q) < 2C'$; hence there are $h_p \in H^p(A_0)$ and $h_q \in H^q(A_1)$ such that $g_p - g_q = h_p - h_q$ and $\|h_p\|_{H^p} + t\|h_q\|_{H^q} < 2C'$. Now if we let $\hat{f} = g_p - h_p = g_q - h_q$, then we find that $\hat{f} \in L^p(A_0) \cap L^q(A_1)$, $f = \hat{f} + H^p(A_0)$ in the space $L^p/H^p(A_0)$, and $f = \hat{f} + H^q(A_1)$ in the space $L^q/H^q(A_1)$; moreover,

$$J_t(\hat{f}, L^p, L^q) \leq \max(\|\hat{f}\|_{L^p}, t\|\hat{f}\|_{L^q}) \leq 1 + 2C'.$$

By homogeneity, this completes the proof of (i)$\Rightarrow$(ii) with $C \leq 1 + 2C'$. To check (ii)$\Rightarrow$(iii), simply write (ii) with $t = (\|f\|_{L^p/H^p(A_0)}) \cdot (\|f\|_{L^q/H^q(A_1)})^{-1}$. □

REMARK. It is well known that the Hilbert transform is a bounded operator on all the (so-called mixed norm) spaces of the form $L^p(l^q)$ for all $1 < p, q < \infty$. (Apparently this goes back to [BB]; we refer to [GR] for more information and references). Therefore, the orthogonal projection from $L^2(l^2)$ onto $H^2(l^2)$ is bounded *simultaneously* on all the spaces $L^p(l^q)$ for $1 < p, q < \infty$. It follows immediately that if $1 < p_0, p_1, q_0, q_1 < \infty$ then there is a constant C' such that, for all $f \in H^{p_0}(l^{q_0}) + H^{p_1}(l^{q_1})$ and for all $t > 0$,

$$K_t(f, H^{p_0}(l^{q_0}), H^{p_1}(l^{q_1})) \leq C' K_t(f, L^{p_0}(l^{q_0}), L^{p_1}(l^{q_1})).$$

PROPOSITION 2. *There is a constant C such that, for all $t > 0$ and all $f \in H^1(l_1) + H^1(l_2)$, we have*

$$K_t(f, H^1(l_1), H^1(l_2)) \leq C K_t(f, L^1(l_1), L^1(l_2)).$$

For the proof of Proposition 2, we will use the following.

SUBLEMMA.

$$H^1(l_{4/3}) \subset (H^1(l_1), H^1(l_2))_{1/2,\infty},$$

and the inclusion is bounded with norm less than a constant K.

Proof. Take a function $f=(f_n)$ in the unit ball of $H^1(l_{4/3})$ and factor it as $f=(g_n h_n)$, with $g=(g_n)$ in the unit ball of $H^2(l_2)$ and $h=(h_n)$ in the unit ball of $H^2(l_4)$. This is easy to do by factoring out the Blaschke product of each component f_n and raising the factor without zero to the appropriate power. More precisely, write $f_n=B_nF_n$, where B_n is a Blaschke product and where F_n does not have zeros in D; let F be an outer function such that $|F|=(\Sigma|F_n|^{4/3})^{3/4}$ on the unit circle; then let

$$g_n=B_n(F_n/F)^{2/3}F^{1/2} \quad \text{and} \quad h_n=(F_n/F)^{1/3}F^{1/2}.$$

This factorisation has the properties claimed for g and h.

Recall the inclusion, which obviously follows from the above remark,

$$H^2(l_2)=(H^2(l_{4/3}), H^2(l_4))_{1/2} \subset (H^2(l_{4/3}), H^2(l_4))_{1/2,\infty}.$$

Then, by interpolation, since the operator of coordinatewise multiplication by $h=(h_n)$ maps $H^2(l_{4/3})$ into $H^1(l_1)$ and $H^2(l_4)$ into $H^1(l_2)$, we obtain the announced inclusion. □

Proof of Proposition 2. Consider $f=(f_n)\in H^1(l_1)+H^1(l_2)$ such that

(1) $$K_t(f, L^1(l_1), L^1(l_2))<1.$$

By classical factorisation theory, each f_n can be factored as $f_n=B_nF_n$, where B_n is a Blaschke product and where F_n does not have zeros in D, so that the analytic function $(F_n)^p$ makes sense for any $p>0$. (Alternatively, we could use the inner-outer factorisation instead.) Let us simply denote by $F^{1/2}$ the sequence of analytic functions $F^{1/2}=(F_n^{1/2})_{n\geq 1}$. Note that any assumption of the form (1) depends only on the values of each $|f_n|$ on the boundary. Now, on the boundary we have $|f_n|^{1/2}=|F_n|^{1/2}$, so that (1) obviously implies

