Plurisubharmonic Extremal Functions and Complex Foliations for the Complement of Convex Sets in $\mathbf{R}^n$

MIROSŁAW BARAN

In this paper we prove some properties of Siciak's extremal function Φ_E in the case of compact subsets of $\mathbf{R}^n$. In particular, we establish an interesting inequality for extremal functions of convex sets and present some corollaries that follow from this result. Moreover, we obtain effective formulas for the extremal function in a few interesting cases of convex symmetric sets and in the case of special nonsymmetric convex polyhedra. Finally, we present an effective continuous complex foliation of the domain $\mathbf{C}^n \setminus E$ (in the cases when we have explicit representation of the extremal function) by the leaves on which the plurisubharmonic extremal function u_E is harmonic.

1. Introduction and Statement of the Main Results

Let E be a compact set in $\mathbf{C}^n$. By $\Phi_E(z)$ ($\Phi(z,E)$) we denote Siciak's extremal function defined as follows:

$$(1.1)\qquad \Phi_E(z)=\sup\{|p(z)|^{1/\deg p}: p\in \mathbf{C}[w],\ \deg p\geq 1,\ \|p\|_E\leq 1\}$$

for $z\in\mathbf{C}^n$, where $\|p\|_E$ denotes the Čebyshev uniform norm $\|p\|_E=\sup|p|(E)$. For definition and applications of the extremal function we refer to Siciak's papers ([12], [13], [14]) and especially to Pawłucki and Pleśniak's papers ([9], [10]). The basic property of the extremal function just defined is contained in the following Zakharyuta–Siciak theorem (see [15] and [13]).

1.2. THEOREM. *If E is a compact subset of $\mathbf{C}^n$ then*

$$\Phi_E(z)=\exp u_E(z)\quad \text{for } z\in\mathbf{C}^n,$$

where $u_E(z)=\sup\{u(z): u\in\mathcal{L}_n,\ u|_E\leq 0\}$ and $\mathcal{L}_n$ is the Lelong class of plurisubharmonic functions in $\mathbf{C}^n$ (briefly, PSH($\mathbf{C}^n$)*) with logarithmic growth: $u(z)\leq \text{const}+\log(1+|z|)$, $z\in\mathbf{C}^n$.*

In this paper we consider the case when E is a compact set in $\mathbf{R}^n$. (Here we treat $\mathbf{R}^n$ as the subset of $\mathbf{C}^n$ such that $\mathbf{C}^n=\mathbf{R}^n+i\mathbf{R}^n$). Let us denote by g the Joukowski transformation: $g(z)=\frac{1}{2}(z+\frac{1}{z})$ for $z\in\mathbf{C}\setminus\{0\}$. Let $h:\mathbf{C}\setminus[-1,1]\to\mathbf{C}$

Received May 29, 1990
Michigan Math. J. 39 (1992).

be its inverse; $h(z)=z+(z^2-1)^{1/2}$ with an appropriate branch of the root function.

The aim of this paper is to prove the following results.

1.3. THEOREM. *For a compact set* $E\subset \mathbf{R}^n$,

$$\Phi_E(z)=\sup\{|h(p(z))|^{1/\deg p}: p\in \mathbf{R}[w],\ \deg p\geq 1,\ \|p\|_E\leq 1\}.$$

1.4. THEOREM. *There exists a sequence of polynomials* $p_k\in \mathbf{R}[w]$ *such that* $\deg p_k\leq \deg p_{k+1}$ *and*

$$\Phi_E(z)=\sup_{k\geq 1}|h(p(z))|^{1/\deg p}=\overline{\lim_{k\to\infty}}|h(p(z))|^{1/\deg p}.$$

In the special case of compact, convex subsets of $\mathbf{R}^n$, the following inequality holds.

1.5. THEOREM. *If* E *is a compact, convex subset of* $\mathbf{R}^n$ *with* $0\in \operatorname{int}(E)$ *and* E^* *is the convex dual set to* E, *then*

$$\Phi_E(z)\leq \inf_{d\in\operatorname{int}(E)}\ \sup_{w\in K}\left|h\left(\frac{(z-d)\cdot w}{1-|d\cdot w+\beta|}\right)\right| \quad \text{for } z\in \mathbf{C}^n,$$

where

$$K=\frac{2}{1+|\alpha|}\operatorname{extr}(E^*),\quad \alpha=\inf\{x\cdot y: x\in E, y\in E^*\},\quad \beta=-\frac{1+\alpha}{1+|\alpha|}.$$

Here $E^*=\{x\in \mathbf{R}^n: x\cdot y\leq 1 \text{ for every } y\in E\}$.

