Localization of Hilbert Modules

XIAOMAN CHEN & R. G. DOUGLAS

In [6] it was shown how to reformulate the study of bounded linear operators on Hilbert spaces into the study of Hilbert modules. Here we study modules over a function algebra which have also a Hilbert space structure relative to which the multiplication is jointly continuous.

Localization is a useful method in the study of Hilbert modules. Many important invariants can be obtained via localization. For example, one can recover the characteristic operator function for the canonical model of Sz.-Nagy and Foiaş as well as the curvature of some hermitian holomorphic vector bundles. Some examples [6] show that localization seems to be related to the index and spectral theory of Hilbert modules.

Localization of Hilbert modules was introduced in [6] using the module tensor product. Tensoring with a finite-dimensional module whose spectrum consists of a single point yields another finite-dimensional module with the same spectrum. Analyzing the latter module yields the sought-after invariants.

A finite-dimensional module with spectrum a single point involves various partial derivatives and their evaluation at that point. One defines the order of such a module to be the order of the highest derivative that is needed.

In this note, we show that zero$^{\text{th}}$ order localization is not enough to get much information about a Hilbert module. In particular, in the several-variables case (which is our main interest in the Hilbert module approach to operator theory), zero$^{\text{th}}$ order localization determines just the first stage of the Koszul complex introduced by Taylor [7]. In [5], first-order localization was considered and some interesting calculations were made. However, much groundwork needs to be developed to obtain efficient localization techniques for finding invariants for Hilbert modules via localization.

In this paper, we confine our attention to Hilbert modules over $\mathcal{A}(\Omega)$, $\mathcal{A}(\Omega)=\{f \mid f \in C(\bar{\Omega}) \cap \operatorname{Hol}(\Omega)\}$, where Ω is a bounded connected domain in $\mathbf{C}^n$. We define higher-order localizations for this kind of Hilbert module. As an application we study the relation between modules being locally unitarily equivalent and globally unitarily equivalent. We obtain an extension of the Cowen–Douglas theory [1; 2] to the context of Hilbert modules.

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1. Preliminaries

A Hilbert module $\mathfrak{M}$ over $A(\Omega)$ is a Hilbert space $\mathfrak{M}$ together with a unital module multiplication

$$A(\Omega)\times\mathfrak{M}\to\mathfrak{M}.$$

We assume that there is a constant $k>0$ such that $\|f\cdot x\|\le k\|f\|\|x\|$ for f in $A(\Omega)$ and x in $\mathfrak{M}$. (By the uniform boundedness principle this is equivalent to joint continuity.)

If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are Hilbert modules over $A(\Omega)$ we let $\mathfrak{M}_1\otimes\mathfrak{M}_2$ denote the Hilbert space tensor product of $\mathfrak{M}_1$ and $\mathfrak{M}_2$. There are two ways of making $\mathfrak{M}_1\otimes\mathfrak{M}_2$ into a Hilbert module over $A(\Omega)$, one using the action on $\mathfrak{M}_1$ and the other using the action on $\mathfrak{M}_2$. By making these two actions equal we obtain the module tensor product. This is accomplished by forming the quotient of $\mathfrak{M}_1\otimes\mathfrak{M}_2$ by the closure $\mathfrak{N}$ of the linear span of the vectors

$$\{fx\otimes y-x\otimes fy\mid f\in A(\Omega),\, x\in\mathfrak{M}_1,\, y\in\mathfrak{M}_2\}.$$

Since the latter is a submodule, the quotient of $\mathfrak{M}_1\otimes\mathfrak{M}_2$ by $\mathfrak{N}$ is a Hilbert module which we denote by $\mathfrak{M}_1\otimes_{A(\Omega)}\mathfrak{M}_2$.

Let $\mathbf{C}_z$ be the local module over $A(\Omega)$ for z in Ω, where $\mathbf{C}_z$ is the Hilbert space $\mathbf{C}$ and module multiplication is defined by $f\cdot\lambda=f(z)\lambda$.

DEFINITION 1.1. If $\mathfrak{M}$ is a Hilbert module over $A(\Omega)$, then $\mathfrak{M}\otimes_{A(\Omega)}\mathbf{C}_z$ is called the *localization* of the Hilbert module $\mathfrak{M}$ at z.

The following lemma is used often in this paper.

LEMMA 1.2 [6, Thm. 5.14]. *Let $\mathfrak{L}$ be a finite-dimensional Hilbert $A(\Omega)$-module with cyclic vector e_0, and set $J=\{f\mid f\in A(\Omega),\ f\cdot e_0=0\}$. If $\mathfrak{M}$ is a Hilbert $A(\Omega)$-module, then $\mathfrak{M}\otimes_{A(\Omega)}\mathfrak{L}$ and the quotient $\mathfrak{M}/[J\mathfrak{M}]^-$ are similar $A(\Omega)$-modules.*

In particular, $\mathfrak{M}\otimes_{A(\Omega)}\mathbf{C}_z$ is unitarily module equivalent to $\mathfrak{M}/[J_z\mathfrak{M}]^-$, where $J_\lambda=\{f\mid f\in A(\Omega),\ f(\lambda)=0\}$ is an ideal of $A(\Omega)$. We set $\mathfrak{M}_\lambda=\mathfrak{M}/[J_{\bar\lambda}\mathfrak{M}]^-$ for $\bar\lambda$ in Ω. The complex conjugate is necessary to make $\mathfrak{M}_\lambda$ holomorphic in λ.