(2) $$K_{t^{1/2}}(F^{1/2}, L^2(l_2), L^2(l_4))<2^{1/2}.$$

Therefore, by the previous Remark,

$$K_{t^{1/2}}(F^{1/2}, H^2(l_2), H^2(l_4))<2^{1/2}C,$$

where C is a numerical (absolute) constant. Hence, there is a decomposition $F^{1/2}=g_0+g_1$ with

(3) $$\|g_0\|_{H^2(l^2)}+t^{1/2}\|g_1\|_{H^2(l^4)}<2^{1/2}C.$$

Let us now return to $f=(f_n)=(B_n(g_{0_n}+g_{1_n})^2)$. Let us simply denote by g_0g_1 the sequence $(g_{0_n}g_{1_n})_{n\geq 1}$; similarly, we denote by g_0^2 and g_1^2 the sequences of squares. Observe that, by (3) and by the Hölder inequality, we have

$$\|g_0g_1\|_{H^1(l_{4/3})}<2C^2t^{-1/2},$$

which implies by the sublemma that

$$t^{-1/2}K_t(g_0g_1, H^1(l_1), H^1(l_2)) < 2C^2Kt^{-1/2}.$$

After simplification, we have

$$K_t(g_0g_1, H^1(l_1), H^1(l_2)) < 2C^2K.$$

On the other hand, by (3) we clearly have

$$K_t(g_0^2+g_1^2, H^1(l_1), H^1(l_2)) \le 2C^2+2C^2=4C^2.$$

Therefore, we conclude by the triangle inequality (and by the fact that Blaschke products are of unit norm in H^∞) that

$$\begin{aligned}K_t(f, H^1(l_1), H^1(l_2)) &\le K_t(g_0^2+g_1^2, H^1(l_1), H^1(l_2)) + K_t(2g_0g_1, H^1(l_1), H^1(l_2))\\ &\le 4C^2+4C^2K.\end{aligned}$$

By homogeneity, this completes the proof. □

COROLLARY. *There is a constant C such that, for all* $1<p<2$ *and all* $f\in L^1/H^1(l_p)$, *we have*

$$\|f\|_{L^1/H^1(l_p)} \le C\|f\|^{\theta}_{L^1/H^1(l_2)}\|f\|^{1-\theta}_{L^1/H^1(l_1)}, \tag{4}$$

where $1/p=\theta/2+(1-\theta)/1$.

Proof. By Proposition 2 and Proposition 1, there is a constant C such that every $f\in L^1/H^1(l_1)$ admits a lifting $\hat{f}\in L^1(l_1)$ such that we have *simultaneously*

$$\|\hat{f}\|_{L^1(l_1)} \le C\|f\|_{L^1/H^1(l_1)} \quad\text{and}\quad \|\hat{f}\|_{L^1(l_2)} \le C\|f\|_{L^1/H^1(l_2)}.$$

Then (4) is an immediate consequence of Hölder's inequality. □

The preceding corollary immediately implies the following proposition.

PROPOSITION 3. *There is a constant C such that, for all Banach spaces Y and for all* $2<q<\infty$ *and all 2-summing operators* $u: A\to Y$, *we have*

$$\pi_q(u) \le C\pi_2(u)^{\theta}\|u\|^{1-\theta}, \tag{5}$$

where $1/q=\theta/2+(1-\theta)/\infty$.

Proof. We first claim that, for any $n>1$ and for any $x_1, x_2, \dots, x_n$ in A, we have

$$\sum_1^n \|u(x_i)\| \le \lambda\|(\sum|x_i|^q)^{1/q}\|_\infty,$$

where $\lambda \le Cn^{1/q'}$. Indeed, let us denote by $\lambda(q,n)$ the best constant in this inequality. Assume, without loss of generality, that u is the adjoint of an operator $v: Y^*\to L^1/H^1$. Let $p=q'$. By duality, we find

$$\lambda(q,n) = \sup\{\|(v(y_i))\|_{L^1/H^1(l_p^n)}\},$$

where the sup runs over all n-tuples (y_i) in Y such that $\sup\|y_i\|\le 1$. Therefore, (4) immediately yields

$$\lambda(q,n)\le C\lambda(2,n)^{\theta}\lambda(\infty,n)^{1-\theta}\le C(n^{1/2}\pi_2(u)^{\theta}(n\|u\|)^{1-\theta}.$$