In addition, we obtain the complex foliation of $\mathbf{C}^n\setminus E$, such that u_E is harmonic for each leaf, in the case of convex, symmetric compact sets in $\mathbf{R}^n$ and in some special nonsymmetric convex cases; this generalizes Lundin's result [7].

2. The Joukowski Function, Its Inverse, and the Proof of Results for Siciak's Extremal Function

In our considerations the crucial role is played by the following holomorphic function, called the Joukowski function:

$$g(z)=\tfrac{1}{2}(z+\tfrac{1}{z}),\quad z\in \mathbf{C}\setminus\{0\}.$$

This function is univalent on $|z|>1$ and on $0<|z|<1$, $g(\{|z|>1\})=\mathbf{C}\setminus[-1,1]$, and the inverse function $h=g^{-1}:\mathbf{C}\setminus[-1,1]\to\mathbf{C}\setminus\bar{B}$ (where $\bar{B}$ denotes the unit disk in $\mathbf{C}$) has the form $h(z)=z+(z^2-1)^{1/2}$ if we choose an appropriate branch of the square root. Note that g is a solution of the equation

$$|g(z)+1|+|g(z)-1|=2g(|z|),\quad z\neq 0. \tag{2.1}$$

On the other hand, equation (2.1) characterizes the Joukowski function in the following sense (see [2]): If $f:\mathbf{C}\setminus\{0\}\to\mathbf{C}$ is a holomorphic function satisfying (2.1), then there exist $a>0$ and p in $\mathbf{N}$ such that

$$f(z) = g((az)^p) \quad \text{for each } z \neq 0.$$

Following (2.1), we obtain the following most important formula for the function h:

(2.2) $$|h(z)| = h(\tfrac{1}{2}|z+1| + \tfrac{1}{2}|z-1|) \quad \text{for } z \in \mathbf{C},$$

where on the right side we have $h(t) = t + (t^2-1)^{1/2}$ $(t \geq 1)$ with the usual arithmetic root. Other properties of functions g and h are contained in the following proposition; its straightforward proof will be omitted.

2.3. PROPOSITION.

(1) $\bar{z}g(z) - z\overline{g(z)} = \frac{1}{2}(|z|^2 - |z|^{-2})$ *for* $z \neq 0$;
(2) $|h(z)| \leq r$, $r > 1$, *if and only if* $|rz - r^{-1}\bar{z}| \leq \frac{1}{2}(r^2 - r^{-2})$;
(3) $|h(z)| = r \geq 1$ *if and only if* $|z+1| + |z-1| = 2g(r)$.

Note also the interesting connection between the Joukowski function g, its inverse h, and the Čebyshev polynomials $T_n(x) = \cos(n \arccos x)$, $x \in [-1, 1]$:

(2.4) $$T_n(z) = g(h^n(z)).$$

Observe that Theorem 1.3 is a generalization of (2.4).

2.5. PROOF OF THEOREM 1.3. Denote the right side in Theorem 1.3 by $\varphi(z)$. The inequality $\geq$ is a simple consequence of Theorem 1.2. The opposite inequality holds true by the following facts:

(2.6) $|h(z)| = \Phi_{[-1,1]}(z) \geq \Phi_{\bar{B}}(z) = \max(1, |z|)$, $z \in \mathbf{C}$, and
(2.7) if $p \in \mathbf{C}[w]$, $\|p\|_E \leq 1$, then there exist $p_1, p_2 \in \mathbf{R}[w]$ with $\|p_1\|_E, \|p_2\|_E \leq 1$ and $p = p_1 + ip_2$.