REMARK 1.3. In the proof of Lemma 1.2 we see that the map Ψ from $[J\mathfrak{M}]^\perp\to\mathfrak{M}\otimes_A\mathfrak{L}$ defined by $\Psi(h)=h\otimes_A e_0$ is an invertible module mapping.

Many examples [4] show that the invariants for Hilbert modules depend on the behavior of the localization in a neighborhood of a point and not just on what happens at the point. So we must consider how the localizations fit together as z varies. Thus we view $z\mapsto\mathfrak{M}\otimes_A\mathbf{C}_z$ as a sheaf over Ω called the *spectral sheaf*. The details can be found in [4]. Here, we just introduce a class of Hilbert modules over $A(\Omega)$ for which the localizations fit together very well and the sheaf is actually a vector bundle.

Let $\mathrm{Gr}(n, \mathfrak{M})$ denote the Grassmann manifold of all n-dimensional subspaces of the Hilbert space $\mathfrak{M}$ and let Ω^* be the conjugate domain of Ω.

DEFINITION 1.4. Let $\mathfrak{M}$ be a Hilbert module over $A(\Omega)$. We say $\mathfrak{M}$ is *locally free* at λ_0 in Ω^* if there exists a neighborhood Δ of λ_0, an integer n with

$$\lambda \mapsto f(\lambda) = \mathfrak{M}_\lambda \quad (\Delta \to \mathrm{Gr}(n, \mathfrak{M})),$$

and holomorphic $\mathfrak{M}$-valued functions $\{\gamma_i(\lambda)\}_{i=1}^n$ such that

$$f(\lambda) = \bigvee\{\gamma_1(\lambda), \dots, \gamma_n(\lambda)\} \quad \text{for } \lambda \text{ in } \Delta.$$

We say that $\mathfrak{M}$ is *locally free* on the open set $\Omega_0^* \subset \Omega^*$ if at each λ_0 in Ω_0^*, $\mathfrak{M}$ is locally free.

EXAMPLE 1.5. Consider $H^2(\mathbf{D}^2)$ as a Hilbert module over $A(\mathbf{D}^2)$ and the submodule, $H^2_{(0,0)}(\mathbf{D}^2) = \{g \mid g \in H^2(\mathbf{D}^2),\ g(0,0) = 0\}$. It is easy to prove that $H^2(\mathbf{D}^2)$ is locally free on $\mathbf{D}^2$. However, while $H^2_{(0,0)}(\mathbf{D}^2)$ is locally free on $\mathbf{D}^2 \backslash \{0,0\}$, $H^2_{(0,0)}(\mathbf{D}^2)$ is not locally free at $(0,0)$.

If I is an ideal in the polynomial ring $\mathbf{C}[z_1, \dots, z_n]$, then the closure $[I]$ of I in $H^2(\mathbf{D}^n)$ is an $A(\mathbf{D}^n)$-module. For $\bar{z}_0$ not in $Z[I] = \{z \mid z \in \mathbf{C}^n,\ f(z) = 0$ for every f in $I\}$, we see that $J_{\bar{z}_0} I$ has codimension 1 in I. Therefore, it follows that $[I]_{z_0}$ has dimension 1 for $\bar{z}_0$ in $Z[I] \cap \mathbf{D}^n$. Moreover, if we choose $p(z)$ in I such that $p(\bar{z}_0) \neq 0$, then $\gamma(\lambda) = p \otimes_A 1_\lambda$ (where 1_λ is 1 in $\mathbf{C}_\lambda$) defines a holomorphic spanning section for $[I]_z$ for z in a neighborhood of $\bar{z}_0$. Thus $[I]$ is locally free on the complement of $Z[I]$ in $\mathbf{D}^n$, and we can use the vector-bundle point of view to try to classify Hilbert modules of the form $[I]$. (Actually we use such sections of the spectral sheaf to define the holomorphic structure.)

PROPOSITION 1.6. *If* $[I_1] \otimes_{A(\mathbf{D}^n)} \mathbf{C}_z \cong [I_2] \otimes_{A(\mathbf{D}^n)} \mathbf{C}_z$ *as hermitian holomorphic vector bundles on some open set* $\Omega_0^* \subset \mathbf{D}^n$, *then* $[I_1] \cong [I_2]$ *as Hilbert modules.*

Using the fact that $\bigvee_{\lambda \in \Omega_0^*} [I]_\lambda = [I]$ for any open set Ω_0 on which $[I]$ is locally free, this proposition is easily proved using ideas from [1]. The spanning property for $[I]$ follows from the analogous result for $H^2(\mathbf{D}^n)$ since projecting a holomorphic section γ for $H^2(\mathbf{D}^n)$ into $[I]$ yields one for $[I]$ off the zero set of γ. This proposition shows that global properties of the spectral sheaf determine the module in the case of $[I]$.

2. Higher-order Derivative Localization

Since local invariants depend on the behavior of $\mathfrak{M}_z$ in a neighborhood of z, naturally we want to define some kind of localization which depends on the higher derivatives of the functions. In the one-variable case, it seems obvious how to define such a Hilbert module structure on $\mathbf{C}^n$ at λ. Namely, one sets

$$f \cdot h = \Lambda f_\lambda \cdot h,$$

where

$$\Lambda f_\lambda \cdot h = \begin{bmatrix} f(\lambda) & f'(\lambda) & \frac{1}{2!}f''(\lambda) & \cdots & \frac{1}{(n-1)!}f^{(n-1)}(\lambda) \\ 0 & f(\lambda) & f'(\lambda) & \cdots & \\ 0 & 0 & f(\lambda) & & \frac{1}{2!}f''(\lambda) \\ \vdots & \vdots & \vdots & \ddots & f'(\lambda) \\ 0 & 0 & 0 & \cdots & f(\lambda) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{bmatrix}.$$

It is easy to show that such a multiplication yields a module action.

