A Nonpermutational Integral Relation Algebra

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1. Introduction

Tarski and Jónsson raised the question whether every integral representable relation algebra (RA) was representable over a group. McKenzie [MK1; MK2] answered this question in the negative. In fact, McKenzie introduced the notion of a permutational relation algebra, and he showed that every group representable RA is permutational, but that not every permutational RA is group representable. Moreover, he showed that the class $\mathcal{G}$ of all group representable RAs is not finitely axiomatizable relative to the class $\mathcal{P}$ of all permutational RAs. He then raised the problem of whether a nonpermutational representable integral relation algebra exists. In this paper, we answer this question in the affirmative. We also prove that the class of all permutational relation algebras is not finitely axiomatizable over the class of representable integral relation algebras.

1.1. NOTATION AND DEFINITIONS. A relation algebra

$$\mathfrak{A}=(A,+,\cdot,-,;,{}^{-1},0,1,1')$$

is a structure of type $(2,2,1,2,1,0,0,0)$, where

R1 $(A,+,\cdot,-,0,1)$ is a Boolean algebra;
R2 $(A,;,{}^{-1},1')$ is an involuted monoid; and
R3 for all $a,b,c\in A$, the conditions

$$(a;b)\cdot c=0,\quad (a^{-1};c)\cdot b=0,\quad (c;b^{-1})\cdot a=0$$

are equivalent.

For history and context of the theory of relation algebras, the reader is invited to consult [TG] or [J].

For a nonempty set U, we set $V=U\times U$ and consider the following operations on $\mathfrak{P}(V)$, the power set of V:

(1) the Boolean operations $\cup$, $\cap$, $-$, together with the constants $\emptyset$ and $U\times U$.

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(2) The relative operations | (relative multiplication) and $^{\vee}$ (conversion), as well as the constant $E = \mathrm{id}\,U$.

Any subset of $\mathfrak{P}(V)$ which is closed under these operations and contains the constants is an RA, if the operations and constants are interpreted naturally. Relation algebras of this kind are called *proper on U.* An RA is called a *representable relation algebra* (RRA) or *relation set algebra* if it is isomorphic to a subdirect product of RAs each of which is proper on some nonempty set.

An RA $\mathfrak{A}$ is called *integral* if for all $R, S \in A$, $R\,;S = 0$ implies $R = 0$ or $S = 0$. This is equivalent to the fact that $1'$ is an atom of $\mathfrak{A}$. It is well known that an integral representable RA is isomorphic to a proper RA on some set U; thus we can speak of an integral RRA as being *representable on a set U,* or just *integral on U.*

A permutation group G on a set U is called

(1) *transitive,* if for all $x, y \in U$ there exists some $\varphi \in G$ such that $y = \varphi x$;
(2) *semiregular,* if the identity on U is the only element of G that fixes a point, or, equivalently, if for all $\varphi, \psi \in G$ the fact that $\varphi x = \psi x$ for some $x \in U$ implies $\varphi = \psi$;
(3) *regular,* if G is semiregular and transitive.

An *orbit* of G is a set of the form $\{\varphi x : \varphi \in G\}$, where $x \in U$, and the *degree* of G is the cardinality of $\{x \in U : \varphi x \neq x$ for some $\varphi \in G\}$. If G is transitive then G has only one orbit, namely U, and its degree is just $|U|$. With some abuse of notation we speak of an *orbit of a permutation* φ when we mean a set containing all elements of a cycle of φ. A subset M of U is called a *block* of G, if for each $\varphi \in G$ we have either $\varphi[M] = M$ or $\varphi[M] \cap M = \emptyset$. If M is a block of G then $\varphi[M]$ is a block of G for every $\varphi \in G$, and two such blocks are called *conjugate.* The system of blocks conjugate to a given block is called a *complete block system.* If G is transitive then a complete block system partitions U. For other group-related notions not defined here, the reader is referred to [W].

For a positive integer n, K_n is the complete graph on n vertices; if n is a prime power then $GF(n)$ is the finite field with n elements. Finally, ω denotes the first infinite ordinal.