Hence

$$\lambda(q,n)\le Cn^{1/q'}\pi_2(u)^{\theta}\|u\|^{1-\theta}. \tag{6}$$

For simplicity, let $B=C\pi_2(u)^{\theta}\|u\|^{1-\theta}$. By (6) we have, for any $x_1,x_2,\dots,x_n$ in A,

$$n^{-1/q'}\sum_1^n\|u(x_i)\|\le B\|(\sum|x_i|^q)^{1/q}\|_\infty. \tag{7}$$

Now let us rewrite (7) for a sequence composed of $x_1/(k_1)^{1/q}$ repeated k_1 times, $x_2/(k_2)^{1/q}$ repeated k_2 times, etc. We obtain

$$(\sum k_i)^{-1/q'}\sum k_i^{1/q'}\|u(x_i)\|\le B\|(\sum|x_i|^q)^{1/q}\|_\infty.$$

Clearly, since the sequences of the form $((\sum k_i)^{-1}k_i)$ are obviously dense in the set of all sequences (α_i) such that $\sum\alpha_i=1$, we obtain

$$\sum(\alpha_i)^{1/q'}\|u(x_i)\|\le B\|(\sum|x_i|^q)^{1/q}\|_\infty.$$

Taking the supremum over all such (α_i), we finally obtain the announced result (5). □

We now recall a classical inequality due to Khintchine, concerning the Rademacher functions $r_1,r_2,\dots,r_n,\dots$ defined on the Lebesgue interval. For every $q>2$ there is a constant B_q such that, for all finite sequences of scalars (α_i), we have

$$\left(\int|\sum\alpha_i r_i|^q\,dt\right)^{1/q}\le B_q(\sum|\alpha_i|^2)^{1/2}.$$

The following is a known result of Maurey [M].

PROPOSITION 4. *Let X be any Banach space. Let Y be a Banach space of cotype* 2, *that is, such that there is a constant C_2 satisfying, for all n and for all n-tuples $y_1,y_2,\dots,y_n$ in Y,*

$$(\sum\|y_i\|^2)^{1/2}\le C_2\left(\int\|\sum r_iy_i\|^2\,dt\right)^{1/2}.$$

Then, for every $q>2$, every q-summing operator $u:X\to Y$ is actually 2-summing, and moreover

$$\pi_2(u)\le B_qC_2\pi_q(u).$$

Proof. Let $x_1,x_2,\dots,x_n$ be a finite subset of X such that $\sum|x^*(x_i)|^2\le 1$ for all x^* in the unit ball of X^*. Then, by the above Khintchine inequality, we have for all x^* in the unit ball of X^* that

$$\left(\int \left|\sum r_i x^*(x_i)\right|^q dt\right)^{1/q} \leq B_q.$$

Hence, by the definition of $\pi_q(u)$ (note that the integral below is actually an average over 2^n choices of signs),

$$\left(\int \left\|\sum r_i u(x_i)\right\|^q dt\right)^{1/q} \leq \pi_q(u) B_q.$$

Hence, by the definition of the cotype 2,

$$\left(\sum \|u(x_i)\|^2\right)^{1/2} \leq C_2 B_q \pi_q(u).$$

By homogeneity, this proves Proposition 4. □

We can now complete the proof of Bourgain's theorem.

Proof of Bourgain's theorem. We use the same general line of attack as Bourgain. This approach is based on an extrapolation trick that originates in the work of Maurey [M], and has been used several times before Bourgain's work (especially by the author) to prove various extensions of Grothendieck's theorem. (The latter theorem corresponds to the case $A = C$, $Y = l_1$ in the above statement; see [P1].) In this approach, the crucial point reduces to showing (5). Indeed, assuming (5), it is easy to conclude: By Proposition 4 we have $\pi_2(u) \leq C_2 B_q \pi_q(u)$; hence, by (5) $\pi_2(u) \leq C C_2 B_q \pi_2(u)^\theta \|u\|^{1-\theta}$, and hence if we assume *a priori* that $\pi_2(u)$ is finite then we obtain

$$\pi_2(u) \leq (C C_2 B_q)^{1/1-\theta} \|u\|, \tag{8}$$

which establishes the announced result in the case of a 2-summing operator. Hence, in particular, (8) holds if u is of finite rank; since A has the metric approximation property, we can easily conclude that (8) actually holds for arbitrary operators. Finally, the last assertion follows from a well-known factorisation property of 2-summing operators, due to Pietsch (cf., e.g., [P1, Chap. 1]). □