In the several-variables case, however, defining a module multiplication on $\mathbf{C}^n$ is more complicated. But the difficulty is more notational than conceptual. Thus we begin by introducing some notation.

Let $I=(i_1, i_2, \ldots, i_n)$ be an n-tuple of positive integers, $I! = i_1! i_2! \cdots i_n!$, $|I| = i_1 + i_2 + \cdots + i_n$, and $I_1 + I_2 = (i_1^1 + i_1^2, i_2^1 + i_2^2, \ldots, i_n^1 + i_n^2)$, where $I_j = (i_1^j, i_2^j, \ldots, i_n^j)$, $j = 1, 2$. If there is a K such that $J+K=I$, we say $J \leq I$.

Set $f^I = \partial^{|I|} f / \partial z^I$ and $s = (i_1+1)!\,(i_2+1)! \cdots (i_n+1)!$ for f in $A(\Omega)$ and $I = (i_1, i_2, \ldots, i_n)$. Since $\partial^{|I|}/\partial z^I$ determines a jet, we sometimes refer to I as a jet. For a jet I and a λ in Ω we want to define a multiplication on $\mathbf{C}^s$, making it into the Hilbert module which we will denote by $\mathbf{C}_\lambda^I$. Let $\{e_J\}_J$ denote an orthogonal basis for $\mathbf{C}^s$, where J ranges over the jets $J \leq I$. We define a matrix Λ_λ^I of differential operators, where the general matrix element a_{HJ} is defined by

$$a_{HJ} = \begin{cases} \dfrac{1}{K!} \dfrac{\partial^{|K|}}{\partial z^K} & \text{if } H+K=J, \\ 0 & \text{if } H \not\leq J. \end{cases}$$

For f in $A(\Omega)$ and λ in Ω we let $\Lambda_\lambda^I f$ denote the action of the operator matrix Λ_λ^I on the function f with the resulting function evaluated at λ. Hence we obtain a multiplication

$$A(\Omega) \times \mathbf{C}^s \to \mathbf{C}^s,$$

that is,

$$f \cdot x = (\Lambda_\lambda^I f) x$$

for $x = \sum_J b_J e_J$ in $\mathbf{C}^s$.

THEOREM 2.1. *The above multiplication makes $\mathbf{C}^s$ into a Hilbert module over $A(\Omega)$.*

Proof. We have to prove the above mapping is a module action. All of this is easy to verify except the associativity $(f \cdot g) \cdot x = f \cdot (g \cdot x)$ for f and g in $A(\Omega)$ and x in $\mathbf{C}^s$.

Fix f and g in $A(\Omega)$ and let the matrix elements for $\Lambda_\lambda^I f$, $\Lambda_\lambda^I g$, and $\Lambda_\lambda^I fg$ be denoted by a_{JK}, b_{JK}, and c_{JK}, respectively. The product matrix is defined by

$$\tilde{c}_{JK} = \sum_L a_{JL} b_{LK},$$

where the sum is over all jets $L \le I$. However, the product $a_{JL} b_{LK} = 0$ unless $J \le L \le K \le I$; since $c_{JK} = 0$ unless this is valid, we can restrict to this case. Using the Leibniz rule we have

$$\begin{aligned}
\tilde{c}_{JK} &= \sum_{J \le L \le K} a_{JL} b_{LK} \\
&= \sum_{J \le L \le K} \left(\frac{1}{(L-J)!} \frac{\partial^{|L-J|}}{\partial z^{(L-J)}} f \right)(\lambda) \left(\frac{1}{(K-L)!} \frac{\partial^{|K-L|}}{\partial z^{(K-L)}} g \right)(\lambda) \\
&= \sum_{J \le L \le K} \frac{1}{(L-J)!} \frac{1}{(K-L)!} \frac{\partial^{|K-J|}}{\partial z^{(K-J)}} (fg)(\lambda) \\
&= \frac{1}{(K-J)!} \frac{\partial^{|K-J|}}{\partial z^{(K-J)}} (fg)(\lambda).
\end{aligned}$$

Using this formula, it is easy to verify $f \cdot e = \Lambda_\lambda^I f \cdot e$ is a module multiplication. □

EXAMPLE 2.2. In the two-variable case, with $I = (2, 1)$, we have the matrix

$$\Lambda = \begin{pmatrix}
\frac{\partial^0}{\partial^0 z} & \frac{\partial^{(1,0)}}{\partial^{(1,0)} z} & \frac{\partial^{(0,1)}}{\partial^{(0,1)} z} & \frac{1}{2}\frac{\partial^{(2,0)}}{\partial^{(2,0)} z} & \frac{\partial^{(1,1)}}{\partial^{(1,1)} z} & \frac{1}{2}\frac{\partial^{(2,1)}}{\partial^{(2,1)} z} \\
0 & \frac{\partial^0}{\partial^0 z} & 0 & \frac{\partial^{(1,0)}}{\partial^{(1,0)} z} & \frac{\partial^{(0,1)}}{\partial^{(0,1)} z} & \frac{\partial^{(1,1)}}{\partial^{(1,1)} z} \\
0 & 0 & \frac{\partial^0}{\partial^0 z} & 0 & \frac{\partial^{(1,0)}}{\partial^{(1,0)} z} & \frac{1}{2}\frac{\partial^{(2,0)}}{\partial^{(2,0)} z} \\
0 & 0 & 0 & \frac{\partial^0}{\partial^0 z} & 0 & \frac{\partial^{(0,1)}}{\partial^{(0,1)} z} \\
0 & 0 & 0 & 0 & \frac{\partial^0}{\partial^0 z} & \frac{\partial^{(1,0)}}{\partial^{(1,0)} z} \\
0 & 0 & 0 & 0 & 0 & \frac{\partial^0}{\partial^0 z}
\end{pmatrix}.$$

Denoting the above Hilbert module by $\mathbf{C}_\lambda^I$, we have the following definition.