By (2.7) we get $|p(z)|^{1/\deg p} \leq (\sqrt{2})^{1/\deg p} \varphi(z)$. Using this inequality for the polynomial p^k completes the proof of theorem. □

2.8. COROLLARY. *If E is a compact subset of* $\mathbf{R}^n$, *then*

$$\Phi_E(z) = \Phi_E(\bar{z}) \quad \textit{for } z \in \mathbf{C}^n.$$

2.9. PROOF OF THEOREM 1.4. By Proposition 4.11 from [13], we have $\Phi_E(z) = \lim_{k\to\infty}(\Phi_k(z))^{1/k}$, where $\Phi_k(z) = \max_{1 \leq j \leq m_k} |L^{(j)}(z, \zeta^{(m_k)})|$. Here $\zeta^{(m_k)} = \{z_1, \ldots, \zeta_{m_k}\}$ is the kth system of extremal points of the rank $m_k = \binom{k+n}{k}$ and $L^{(j)}(z, \zeta^{(m_k)})$ is the system of Lagrange interpolation polynomials with nodes ζ_j. Every such polynomial has degree $\leq k$ and Čebyshev norm on E not greater than 1. Hence

$$(\Phi_k(z))^{1/k} \leq \max_{1 \leq j \leq m_k} |L^{(j)}(z, \zeta^{(m_k)})|^{1/\deg L^{(j)}} \leq \Phi_E(z).$$

Let l_k be the smallest common multiple of the numbers $1, \ldots, k$, and let $q_{k,j}(z) = L^{(j)}(z, \zeta^{(m_k)})^{l_k/\deg L^{(j)}}$. Note that $\deg q_{k,j} = l_k$ and

$$\Phi_E(z) = \lim_{k\to\infty} \max_{1 \leq j \leq m_k} |q_{k,j}(z)|^{1/l_k}.$$

If we order these polynomials with respect to degree we obtain the sequence required in Theorem 1.4. □

2.10. COROLLARY. *Let* (α_k) *be a sequence of real numbers such that* $0<\alpha_k\le 1$ *and* $\lim_{k\to\infty}\sqrt[k]{\alpha_k}=1$. *Then*

$$\begin{aligned}\Phi_E(z)&=\sup_{k\ge 1}\sup\{|p(z)|^{1/\deg p}: p\in \mathbf{R}[w],\ \deg p=k,\ \|p\|_E\le\alpha_k\}\\&=\sup_{k\ge 1}\sup\{|h(p(z))|^{1/\deg p}: p\in \mathbf{R}[w],\ \deg p=k,\ \|p\|_E\le\alpha_k\}.\end{aligned}$$

2.11. COROLLARY. *If* $x_0\in E$ *then*

$$\Phi_E(z)=\sup\{|h(p(z))|^{1/\deg p}: p\in\mathbf{R}[w],\ \deg p\ge 1,\ \|p\|_E\le 1,\ p(x_0)=0\}.$$

Proof. Observe that if p is a real polynomial with $\|p\|_E\le\frac14$ and $q(z)=p(z)-p(x_0)$, then

$$\begin{aligned}|h(p(z))|&=h(\tfrac12|p(z)+1|+\tfrac12|p(z)-1|)\\&\le h(\tfrac12(\tfrac12(|2q(z)+1|+|2q(z)-1|)+1))\\&\le h(\tfrac12(|2q(z)+1|+|2q(z)-1|))=|h(2q(z))|,\end{aligned}$$

which completes the proof. □

2.12. COROLLARY. *If* $E=-E$ *then*

$$\begin{aligned}\Phi_E(z)=\sup\{&|h(p(z))|^{1/\deg p}: p\in\mathbf{R}[w],\ \deg p\ge 1,\ \|p\|_E\le 1,\\&p(z)=p(-z)\ \textit{for each } z \textit{ or } p(-z)=-p(z)\ \textit{for every } z\}.\end{aligned}$$

Proof. Let $p\in\mathbf{R}[w]$, $\|p\|_E\le\frac14$. We have

$$p(z)=\tfrac12(p(z)+p(-z))+\tfrac12(p(z)-p(-z))=q(z)+r(z),$$
$$q(z)=q(-z),\quad r(-z)=-r(z).$$

The next steps are similar to those in the earlier corollary. □

2.13. PROOF OF THEOREM 1.5. In the proof, the crucial role is played by the following.

2.14. LEMMA [1]. *Let* E *be a compact subset of* $\mathbf{C}^n$ *and let* $f:\bar D\to\mathbf{C}^n$ *be a continuous mapping, holomorphic in the domain* $D\subset\mathbf{C}$. *Let a continuous function* $\varphi: f(\bar D)\to[1,+\infty)$ *satisfy the following conditions:*

(i) $\varphi|_{f(\partial D)}=1$;
(ii) $\log(\varphi\circ f)$ *is harmonic in* D;
(iii) $|z|\le M\varphi(z)$ *for* $z\in f(\bar D)$ *with some constant* M.

Then $\Phi_E(z)\le\varphi(z)$ *for* $z\in f(D)$.