REMARK 5. There is also a slightly different way to prove (8). One can use a simple interpolation argument to prove that, for any $n > 1$ and for any $x_1, x_2, \ldots, x_n$ in A, we have

$$\|(u(x_i))\|_{l_{q,\infty}(Y)} \leq B \|(x_i)\|_{L_\infty(l_{q,\infty})},$$

where B is as above. We may then apply this, replacing $x_1, x_2, \ldots, x_n$ by the 2^n-tuple formed by the 2^n choices of signs $\sum r_i(t) x_i$. After normalisation by a factor $2^{-n/q}$, we obtain

$$\left\|\left(\sum r_i u(x_i)\right)\right\|_{L_{q,\infty}(dt;Y)} \leq B \left\|\sum r_i x_i\right\|_{L_\infty(L_{q,\infty}(dt))}.$$

But then, we observe that Khintchine's inequality implies *a fortiori* the equivalence of $\|\sum r_i x_i\|_{L_\infty(L_{q,\infty}(dt))}$ with $\|(x_i)\|_{L_\infty(l_2)}$. This immediately leads to (8) by the same argument as above.

REMARK 6. As is well known, it follows from Bourgain's theorem as stated above that L_1/H^1 is a cotype-2 space. This can be derived as in [B1] from a result of Wojtaszczyk which ensures that L_1/H^1 is isomorphic to $L_1/H^1(l_1)$. Alternatively, if one wishes to avoid the use of the latter result, one can observe that our proof of Bourgain's theorem is valid with essentially the same argument, using $L_1/H^1(l_1)$ instead of L_1/H^1.

Actually, we can generalize Bourgain's theorem as follows.

THEOREM 7. *Let $0<r<1$. Then any operator $u: c_0 \to L_r/H^r$ is 2-summing. Moreover, there is a constant C_r such that every operator $u: c_0 \to L_r/H^r$ is 2-summing and satisfies $\pi_2(u) \le C_r\|u\|$. Finally, L_r/H^r is of cotype 2.*

Proof. We only sketch the argument. (It might very well be that this result follows from the other proofs; however, it seems to have passed unnoticed so far.) Consider an operator $u: l_\infty^n \to L_r/H^r$. We will show that there is a constant C_r independent of n such that

$$(9) \qquad \forall m,\ \forall x_1, x_2, \ldots, x_m \in l_\infty^n, \quad \|(u(x_i))\|_{L_r/H^r(l_2^m)} \le C_r\|u\|\|(x_i)\|_{l_\infty^n(l_2^m)}.$$

We argue similarly as above, but in a dual setting. Let $r \le p \le \infty$. We denote by $C_p(u)$ the smallest constant C such that

$$\forall m, \forall x_1, x_2, \ldots, x_m \in l_\infty^n, \quad \|(u(x_i))\|_{L_r/H^r(l_p^m)} \le C\|(x_i)\|_{l_p^m(l_\infty^n)}.$$

Obviously, we have $C_r(u) = \|u\|$. Choose p so that $r<1<p<2$. Let θ be chosen so that $1/p = (1-\theta)/r + \theta/2$. A simple adaptation of Propositions 1 and 2 yields a simultaneous "good" lifting for the couple $L_r/H^r(l_r), L_r/H^r(l_2)$, and the corresponding extension of (4). It follows that we have, for some constant C' (independent of m),

$$\|(u(x_i))\|_{L_r/H^r(l_p^m)} \le C'C_2(u)^\theta\|u\|^{1-\theta}m^{1/p}\sup\|x_i\|_{l_\infty^n}.$$

As a consequence, if $B' = C'C_2(u)^\theta\|u\|^{1-\theta}$, we have

$$(10) \qquad \|(u(x_i))\|_{L_r/H^r(l_p^m)} \le B'\|(x_i)\|_{l_{p,r}^m(l_\infty^n)}.$$

It is easy to check that, for some constant C'' (independent of m or n), we have

$$\|(x_i)\|_{l_{p,r}^m(l_\infty^n)} \le C''m^{1/p-1/2}\|(x_i)\|_{l_2^m(l_\infty^n)},$$

so that (10) gives, after normalisation,

$$(11) \qquad \|(u(x_i))\|_{L_r/H^r(L_p^m)} \le B'C''\|(x_i)\|_{L_2^m(l_\infty^n)},$$

where L_p^m denotes the L_p-space relative to $\{1, 2, \ldots, m\}$ equipped with the uniform probability measure.