DEFINITION 2.3. Suppose $\mathfrak{M}$ is a Hilbert module over $A(\Omega)$. We call $\mathfrak{M} \otimes_{A(\Omega)} \mathbf{C}_\lambda^I$ the *I-localization* at λ.

COROLLARY 2.4. *For $\mathfrak{M}$ a Hilbert module over $A(\Omega)$, $\mathfrak{M} \otimes_{A(\Omega)} \mathbf{C}_\lambda^I$ and $\mathfrak{M}/[J_\lambda^I \mathfrak{M}]^-$ are similar $A(\Omega)$-modules, where $J_\lambda^I = \{f \mid f \in A(\Omega),\ f^{I'}(\lambda) = 0$ for all $I' \le I\}$.*

Proof. We only need to observe that e_I is a cyclic vector in $\mathbf{C}_\lambda^I$. □

REMARK 2.5. We don't know whether $\mathfrak{M} \otimes_{A(\Omega)} \mathbf{C}_\lambda^I$ and $\mathfrak{M}/[J_\lambda^I \mathfrak{M}]^-$ are unitarily module equivalent.

DEFINITION 2.6. Denote $\mathcal{M}/[J_\lambda^I \mathcal{M}]^-$ by $\mathcal{M}_\lambda^I$. A *quasi-vector bundle* for the Hilbert module $\mathcal{M}$ is defined by $\mathcal{M}^I = \bigcup_{\lambda \in \Omega^*} \mathcal{M}_\lambda^I$, which is the disjoint union of the $\mathcal{M}_\lambda^I$.

DEFINITION 2.7. For f in $A(\Omega)$ and h in $H^2(\Omega)$ we set $f^*h = T_f^* h$, where T_f is the operator on $H^2(\Omega)$ defined by $T_f h = f \cdot h$.

3. Cowen–Douglas Theory

According to the Cowen–Douglas theory [1; 2], the higher-order local operators determine completely a global operator in the $B_n(\Omega)$-class. Since we have now introduced higher-order localization, we want to know how the higher-order localizations are related to the global Hilbert module. There is a difference, however, from the Cowen–Douglas theory, because in general $\mathcal{M}_\lambda^I$ is not the kernel of some commuting n-tuple of operators. One reason is that the function algebra $A(\Omega)$ may not be generated by the coordinate functions $z_1, z_2, \ldots, z_n$.

THEOREM 3.1. *Suppose $\bigcup_{\lambda \in \Omega^*} \mathcal{M}_\lambda$ is locally free at λ_0 and $\{\gamma_i(\lambda)\}_{i=1}^k$ is a frame on a neighborhood of λ_0. Then for any I, $\bigcup_{\lambda \in \Omega^*} \mathcal{M}_\lambda^I$ is locally free at λ_0 and $\{\gamma_i^{I'}\}_{i=1, I' \leq I}^k$ is a frame for $\bigcup_{\lambda \in \Omega^*} \mathcal{M}_\lambda^I$ on the same neighborhood of λ_0.*

COROLLARY 3.2. *If $\bigcup_{\lambda \in \Omega^*} \mathcal{M}_\lambda$ is locally free at λ_0, then* $\dim \mathcal{M}_\lambda^I = (I+\mathbf{1})! \dim \mathcal{M}_\lambda$, *where $I = (i_1, \ldots, i_n)$ and $\mathbf{1} = (1, 1, \ldots, 1)$.*

The assumption that $\mathcal{M}$ is locally free is necessary. For example, if we take $\mathcal{M} = H^2_{(0,0)}(\mathbf{D}^2) = \{f \mid f \in H^2(\mathbf{D}^2),\ f(0,0) = 0\}$, then $\dim \mathcal{M}_{(0,0)} = 2$ while $\dim \mathcal{M}_{(0,0)}^{(1,0)} = 3$. From Example 1.5 we know that $\mathcal{M}$ is not locally free at $(0,0)$.

Proof of Theorem 3.1. We prove this theorem by induction on $|I|$. Assume Theorem 3.1 holds when $|I| \leq m$. We show Theorem 3.1 holds when $|I| = m+1$.

Since $(f - f(\bar{\lambda}))^* \gamma_i(\lambda) = 0$, $i = 1, 2, \ldots, k$, for any f in $A(\Omega)$ and λ in Ω^*, it follows that

$$f^* \gamma_i^I(\lambda) = \sum_{I_1 + I_2 = I} \frac{I!}{I_1! \, I_2!} \overline{f^{I_1}(\bar{\lambda})} \gamma_i^{I_2}(\lambda) \tag{3.1}$$

for any jet I. Hence, we claim that $\gamma_i^{I'}(\lambda)$ is orthogonal to $J_\lambda^I \mathcal{M}$ for $i = 1, 2, \ldots, k$, $I' \leq I$. In particular,

(1) for $I_1 \leq I_2$, $(z - \bar{\lambda})^{(I_1)*} \gamma_i^{I_2}(\lambda) = (I_2!/(I_2 - I_1)!) \gamma_i^{I_2 - I_1}(\lambda)$; and
(2) for $I_1 \not\leq I_2$, $(z - \bar{\lambda})^{(I_1)*} \gamma_i^{I_2}(\lambda) = 0$,

where $(z - \bar{\lambda})^{(I)} = (z_1 - \bar{\lambda}_1)^{i_1} (z_2 - \bar{\lambda}_2)^{i_2} \cdots (z_n - \bar{\lambda}_n)^{i_n}$.