Now, let E be a compact, convex subset of $\mathbf{R}^n$ with $0\in\operatorname{int}(E)$. Define K, α, and β as before. Then we can write $E=\{z\in\mathbf{C}^n: z\cdot w+\beta\in[-1,1]$ for each $w \in K\}$. Fix $z \in \mathbf{C}^n \backslash E$, $d \in \operatorname{int}(E)$, and $c \in \mathbf{C}^n \backslash \{0\}$. Define $f(\zeta) = \frac{1}{2}(\zeta c + \zeta^{-1}\bar{c}) + d$ for $\zeta \in \bar{D}$, where $D = \mathbf{C} \backslash \bar{B}$. Observe that the condition

$$(*) \qquad |c \cdot w| + |d \cdot w + \beta| \leq 1 \quad \text{for every } w \in K$$

implies that $f(\partial D) \subset E$. Indeed, $f(e^{i\theta}) \in E$ if and only if

$$|\operatorname{Re}(e^{i\theta}(c \cdot w)) + d \cdot w + \beta| \leq 1 \quad \text{for every } w \in K,$$

which is equivalent to (*). It is easy to verify that we can apply Lemma 2.14 to the function $\varphi(z) = |\zeta|$ if $z = f(\zeta)$ ($\zeta \in \bar{D}$), and we obtain the inequality $\Phi_E(f(\zeta)) \leq |\zeta|$ for $\zeta \in D$. Choose $r > 1$ and $c \in \mathbf{C}^n \backslash \{0\}$ such that $z = f(r)$ and $f(\partial D) \subset E$. The first assumption implies that

$$c = 2(r^2 - r^{-2})^{-1}(r(z-d) - r^{-1}(\bar{z} - d)).$$

Hence, due to the second condition and (*), we have

$$\left| h\left(\frac{(z-d) \cdot w}{1 - |d \cdot w + \beta|} \right) \right| \leq r \quad \text{for every } w \in K.$$

Set

$$r = \max\left\{ \left| h\left(\frac{(z-d) \cdot w}{1 - |d \cdot w + \beta|} \right) \right| : w \in K \right\}.$$

We have

$$\Phi_E(z) \leq \max\left\{ \left| h\left(\frac{(z-d) \cdot w}{1 - |d \cdot w + \beta|} \right) \right| : w \in K \right\},$$

and—because $d \in \operatorname{int}(E)$ was arbitrary—Theorem 1.5 is proved. □

3. Some Applications of Theorem 1.5

If E is a compact, convex and symmetric subset of $\mathbf{R}^n$ with nonempty interior, then, due to Theorem 1.5, we easily obtain

$$(3.1) \qquad \Phi_E(z) = \sup\{|h(z \cdot w)| : w \in \operatorname{extr}(E^*)\}$$

for $z \in \mathbf{C}^n$ (see [1]; see also [7] and [4], where the result is less precise).

3.2. REMARK. A stronger theorem was proved in [1] under the following assumption: If $\Phi_E(z_0) = |h(z_0 \cdot w_0)|$ with some $w_0 \in \operatorname{extr}(E^*)$, then $\Phi_E(f(\zeta)) = |\zeta|$ for $|\zeta| \geq 1$, where $f(\zeta) = \frac{1}{2}(\zeta c + \zeta^{-1}\bar{c})$ and the vector c is given by the conditions $z_0 = f(\zeta_0)$, $\zeta_0 = h(z_0 \cdot w_0)$. We will apply this result in the last section of this paper.

3.3. PROPOSITION. *If $E_1, \ldots, E_N$ are compact, convex and symmetric subsets of $\mathbf{R}^n$ with nonempty interior, and if $E = \bigcap_{k=1}^N E_k$, then*

$$\Phi_E = \max_{1 \leq k \leq N} \Phi_{E_k}.$$

The proof easily follows from (3.1).

The next proposition is also simple, but less trivial.