Let $K = B'C''$. We now take $m = 2^k$, replace (x_i) by the 2^k "choices of signs" $x_t = \sum_1^k r_i(t)x_i$, and use the dualisation of Khintchine's inequality in L_p, which states that the quotient of L_p $(p>1)$ by the orthogonal of the Rademacher functions can be identified with l_2. If we simply denote by $Q(n)$

the quotient space of $L_2(l_\infty^n)$ by the subspace of all functions "orthogonal" to the Rademacher functions (i.e., that have a zero integral against any Rademacher function), we can deduce from (11) that

$$\|(u(x_i))\|_{L_r/H^r(l_2^k)} \le K\|(x_i)\|_{Q(n)}. \tag{12}$$

On the other hand, by a known reformulation of Grothendieck's theorem (see [P1, Cor. 6.7, p. 77]), we have

$$\|(x_i)\|_{Q(n)} \le K'\|(x_i)\|_{l_\infty^n(l_2^k)}, \tag{13}$$

where K' is a numerical constant. Therefore, (12) implies $C_2(u) \le KK'$. Recalling the value of K and B', we conclude that

$$C_2(u) \le K'C''C'C_2(u)^\theta \|u\|^{1-\theta},$$

so that we again conclude by "extrapolation" that $C_2(u) \le K''\|u\|$ for some constant K'' depending only on r. Combining (12) and (13) with this last estimate, we obtain the announced result (9) with $C_r = KC'C''$. Since there is obviously a norm-1 inclusion of $L_r/H^r(l_2^m)$ into $l_2^m(L_r/H^r)$, we have $\pi_2(u) \le C_2(u) \le C_r\|u\|$, and this completes the proof for $X = l_\infty^n$ (with a constant C_r bounded independently of n). By density, this is enough to prove the case of an operator defined on c_0. Finally, the cotype-2 property can be proved as indicated in Remark 6, by observing that the first part of Theorem 7 remains valid with $L_r/H^r(l_r)$ (or equivalently $l_r(L_r/H^r)$) in the place of L_r/H^r. We then follow a standard argument: Given elements $x_1, x_2, \ldots, x_n \in L_r/H^r$, we consider the operator $u: l_\infty^n \to L_r(L_r/H^r)$ defined by $u(a_1, a_2, \ldots, a_n) = \sum a_i r_i x_i$, where $r_1, r_2, \ldots, r_n$ are the Rademacher functions as before. We have

$$(\sum \|x_i\|^2)^{1/2} \le \pi_2(u) \le C_r\|u\|, \tag{14}$$

but it is well known that there is a constant B_r depending only on r such that

$$\|u\| \le B_r \|\sum r_i(t) x_i\|_{L_r(dt; L_r/H^r)}.$$

Therefore, (14) implies that L_r/H^r is a cotype-2 space. □

FINAL REMARKS. (1) As a corollary, one obtains that every rank-n operator on A extends to the whole of $C(\mathbf{T})$ with norm at most $C \operatorname{Log} n$ for some constant C. This follows from Bourgain's theorem and a previous result of Mityagin and Pelczyński (see [B1] for the deduction).

(2) The preceding argument shows that

$$H^\infty(l_{p,\infty}) = (H^\infty(l_1), H^\infty(l_\infty))_{\theta,\infty}, \tag{15}$$

where $1/p = 1-\theta$ and $0 < \theta < 1$. But this kind of result is not really new. It can be derived from the remarks on interpolation spaces included in [B1] using a rather simple factorisation argument, such as the one used for Theorem 2.7 in [HP]. More results along this line have been obtained by Xu [X]. In [P2], we will give a more systematic treatment of results such as (15) in more general cases for the real interpolation method with arbitrary parameters.

(3) We should mention that while Kisliakov's recent proof of Bourgain's theorem seems more complicated than the above, it also yields more information on the so-called (p,q)-summing operators which does not follow from our approach (cf. [K2; K3]). Moreover, although the above argument applies also for an operator defined on H^∞ and with values in a cotype-2 space Y with the bounded approximation property, it is a well-known drawback of the "extrapolation method" that it does not apply to the case of a linear operator from H^∞ into its dual, although that case was settled in [B2].

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