Using (3.1), it is easy to show the $\{\gamma_i^{I'}\}_{I' \leq I, i=1}^{k}$ are independent at each point λ belonging to the neighborhood of λ_0 and that $\bigvee\{\gamma_i^{I'}(\lambda)\}$ is orthogonal to $J_\lambda^I \mathcal{M}$. We must prove that

$$\bigvee_{\substack{I' \le I \\ i=1}}^{k} \{\gamma_i^{I'}\} = \mathfrak{M}_\lambda^I.$$

Without loss of generality, assume $I=(i_1+1, i_2, \ldots, i_n)$ and $I'=(i_1, i_2, \ldots, i_n)$. It is sufficient to prove that for any x in $\mathfrak{M}\ominus[J_\lambda^I\mathfrak{M}]^-$, we can choose constants $b_i^{I''}$ such that $x-\sum_{I''} b_i^{I''}\gamma_i^{I''}(\lambda_0)$ is orthogonal to $J_\lambda^{I'}\mathfrak{M}$, where $I''=(i_1+1, j_2, \ldots, j_n)$, $j_k \le i_k$ and $k=2, \ldots, n$.

It is clear that $(z_1-\bar{\lambda}_{01})^*x$ is in $\mathfrak{M}_\lambda^{I'}$, since for any f in $J_{\lambda_0}^{I'}$ we have $(z_1-\bar{\lambda}_{01})f$ in $J_{\lambda_0}^I$. Hence, by the induction hypothesis,

$$(z_1-\bar{\lambda}_{01})^*x = \sum_{\substack{K \le I' \\ i=1}}^{k} a_i^K \gamma_i^K(\lambda_0).$$

At first, we want to choose $b_i^{I''}$ such that $x-\sum_{I'', i=1}^{k} b_i^{I''}\gamma_i^{I''}(\lambda_0)$ is in $\operatorname{Ker}(z_1-\bar{\lambda}_{01})^{*i_1+1}$. Obviously,

$$(z_1-\bar{\lambda}_{01})^{*i_1+1}x = \sum_{\substack{K=(i_1, K_2, \ldots, K_n) \le I' \\ i=1}}^{k} a_i^K \gamma_i^{(0, K_2, \ldots, K_n)}(\lambda_0) i_1!.$$

But

$$(z_1-\bar{\lambda}_{01})^{*i_1+1} \sum_{\substack{I'' \\ i=1}}^{k} b_i^{I''}\gamma_i^{I''}(\lambda_0) = \sum_{\substack{I'' \\ i=1}}^{k} (i_1+1)!\, b_i^{I''}\gamma_i^{(0, j_2, \ldots, j_n)}(\lambda_0);$$

hence we take

$$b_i^{(i_1+1, j_2, \ldots, j_n)} = \frac{1}{i_1+1}\, a_i^{(i_1, j_2, \ldots, j_n)},$$

which yields that $x-\sum_{I'', i=1}^{k} b_i^{I''}\gamma_i^{I''}(\lambda_0)$ is in $\operatorname{Ker}(z_1-\bar{\lambda}_{01})^{*i_1+1}$. It is also clear that $x-\sum_{I'', i=1}^{k} b_i^{I''}\gamma_i^{I''}(\lambda_0)$ is in $\mathfrak{M}_{\lambda_0}^I$. But for any f in $J_{\lambda_0}^{I'}$, we can construct a function g in $J_{\lambda_0}^I$ as follows:

$$g = f - \sum_{(j_2, \ldots, j_n) \le (i_2, \ldots, i_n)} \frac{1}{(i_1+1)!\, j_2! \cdots j_n!} f^{(i_1+1, j_2, \ldots, j_n)}(\bar{\lambda}_0) \\ (z_1-\bar{\lambda}_{01})^{i_1+1}(z_1-\bar{\lambda}_{02})^{j_2} \cdots (z_n-\bar{\lambda}_{0n})^{j_n};$$

hence

$$g^*\left(x - \sum_{\substack{I'' \\ i=1}}^{k} b_i^{I''}\gamma_i^{I''}(\lambda_0)\right) = 0.$$

This implies

$$f^*\left(x - \sum_{\substack{I'' \\ i=1}}^{k} b_i^{I''}\gamma_i^{I''}(\lambda_0)\right) = 0,$$

since

$$(z_1-\bar{\lambda}_{01})^{*i_1+1}\left(x - \sum_{\substack{I'' \\ i=1}}^{k} b_i^{I''}\gamma_i^{I''}(\lambda_0)\right) = 0.$$

Using the hypothesis of induction, we get

$$x=\sum_{\substack{I''\\ i=1}}^{k} b_i^{I''}\gamma_i^{I''}(\lambda_0)+\sum_{\substack{K\le I'\\ i=1}}^{k} c_i^K\gamma_i^K(\lambda_0). \qquad \square$$