3.4. PROPOSITION. *If a set E satisfies the hypotheses of Theorem 1.5, then there exists a constant $M>0$ such that $\Phi_E(z)\le h(1+M\operatorname{dist}(z,E))$ for each $z\in\mathbf{C}^n$. In every case we can take $M=1/\operatorname{dist}(\frac{1}{2}E,\partial E)$, and if a set E is symmetric then we may take $M=2/\delta(E)$, where $\delta(E)$ denotes the diameter of the set E.*

Proof. First, observe that there exists a compact set $E_0\subset\operatorname{int}(E)$ such that the following condition is fulfilled:

(*) for every $a\in E$ there exists $d\in E_0$ such that for each $w\in K$ the following inequality holds:

$$|(a-d)\cdot w|+|d\cdot w+\beta|\le 1.$$

Indeed, fix $b\in\operatorname{int}(E)$ and define $E_0=\frac{1}{2}E+\frac{1}{2}b$. It is obvious that $E_0\subset\operatorname{int}(E)$ and the set E_0 is compact. For $a\in E$ put $d=\frac{1}{2}(a+b)\in E_0$ and let $\alpha_w=a\cdot w+\beta$, $\gamma_w=b\cdot w+\beta$ for $w\in K$. Then

$$|(a-d)\cdot w|+|d\cdot w+\beta|=\tfrac{1}{2}|\alpha_w-\gamma_w|+\tfrac{1}{2}|\alpha_w+\gamma_w|=\max(|\alpha_w|,|\gamma_w|)\le 1.$$

The set E_0 may be also defined in the following manner. For $a\in E$ let $\delta(a)=\sup\{\operatorname{dist}(\frac{1}{2}(a+b),\partial E): b\in E\}$. There exists $b\in E$ such that $\delta(a)=\operatorname{dist}(\frac{1}{2}(a+b),\partial E)$. Put $d(a)=\frac{1}{2}(a+b)$, $F_0=\{d(a): a\in E\}$, and $E_0=\bar{F}_0$. The set E_0 then has the property required in (*). Moreover, if E is symmetric then we obtain $E_0=\{0\}$. Now observe that, if S_w denotes the strip $S_w=\{x\in\mathbf{R}^n: x\cdot w+\beta\in[-1,1]\}$, then $1-|d\cdot w+\beta|=|w|\operatorname{dist}(d,\partial S_w)$. Fix $z\in\mathbf{C}^n$. For $a\in E$ choose $d\in E_0$ such that condition (*) holds. Then

$$\begin{aligned}
&\frac{1}{2}\left(\left|\frac{(z-d)\cdot w}{1-|d\cdot w+\beta|}+1\right|+\left|\frac{(z-d)\cdot w}{1-|d\cdot w+\beta|}-1\right|\right)\\
&\quad\le\frac{|z-a|}{\operatorname{dist}(d,\partial S_w)}+\frac{1}{2}\left(\left|\frac{(a-d)\cdot w}{1-|d\cdot w+\beta|}+1\right|+\left|\frac{(a-d)\cdot w}{1-|d\cdot w+\beta|}-1\right|\right)\\
&\quad=\frac{|z-a|}{\operatorname{dist}(d,\partial S_w)}+1\le 1+M|z-a|,
\end{aligned}$$

where $M=\max\{1/\operatorname{dist}(d,\partial E): d\in E_0\}$. By Theorem 1.5 we obtain

$$\begin{aligned}
\Phi_E(z)&\le h\left(\max_{w\in K}\frac{1}{2}\left(\left|\frac{(z-d)\cdot w}{1-|d\cdot w+\beta|}+1\right|+\left|\frac{(z-d)\cdot w}{1-|d\cdot w+\beta|}-1\right|\right)\right)\\
&\le h(1+M|z-a|).
\end{aligned}$$

Hence $\Phi_E(z)\le h(1+M\operatorname{dist}(z,E))$, which completes the proof. □

3.5. REMARK. From the above Proposition 3.4 it immediately follows that there exists a constant M_1 such that

$$\Phi_E(z)\le 1+M_1\delta^{1/2}\quad\text{for }\operatorname{dist}(z,E)\le\delta\le 1.$$

This implies the well-known fact that E satisfies the HCP condition (see [9]) with a constant $\mu = \frac{1}{2}$.

3.6. REMARK. If E is a compact, convex subset of $\mathbf{R}^n$ with nonempty interior, then for every $b \in \operatorname{int}(E)$, $0 \in \operatorname{int}(E-b)$. It is easily seen that $M = \inf\{\operatorname{dist}(\frac{1}{2}(E+b), \partial E) : b \in \operatorname{int}(E)\}$.

At the end of this section we will give some effective formulas for Siciak's extremal function. For details the reader is referred to [1].