1.2. BACKGROUND. Let us denote the class of all integral RRAs by $\mathfrak{I}$. If $\mathfrak{A}$ is an integral proper RA on U, then we denote by $\mathrm{Aut}_0\mathfrak{A}$ the group of base automorphisms of $\mathfrak{A}$, that is, the set of all permutations φ of U such that for all $R \in A$ and all $x, y \in U$,

$$(x, y) \in R \text{ if and only if } (\varphi x, \varphi y) \in R.$$

If $\mathfrak{A}$ is a proper RA on U, then $\mathfrak{A}$ is called *c-permutational on U* if $\mathrm{Aut}_0\mathfrak{A}$ is transitive. More generally, an algebra $\mathfrak{A} \in \mathfrak{I}$ is called *permutational* if it is isomorphic to a *c*-permutational one. Following McKenzie, we denote the class of these algebras by $\mathcal{P}$.

An integral RRA which is not c-permutational on some set can still be permutational, as the following example from [ADN] shows. Let

$$U=\{0,1,\dots,6\} \quad \text{and} \quad S=\{0,1,2\}\times\{0,1,2\}\cup\{3,4,5,6\}\times\{3,4,5,6\};$$

that is, S is the disjoint union of a K_3 and a K_4. S generates a three-atom integral proper RA $\mathfrak{A}$ on U. Since, for example, 0 is an even vertex and 3 is an odd vertex, $\mathrm{Aut}_0\mathfrak{A}$ cannot be transitive. Since $S\cup E$ is an equivalence relation and $\mathfrak{A}$ is integral, and both properties can be read off the multiplication table of $\mathfrak{A}$, S must be a union of complete graphs each of which has at least three vertices. Hence no representation of $\mathfrak{A}$ on a seven-element set can be c-permutational. On the other hand, $\mathfrak{A}$ is isomorphic to a c-permutational RA on $U=\{0,1,\dots,5\}$: Just let S be the union of two disjoint K_3s and let $\mathfrak{B}$ be the proper integral RA on U generated by S; then $\mathfrak{A}\cong\mathfrak{B}$. (This algebra is also considered in [TG] in a different context.)

McKenzie [MK1; MK2] raised the problem of whether there is a nonpermutational integral RRA, that is, if $\mathcal{P}\neq\mathcal{I}$. This question arose from his negative solution to a problem of Tarski and Jónsson: Is every integral RRA representable over a group? In fact, he showed that each group representable RA is permutational, but that not every permutational RA is group representable. To complete the picture, we shall exhibit many $\mathfrak{A}\in\mathcal{I}$ that are not permutational.

The rest of the paper is organized as follows: In Section 2 we present an example of a nonpermutational integral RRA. Section 3 generalizes this to show that $\mathcal{P}$ is not finitely axiomatizable over $\mathcal{I}$. This complements a result by McKenzie [MK2], where he proves that the class of group representable RAs is not finitely axiomatizable over $\mathcal{P}$. There, as well as in what follows, the construction is made via ultraproducts.

The main result of this paper has certain logical consequences; these are easier to explain in terms of certain cylindric algebra investigations that are strongly related to the present result. They are based on the strong and well-investigated connection between cylindric algebras and relation algebras, as summarized in [HMT, §5.3] and [N]. The connections among relation algebras, cylindric algebras and logic are discussed, for example, in [HMT, §§4.3, 5.6], [N], [S], and [V]. Since the applications we have in mind are best presented after some discussion of cylindric algebras - which would be beyond the scope of the present article - they will be described in a separate paper. Here, we restrict ourselves to mentioning that some of the applications are related to questions investigated in the expanded version of [S].

2. The Example

THEOREM 1. *There is a nonpermutational integral relation set algebra.*

Proof. We shall proceed in two steps: First, we construct an $\mathfrak{A}\in\mathcal{I}$ on a base set of 45 elements which is not c-permutational; then we show that no representation of $\mathfrak{A}$ is c-permutational.

To facilitate notation, we let $\mathbf{n}=\{0,1,\ldots,n-1\}$ for each positive integer n. Also, if $(x,y)\in\omega\times\omega$ then we shall usually just write xy instead of (x,y). If $R,S\in\operatorname{Re}U$, and if no confusion can arise, we shall write RS instead of $R\,|\,S$. Finally, let $U=\mathbf{5}\times\mathbf{9}$.