PROPOSITION 3.3. *Suppose $\mathfrak{M}$ is locally free at λ_0, that is, there is a holomorphic frame $\{\gamma_i(\lambda)\}_{i=1}^k$ on a neighborhood of λ_0. Then $\mathfrak{M}\otimes_{A(\Omega)}\mathbf{C}_{\bar\lambda_0}^I$ $(=\mathfrak{N}_{\bar\lambda_0}^{\perp}\subset\mathfrak{M}\otimes\mathbf{C}_{\bar\lambda_0}^I)$ has the holomorphic frame*

$$h_i^{I'}(\lambda)=\sum_{K\le I'}\frac{1}{K!}\gamma^K(\lambda)\otimes e_K,\quad I'\le I,\ i=1,2,\dots,k.$$

Proof. Obviously, the $\{h_i^{I'}\}_{i=1,I'\le I}^k$ are linearly independent. We only need to show $h_i^{I'}\perp\mathfrak{N}_{\bar\lambda_0}$, since $\mathfrak{M}\otimes_{A(\Omega)}\mathbf{C}_{\bar\lambda_0}^I$ and $\mathfrak{M}_{\lambda_0}^I$ are similar. We see that

$$\mathfrak{N}_{\bar\lambda_0}=\{fx\otimes e-x\otimes fe\mid f\in A(\Omega),\, x\in\mathfrak{M},\text{ and } e\in\mathbf{C}_{\lambda_0}^I\}^-.$$

It is sufficient to show

$$\langle h_i^{I'}(\lambda_0), fx\otimes e_H-x\otimes fe_H\rangle=0\quad\text{for } H\le I.$$

But we have

$$fx\otimes e_H-x\otimes fe_H=(f-f(\bar\lambda_0))x\otimes e_H-\sum_{\substack{F+G=H\\ F\ne 0}}\frac{1}{G!}f^G(\bar\lambda_0)x\otimes e_F,$$

and by (3.1) we have

$$\langle\gamma_i^{I'}(\lambda_0),(f-f(\bar\lambda_0))x\rangle=\sum_{\substack{K+J=I'\\ K\ne 0}}\frac{I'!}{K!\,J!}\langle\gamma_i^J(\lambda_0),f^K(\bar\lambda_0)x\rangle.$$

Hence

$$\langle h_i^{I'}(\lambda_0),(f-f(\bar\lambda_0))x\otimes e_H\rangle=\sum_{\substack{K+J=H\\ K\ne 0}}\frac{1}{K!\,J!}\langle\gamma_i^J(\lambda_0),f^K(\bar\lambda_0)x\rangle.$$

$$\left\langle h_i^{I'}(\lambda_0),\sum_{\substack{F+K=H\\ F\ne 0}}\frac{1}{K!}f^K(\bar\lambda_0)x\otimes e_F\right\rangle=\sum_{\substack{K+J=H\\ K\ne 0}}\frac{1}{K!\,J!}\langle\gamma_i^J(\lambda_0),f^K(\bar\lambda_0)x\rangle,$$

which means

$$\langle h_i^{I'}(\lambda_0), fx\otimes e_H-x\otimes fe_H\rangle=0\quad\text{for } H\le I'. \qquad \square$$

DEFINITION 3.4 [1]. Let $\Omega\subset\mathbf{C}^n$ be a connected open subset. Two holomorphic curves $f,\tilde f:\Omega\to\mathrm{Gr}(n,\mathfrak{M})$ have order of contact I if for each λ_0 in Ω there exists a unitary U on $\mathfrak{M}$ such that Uf and $\tilde f$ agree to order I at λ_0. That is, if $\gamma_1,\dots,\gamma_k$ are holomorphic spanning cross-sections for E_f at λ_0, then there exist holomorphic spanning cross-sections $\tilde\gamma_1,\dots,\tilde\gamma_K$ for $\tilde E_f$ at λ_0 such that

$$U\gamma_i^{I'}(\lambda_0)=\tilde\gamma_i^{I'}(\lambda_0),\quad I'\le I,\ i=1,2,\dots,k.$$

LEMMA 3.5 [1]. *If* $f:\Omega\to \mathrm{Gr}(n,\mathfrak{M})$ *is a holomorphic curve and* $\gamma_1,\gamma_2,\dots,\gamma_k$ *are holomorphic cross-sections of the vector bundle* E_f *defined over* Ω *such that* $\gamma_1(\lambda_0),\dots,\gamma_k(\lambda_0)$ *is an orthonormal basis for* $f(\lambda_0)$, *then there exist holomorphic cross-sections* $\hat\gamma_1,\dots,\hat\gamma_k$ *of* E_f *defined on some open set* Δ *about* λ_0 *such that* $\hat\gamma_i(\lambda_0)=\gamma_i(\lambda_0)$ *for* $i=1,\dots,k$ *and*

$$\langle\hat\gamma_i^I(\lambda_0),\hat\gamma_j(\lambda_0)\rangle=0 \quad \textit{for } 1\le i,\ j\le k, \textit{ and } I\ne 0.$$

The proof given for Lemma 2.4 in [1] goes through without change.

PROPOSITION 3.6. *Suppose* $\mathfrak{M}$ *and* $\tilde{\mathfrak{M}}$ *are Hilbert modules over* $A(\Omega)$ *such that both are locally free at* $\bar\lambda_0$ *in* Ω. *If we set* $t(\lambda)=\mathfrak{M}_\lambda$ *and* $\tilde t(\lambda)=\tilde{\mathfrak{M}}_\lambda$, *then* $t(\lambda)$ *and* $\tilde t(\lambda)$ *have contact of order* I *if and only if* $\mathfrak{M}_\lambda^{I'}$ *and* $\tilde{\mathfrak{M}}_\lambda^{I'}$ *are unitarily module equivalent at* λ_0, *for any* $I'\le I$.