3.7. EXAMPLE. Let E be a convex symmetric polyhedron that has the following representation:

$$E = \bigcap_{k=1}^{m} \{x \in \mathbf{R}^n : -\alpha_k \le x_1 \beta_1^{(k)} + \cdots + x_n \beta_n^{(k)} \le \alpha_k\},$$

where $\alpha_k > 0$ and $\operatorname{lin}(\beta^{(1)}, \dots, \beta^{(m)}) = \mathbf{R}^n$. We have $\operatorname{extr}(E^*) \subset A \cup (-A)$, where $A = \{(1/\alpha_1)\beta^{(1)}, \dots, (1/\alpha_m)\beta^{(m)}\}$. By 3.1 we get

$$\Phi_E(z) = h\left(\max_{1 \le k \le m} \frac{1}{2} \left(\left| \frac{z \cdot \beta^{(k)}}{\alpha_k} + 1 \right| + \left| \frac{z \cdot \beta^{(k)}}{\alpha_k} - 1 \right| \right) \right) \quad \text{for } z \in \mathbf{C}^n.$$

3.8. EXAMPLE (Lundin [8]; short proof, due to 3.2, is contained in [1]). Let B_n be the unit Euclidian ball in $\mathbf{R}^n$. Then

$$\Phi_{\bar{B}_n}(z) = (h(|z|^2 + |z^2 - 1|))^{1/2}.$$

3.9. EXAMPLE. Let $N \ge n+1$, $y_1, \dots, y_N \in \mathbf{R}^n$, $\operatorname{lin}\{y_1, \dots, y_N) = \mathbf{R}^n$, and $y_k = \sum_{l=1}^{n} \alpha_{k,l} y_l$, where $\alpha_{k,l} = \delta_{k,l}$ for $1 \le k, l \le n$ and $\alpha_{k,l} > 0$ for $1 \le l \le n$, $n+1 \le k \le N$. For real numbers b_k such that $b_k = -1$ for $1 \le k \le n$ and $b_k \in [-1, 1]$ for $n+1 \le k \le N$, define: $\beta_k = \frac{1}{2}(1 - b_k)$. Consider the subset E of $\mathbf{R}^n$ which has the representation

$$E = \{z \in \mathbf{C}^n : 2z \cdot y_k + b_k \in [-1, 1] \text{ for } 1 \le k \le N\}.$$

Then we have

$$\Phi_E(z) = h\left(\max_{1 \le k \le N} \frac{1}{\beta_k} \left(\sum_{l=1}^{n} \alpha_{k,l} |z \cdot y_l| + |z \cdot y_k - \beta_k| \right) \right)$$

for $z \in \mathbf{C}^n$. The proof of the above formula is an immediate consequence of Proposition 3.3, Example 3.8, and Klimek's beautiful theorem [6], which we apply to the mapping $f(z) = (z_1^2, \dots, z_n^2)$. (For details we refer to [1], where the above formula is presented in an implicit form.) Note that in the case of the standard simplex S_n in $\mathbf{R}^n$, where $S_n = \{x \in \mathbf{R}^n : x_1, \dots, x_n \ge 0, x_1 + \cdots + x_n \le 1\}$, we get

$$\phi_{S_n}(z) = h(|z_1| + \cdots + |z_n| + |z_1 + \cdots + z_n - 1|).$$

4. Complex Foliation of the Complement of a Compact Subset of $\mathbf{R}^n$

In this section we will give two examples of the complex foliation of the open set $\mathbf{C}^n \setminus E$ by leaves such that on each leaf the extremal function u_E is harmonic. These examples are strictly connected with Remark 3.2 and Example 3.9. We start with the definition of a continuous complex foliation (see [3], [5]).

4.1. Let M be a complex manifold of the complex dimension n. A family $\mathfrak{F}$ of disjoint submanifolds L of constant degree p is said to be a *continuous foliation* of M if the following two conditions are satisfied:

(i) $\bigcup_{L\in\mathfrak{F}} L = M$;

(ii) for every point $z \in M$ there exists a neighborhood U and a continuous map $\varphi = (\varphi_1, \varphi_2)\colon U \to \mathbf{C}^p \times \mathbf{C}^{n-p}$ such that if $L \in \mathfrak{F}$ then each connected component of the set $L \cap U$ is given by the equation $\varphi_2 = \text{const}$.

Any element of the family $\mathfrak{F}$ is called a *leaf* of the foliation.