2.1. AUXILIARY RELATIONS. We shall start the construction by defining certain relations on the set $\mathbf{9}\times\mathbf{9}$, and deriving their basic properties. These then will be used to define the atoms of $\mathfrak{A}$. The auxiliary relations fall into two groups.

A. Permutations of **9** *(in cycle form)*:

$$S=(012)\,(345)\,(678),\quad S^{\vee}=(021)\,(354)\,(687),$$
$$G=(036)\,(147)\,(258),\quad G^{\vee}=(063)\,(174)\,(285),$$
$$H=(048)\,(156)\,(237),\quad H^{\vee}=(084)\,(165)\,(273),$$
$$K=(057)\,(138)\,(246),\quad K^{\vee}=(075)\,(183)\,(264).$$

Set

$$\mathfrak{F}=\{S,S^{\vee},G,G^{\vee},H,H^{\vee},K,K^{\vee},\operatorname{id}\mathbf{9}\}.$$

Observe that the nontrivial orbits of these permutations are the lines of the affine plane over $GF(3)$. Consequently:

P1 $\mathfrak{F}$ is a partition of $\mathbf{9}\times\mathbf{9}$.

P2 Each two orbits from different elements of $\{S,G,H,K\}$ intersect in exactly one element; each two-element subset of **9** is contained in exactly one such orbit.

The basic connections among these permutations are as follows:

P3 For all $T\in\mathfrak{F}$, $TT=T^{\vee}$ and $TTT=\operatorname{id}\mathbf{9}$.

P4 $H=GS=SG$ and $K=SH=HS$.

P5 $\mathfrak{F}$ is a regular abelian subgroup of $\operatorname{Sym}(\mathbf{9})$.

B. Relations derived from the permutations. In defining a relation T, the notation

$$ijk\to lmn$$

means that if $a\in\{i,j,k\}$ and $b\in\{l,m,n\}$ then $(a,b)\in T$. Now define

$$R_0\colon\ 036\to 012,\ 147\to 345,\ 258\to 678,$$
$$R_1\colon\ 036\to 345,\ 147\to 678,\ 258\to 012,$$
$$R_2\colon\ 036\to 678,\ 147\to 012,\ 258\to 345;$$
$$B_0\colon\ 048\to 057,\ 156\to 138,\ 237\to 246,$$
$$B_1\colon\ 048\to 138,\ 156\to 246,\ 237\to 057,$$
$$B_2\colon\ 048\to 246,\ 156\to 057,\ 237\to 138.$$

Observe that R_0,R_1,R_2 take orbits of G into orbits of S, and B_0,B_1,B_2 take orbits of H into orbits of K. Also,

P6 $\{R_0,R_1,R_2\}$ and $\{B_0,B_1,B_2\}$ are partitions of $\mathbf{9}\times\mathbf{9}$.

2.2. THE CONSTRUCTION OF $\mathfrak{A}$. Recall that $U=5\times 9$. In what follows we shall use $+$ and $-$ mod 5. Now define

$$s=\{(ij, iS(j)): i\in 5, j\in 9\},$$
$$g=\{(ij, iG(j)): i\in 5, j\in 9\},$$
$$h=\{(ij, iH(j)): i\in 5, j\in 9\},$$
$$k=\{(ij, iK(j)): i\in 5, j\in 9\},$$

and, for each $m\in 3$,

$$r_m=\{(ij, (i+1)k): i\in 5, jk\in R_m\},$$
$$b_m=\{(ij, (i+2)k): i\in 5, jk\in B_m\}.$$

Let

$$W_0=\{s, g, h, k, s^{\vee}, g^{\vee}, h^{\vee}, k^{\vee}\},$$
$$W_1=\{r_i: i\in 3\}\cup\{b_i: i\in 3\},$$
$$W_2=\{r_i^{\vee}: i\in 3\}\cup\{b_i^{\vee}: i\in 3\}.$$

$\mathfrak{A}$ is now defined to be the relation algebra on U which is generated by $W_0\cup W_1\cup W_2$; it turns out that $\mathrm{At}\,\mathfrak{A}=W_0\cup W_1\cup W_2\cup\{\mathrm{id}\,U\}$. A complete list of products of atoms is given in Table 1. Note that $\{g, s, r_0, b_0\}$ is a generating set of $\mathfrak{A}$. Since $\mathrm{id}\,U\in\mathrm{At}\,\mathfrak{A}$, $\mathfrak{A}$ is integral.