Proof. Assume that t and $\tilde t$ have contact of order I at λ_0. Then there exist a unitary U on $\mathfrak{M}$, holomorphic spanning cross-sections $\gamma_1,\dots,\gamma_k$ for $\mathfrak{M}_{\lambda_0}$, and holomorphic spanning cross-sections $\tilde\gamma_1,\dots,\tilde\gamma_k$ for $\tilde{\mathfrak{M}}_{\lambda_0}$ such that

$$U\gamma_i^{I'}(\lambda_0)=\tilde\gamma_i^{I'}(\lambda_0),\quad i=1,2,\dots,k,\ I'\le I.$$

For f in $A(\Omega)$, by (3.1) we have

$$\begin{aligned} Uf^*\gamma_i^{I'}(\lambda_0)&=U\sum_{J+K=I'}\frac{I'!}{J!K!}\overline{f^J(\bar\lambda_0)}\gamma_i^K(\lambda_0)\\ &=f^*U\gamma_i^{I'}(\lambda_0);\end{aligned}$$

hence $Uf^*|_{\mathfrak{M}_{\lambda_0}^I}=f^*U|_{\mathfrak{M}_{\lambda_0}^I}$. Let $P_{I\lambda_0}$ ($\tilde P_{I\lambda_0}$) denote the projection from $\mathfrak{M}$ to $\mathfrak{M}_{\lambda_0}^I$ (from $\tilde{\mathfrak{M}}$ to $\tilde{\mathfrak{M}}_{\lambda_0}^I$). We have $UP_{I\lambda_0}f\cdot h=\tilde P_{I\lambda_0}fUh$, since

$$\langle UP_{I\lambda_0}f\cdot\gamma_i^{I'}(\lambda_0),\tilde\gamma_j^{I''}(\lambda_0)\rangle=\langle\tilde P_{I,\lambda_0}fU\gamma_i^{I'},\tilde\gamma_j^{I''}(\lambda_0)\rangle$$

for $I',I''\le I$ and $1\le i,\ j\le k$. Therefore, for any h in $\mathfrak{M}_{\lambda_0}^{I'}$,

$$Uf\cdot h=fUh.$$

This shows that the unitary U defines a module mapping between $\mathfrak{M}_{\lambda_0}^{I'}$ and $\tilde{\mathfrak{M}}_{\lambda_0}^{I'}$.

Conversely, if $\gamma_1(\lambda_0),\dots,\gamma_k(\lambda_0)$ is an orthonormal basis for $\mathfrak{M}_{\lambda_0}$, then $\tilde\gamma_i(\lambda_0)=U\gamma_i(\lambda_0)$, $i=1,2,\dots,k$, defines an orthonormal basis for $\tilde{\mathfrak{M}}_{\lambda_0}$. Using Lemma 3.5, we can choose holomorphic cross-sections $\gamma_1,\dots,\gamma_k$ for $\bigcup_{\lambda\in\Delta}\mathfrak{M}_\lambda$ and $\tilde\gamma_1,\dots,\tilde\gamma_k$ for $\bigcup_{\lambda\in\Delta}\tilde{\mathfrak{M}}_\lambda$ on some open set Δ containing λ_0 and satisfying

$$\langle\gamma_i^I(\lambda_0),\gamma_l(\lambda_0)\rangle=\langle\tilde\gamma_i^I(\lambda_0),\tilde\gamma_l(\lambda_0)\rangle=0$$

for $I\ne 0$, $1\le i$, $l\le k$. We claim that

$$U\gamma_i^{I'}(\lambda_0)=\tilde\gamma_i^{I'}(\lambda_0)\quad\text{for } i=1,2,\dots,k,\ I'\le I.$$

The above statement is valid for $I'=0$, and we assume it holds for $|I'|\le k_0$. Suppose $I'=(i_1,\dots,i_n)$, $|I'|=k_0$, and $I''=(i_1+1,\dots,i_n)$. Then for any f in $J_{\bar\lambda_0}$ we have

$$\begin{aligned} f^*\{U\gamma_i^{I''}(\lambda_0)-\tilde{\gamma}_i^{I''}(\lambda_0)\} &= Uf^*\gamma_i^{I''}(\lambda_0)-f^*\tilde{\gamma}_i^{I''}(\lambda_0) \\ &= U\Big(\sum_{K+J=I''}\frac{I''!}{K!\,J!}\overline{f^K(\bar{\lambda}_0)}\gamma_i^J(\lambda_0)\Big)-\Big(\sum_{K+J=I''}\frac{I''}{K!\,J!}\overline{f^K(\bar{\lambda}_0)}\tilde{\gamma}_i^J(\lambda_0)\Big) \\ &= 0, \end{aligned}$$

since $f(\bar{\lambda}_0)=0$. This means that $U\gamma_i^{I''}(\lambda_0)-\tilde{\gamma}_i^{I''}(\lambda_0)$ is in $\tilde{\mathfrak{M}}_{\lambda_0}$. But

$$(U\gamma_i^{I''}(\lambda_0)-\tilde{\gamma}_i^{I''}(\lambda_0),\tilde{\gamma}_j(\lambda_0))=0, \quad 1\le i,\ j\le k,$$

and hence

$$U\gamma_i^{I''}(\lambda_0)=\tilde{\gamma}_i^{I''}(\lambda_0). \qquad \square$$

The following proposition does not involve the language of Hilbert modules. Its proof is an extension of that of Proposition 2.18 in [1] to the several-variable case. We begin by recalling a definition from [3].