Let E be a compact, convex and symmetric subset of $\mathbf{R}^n$ with nonempty interior. As before, let $K = \overline{\text{extr}(E^*)}$. Define $S(E) = \{z \in \mathbf{C}^n : \max_{w\in K}|z\cdot w| = 1\}$. Then $S(E)$ is the unit sphere in $\mathbf{C}^n$ for the norm $\max_{w\in K}|z\cdot w|$. Let $J_n = \bigcup_{k=1}^n F_k$, where $F_k = \bar{B}^{k-1} \times \{1\} \times \bar{B}^{n-k}$. Next, let ω denote the homeomorphism from $S(I_n)$ to $S(E)$ given by $\omega(z) = z/\|z\|$ for $z \in S(I_n)$, where $I_n = [-1,1]^n$ and $\|z\| = \max_{w\in K}|z\cdot w|$. Let "$\simeq$" be the equivalence relation in $\mathbf{C}^n\setminus\{0\}$ which defines projective space $\mathbf{P}_n$; that is, $z \simeq w$ if and only if there exists $\tau \in \mathbf{C}$ with $|\tau| = 1$ such that $z = \tau w$. Then the compact and connected set J_n is a selector of the family $\pi(I_n)$ of abstract classes of the quotient space $\mathbf{P}_n$, where $\pi\colon \mathbf{C}^n\setminus\{0\} \to \mathbf{P}_n$ is the canonical surjection and $\pi|_{J_n}\colon J_n \to \mathbf{P}_n$ is a homeomorphism. Since the homeomorphism ω is compatible with the relation $\simeq$, the set $\omega(J_n)$ is a selector for $\pi(S(E))$. Moreover, $\omega(J_n)$ is a continuum, and $\pi|_{\omega(J_n)}\colon \omega(J_n) \to \mathbf{P}_n$ is also a homeomorphism.

Denote $\sigma(E) = \omega(J_n)$. For $c \in \sigma(E)$ define $f_c\colon D \to \mathbf{C}^n$ (here $D = \mathbf{C}\setminus\bar{B}$) by the formula: $f_c(\zeta) = \frac{1}{2}(\zeta c + \zeta^{-1}\bar{c})$ and put $L_c = f_c(D)$, $\mathfrak{F} = \{L_c\}_{c\in\sigma(E)}$. It is easily seen that for each $c \in \sigma(E)$, L_c is an analytic curve (complex manifold of dimension 1). Now we can formulate the main result.

4.2. THEOREM. *The family $\mathfrak{F}$ is a continuous complex foliation of the domain $\Omega = \mathbf{C}^n\setminus E$ such that on every leaf the extremal function u_E is harmonic.*

Proof. By (3.1) and Remark 3.2 it follows that family $\mathfrak{F}$ is a cover of Ω and on each curve L_c the function u_E harmonic (since $u_E(f_c(\zeta)) = \log|\zeta|$). Now we will prove that leaves L_c are disjoint. Assume that $z \in L_c \cap L_{c'}$. Then there exist $\zeta_1, \zeta_2 \in D$ such that $z = \frac{1}{2}(\zeta_1 c + \zeta_1^{-1}\bar{c}) = \frac{1}{2}(\zeta_2 c' + \zeta_2\bar{c}')$. Let w be a vector from K such that $|c\cdot w| = 1$. Set $\zeta = \zeta_1 c\cdot w/|c\cdot w|$, $\zeta' = \zeta_2 c'\cdot w/|c'\cdot w|$. According to (2.1) we obtain the equality

$$2g(|\zeta|) = \big||c'\cdot w|g(\zeta') + 1\big| + \big||c'\cdot w|g(\zeta') + 1\big|.$$

Since $|c'\cdot w| \le 1$, the right side in the above expression is not greater than $|g(\zeta') + 1| + |g(\zeta') - 1| = 2g(|\zeta'|)$. Hence $|\zeta| \le |\zeta'|$. By the same reasoning we obtain the opposite inequality. Thus $|\zeta| = |\zeta'|$. Without loss of generality we can assume that $\zeta = \zeta_1$, $\zeta_2 = \tau\zeta$, $|\tau| = 1$. Then we obtain $\zeta^2(c - \tau c') = -\overline{(c - \tau c')}$ and thus $c = \tau c'$. This means that $c \simeq c'$, which implies $c = c'$ and $L_c = L_{c'}$.