DEFINITION 3.7. Let E and $\tilde{E}$ be hermitian vector bundles over Ω, an open subset of $\mathbf{C}^k$, with metric-preserving connections D and $\tilde{D}$. Let j be a positive integer. Then E and $\tilde{E}$ are equivalent to order j at a point z in Ω if there exists an isometry Φ_λ from E_λ to $\tilde{E}_\lambda$ such that

$$\Phi_\lambda\circ\chi\,|_\lambda=\tilde{\chi}\,|_\lambda\circ\Phi_\lambda$$

for each covariant derivative χ and $\tilde{\chi}$ of the curvature of E and $\tilde{E}$ (respectively) of total order less than or equal to j.

PROPOSITION 3.8. *Let f and $\tilde{f}$ be holomorphic maps from Ω into* $\mathrm{Gr}(n,\mathfrak{M})$, *and let E_f and $E_{\tilde{f}}$ be the associated hermitian holomorphic vector bundles with canonical connections D and $\tilde{D}$, respectively. The holomorphic curves f and $\tilde{f}$ have contact of order I at λ_0 for all I if and only if E_f and $E_{\tilde{f}}$ are equivalent to order j at λ_0, $0\le j<+\infty$.*

EQUIVALENCE THEOREM [3]. *Let E and $\tilde{E}$ be n-dimensional hermitian vector bundles over Ω, an open subset of $\mathbf{C}^k$, with metric-preserving connections D and $\tilde{D}$ which are equivalent to order n on Ω. Then, on an open dense subset of Ω, the bundles are locally equivalent.*

Now we give the main result of this section. Using the above propositions and theorems, we obtain the following theorem.

THEOREM 3.9. *Suppose that $\mathfrak{M}$ and $\tilde{\mathfrak{M}}$ are Hilbert modules over $A(\Omega)$, and that $\bigcup_{\lambda\in\Omega_0}\mathfrak{M}_\lambda$ and $\bigcup_{\lambda\in\Omega_0}\tilde{\mathfrak{M}}_\lambda$ are locally free on $\Omega_0\subset\Omega^*$. Then, if for any λ_0 in Ω_0, $\mathfrak{M}^I_{\lambda_0}\cong\tilde{\mathfrak{M}}^I_{\lambda_0}$ as Hilbert modules for all jets I, then there exists a dense open subset $\Omega_{00}\subset\Omega_0$ such that $\bigcup_{\lambda\in\Omega_{00}}\mathfrak{M}_\lambda$ and $\bigcup_{\lambda\in\Omega_{00}}\tilde{\mathfrak{M}}_\lambda$ are locally equivalent as hermitian holomorphic vector bundles.*

LEMMA 3.10. *Suppose $\mathcal{M}$ is a Hilbert module over $A(\Omega)$ and $\mathcal{M}$ is locally free on an open subset $\Omega_0 \subset \Omega^*$. If $\bigvee_{\lambda \in \Omega_0} \mathcal{M}_\lambda = \mathcal{M}$, then for any open subset $\Omega_{00} \subset \Omega_0$,*

$$\bigvee_{\lambda \in \Omega_{00}} \mathcal{M}_\lambda = \mathcal{M}.$$

The proof is just like the proof of Corollary 1.13 in [1].

COROLLARY 3.11. *Suppose the conditions of Theorem 3.9 hold for Hilbert modules $\mathcal{M}$ and $\tilde{\mathcal{M}}$. Furthermore, assume that*

$$\bigvee_{\lambda \in \Omega_0} \mathcal{M}_\lambda = \mathcal{M} \quad \text{and} \quad \bigvee_{\lambda \in \Omega_0} \tilde{\mathcal{M}}_\lambda = \tilde{\mathcal{M}}.$$

If $\mathcal{M}^I_{\lambda_0} \cong \tilde{\mathcal{M}}^I_{\lambda_0}$ as Hilbert modules for any λ_0 in Ω_0 and for all jets I, then $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are unitarily module equivalent.

Just as in the geometrical context, one does not need local unitary equivalence for jets of all orders. A careful examination of the arguments above yields the following refinement.

THEOREM 3.12. *Suppose $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are Hilbert modules over $A(\Omega)$ such that $\bigcup_{\Omega_0} \mathcal{M}_\lambda$ and $\bigcup_{\Omega_0} \tilde{\mathcal{M}}_\lambda$ are locally free on some connected open set $\Omega_0 \subset \Omega^*$ which satisfies $\bigvee_{\lambda \in \Omega_0} \mathcal{M}_\lambda = \mathcal{M}$ and $\bigvee_{\lambda \in \Omega_0} \tilde{\mathcal{M}}_\lambda = \tilde{\mathcal{M}}$. Let s be the dimension of $\mathcal{M}_\lambda$ for λ in Ω_0. If $\mathcal{M}^I_{\lambda_0} \cong \tilde{\mathcal{M}}^I_{\lambda_0}$ as Hilbert modules for λ_0 in Ω_0 and all jets I, $|I| \leq s$, then $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are unitarily module equivalent.*

One advantage to this coordinate-free version of the Cowen–Douglas theory is that it can be extended to modules that are not locally free. In particular, for a quotient module $H^2(\mathbf{D}^n)/[I]$, the spectrum coincides with the zero variety $Z(I)$, to which the approach presented in this paper can be applied. This will be considered in future work.

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Xiaoman Chen
Department of Mathematics
Fudan University
200433 Shanghai
People's Republic of China

R. G. Douglas
Department of Mathematics
SUNY at Stony Brook
Stony Brook, NY 11794