Now we will verify condition (ii). First, observe that the mapping χ: $D\times\sigma(E)\to\Omega$, $\chi(\zeta,c)=\frac{1}{2}(\zeta c+\zeta^{-1}\bar{c})$ is a homeomorphism. Indeed, χ is a continuous bijection and its inverse is given by the formula

$$\chi^{-1}(z)=(\Phi_E(z)c_1(z)\cdot\overline{c(z)}/|c(z)|^2, c(z)),$$

where

$$c_1(z)=2(\Phi_E(z)^2-\Phi_E(z)^{-2})(\Phi_E(z)z-\Phi_E(z)^{-1}\bar{z})$$

and $c(z)=(\pi|_{\sigma(E)})^{-1}(\pi(c_1(z))$. It is easily seen that χ^{-1} is continuous. Let a mapping $\psi: J_n\to\mathbf{C}^{n-1}$ be given as follows: If $z\in F_k$ then let $\psi(z)=e^{i\pi k/n}(z_1-1,\dots,z_{k-1}-1,z_{k+1}-1,\dots,z_n-1)$. It is easy to check that ψ is a homeomorphism onto its image. Now define a map $\varphi:\Omega\to\mathbf{C}\times\mathbf{C}^{n-1}$ by the formula $\varphi=(\mathrm{id}_{\mathbf{C}},\psi\circ\omega^{-1})\circ\chi^{-1}$. It is clear that the map φ is a homeomorphism onto its image, and it is easily seen that for every $z\in\Omega$ the condition (ii) is satisfied with $U=\Omega$. This completes the proof. □

4.3. REMARK. Lundin's paper [7] contains a remark concerning the possibility of a similar foliation in the case of a compact, convex and symmetric subset of $\mathbf{R}^n$ with smooth boundary. Our theorem gives the explicit formula of such a foliation without any assumption on the boundary of the set E.

4.4. REMARK. For every $c\in\mathbf{C}^n$, the following equality holds,

$$\max\{|c\cdot w|: w\in S^{n-1}\}=(\tfrac{1}{2}(|c|^2+|c^2|))^{1/2}.$$

Hence

$$S(\bar{B}_n)=\{c\in\mathbf{C}^n: (\tfrac{1}{2}(|c|^2+|c^2|))^{1/2}=1\}.$$

At the end we will present a similar construction of the foliation for the case of polyhedra whose extremal functions are given in Example 3.9. Let E be a fixed convex polyhedron from Example 3.9. As before, denote by $S(E)$ the unit sphere in $\mathbf{C}^n$ with an appropriate norm $S(E)=\{z\in\mathbf{C}^n: \|z\|=1\}$, where $\|z\|=\max_{1\le k\le N}(1/\beta_k)(\sum_{l=1}^n\alpha_{k,l}|z_l|+|\sum_{l=1}^n\alpha_{k,l}z_l|)$. Denote by ω the homeomorphism from $S(I_n)$ to $S(E)$ given (as before) by the formula $\omega(z)=z/\|z\|$, and let $\sigma(E)=\omega(I_n)$. Next, for a vector $c\in\sigma(E)$ let f_c denote the following mapping from D (D as before) to $\mathbf{C}^n$:

$$f_c(\zeta)=\varphi(\tfrac{1}{2}(\zeta c+\zeta^{-1}\bar{c})+(|c_1|,\dots,|c_n|)),$$

where φ is the linear automorphism of $\mathbf{C}^n$ defined by $\varphi(z)=z_1a_1+\cdots+z_na_n$ (the vectors a_k are given by the condition $y_k\cdot a_l=\delta_{k,l}$). Let $L_c=f_c(D)$, $\mathfrak{F}=\{L_c\}_{c\in\sigma(E)}$. Now we can formulate the second theorem.

4.5. THEOREM. *Let E be a convex polyhedron from Example 3.9. Then the family $\mathfrak{F}$ is a continuous complex foliation of the domain $\mathbf{C}^n\setminus E$ on analytic curves and on each of these curves the extremal function u_E is harmonic. Precisely, $u_E(f_c(\zeta))=\log|\zeta|$ for $\zeta\in D$.*

Proof. It is possible to prove this theorem in a way that is analogous to the proof of Theorem 4.2, by applying the identity

$$\tfrac{1}{2}(\zeta c_k+\zeta^{-1}\bar{c}_k)+|c_k|=\tfrac{1}{2}(\zeta^{1/2}c_k^{1/2}+\zeta^{-1/2}\overline{c_k^{1/2}})^2.$$

Therefore we omit the details.

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Mirosław Baran
Department of Mathematics
Pedagogical University of Cracow
Podchorążych 2
30-084 Cracow
Poland