Let $W=W_0\cup\{\mathrm{id}\,U\}$. For things to come, it is worthwhile to note the following general facts concerning W:

P7 (a) W is a semiregular abelian subgroup of $\mathrm{Sym}\,U$, isomorphic as a group to $\mathfrak{F}$.
(b) The centralizer $Z(W)$ of W in $\mathrm{Sym}(U)$ is transitive and contains W as a normal subgroup. Consequently, the orbits of W form a complete block system of $Z(W)$ [W, 7.1] and thus for every subgroup of $Z(W)$.
(c) The set $e=\bigcup W$ is the corresponding equivalence relation on U, and the classes of e are the blocks of W.
(d) If L is a class of e, then the action of W on L is a regular subgroup of $\mathrm{Sym}\,L$ isomorphic to $\mathfrak{F}$. Thus, if $f\in Z(W)$ agrees with $t\in W$ on one point on L, then $f|L=t|L$.

Note that P7 follows only from $W\subseteq\mathfrak{A}$ and the group structure of W, and does not depend on the representation of $\mathfrak{A}$.

2.3. $\mathfrak{A}$ IS NOT *c*-PERMUTATIONAL. Let $f\in\mathrm{Aut}_0\mathfrak{A}$. Then, as a relation on U, f commutes with every element of $\mathfrak{A}$; in particular, $\mathrm{Aut}_0\mathfrak{A}$ is a subgroup of $Z(W)$. By (P7), every block of W then is a block of $\mathrm{Aut}_0\mathfrak{A}$. Since no element of $W\setminus\{\mathrm{id}\,U\}$ commutes with every element of $\mathfrak{A}$, $\mathrm{Aut}_0\mathfrak{A}\cap W=\{\mathrm{id}\,U\}$.

Let $q: U\to U$ be defined by $ij\mapsto(i+1)j$; then $q\in\mathrm{Aut}_0\mathfrak{A}$. Let $\mathbf{Q}$ be the subgroup of $\mathrm{Aut}_0\mathfrak{A}$ generated by q. Since every element of U is moved by q, the degree of $\mathrm{Aut}_0\mathfrak{A}$ is equal to $|U|$.

;	s	g	h	k	$\breve{s}$	$\breve{g}$	$\breve{h}$	$\breve{k}$	r_0	r_1	r_2	b_0	b_1	b_2	$\breve{r}_0$	$\breve{r}_1$	$\breve{r}_2$	$\breve{b}_0$	$\breve{b}_1$	$\breve{b}_2$
s	$\breve{s}$	h	k	g	$1'$	$\breve{k}$	$\breve{g}$	$\breve{h}$	r_1	r_2	r_0	b_1	b_2	b_0	$\breve{r}_0$	$\breve{r}_1$	$\breve{r}_2$	$\breve{b}_2$	$\breve{b}_0$	$\breve{b}_1$
g	h	$\breve{g}$	$\breve{k}$	$\breve{h}$	k	$1'$	$\breve{s}$	s	r_0	r_1	r_2	b_2	b_0	b_1	$\breve{r}_2$	$\breve{r}_0$	$\breve{r}_1$	$\breve{b}_2$	$\breve{b}_0$	$\breve{b}_1$
h	k	$\breve{k}$	$\breve{h}$	$\breve{g}$	g	s	$1'$	$\breve{s}$	r_1	r_2	r_0	b_0	b_1	b_2	$\breve{r}_2$	$\breve{r}_0$	$\breve{r}_1$	$\breve{b}_1$	$\breve{b}_2$	$\breve{b}_0$
k	g	$\breve{h}$	$\breve{g}$	$\breve{k}$	h	$\breve{s}$	s	$1'$	r_2	r_0	r_1	b_1	b_2	b_0	$\breve{r}_2$	$\breve{r}_0$	$\breve{r}_1$	$\breve{b}_0$	$\breve{b}_1$	$\breve{b}_2$
$\breve{s}$	$1'$	k	g	h	s	$\breve{h}$	$\breve{k}$	$\breve{g}$	r_2	r_0	r_1	b_2	b_0	b_1	$\breve{r}_0$	$\breve{r}_1$	$\breve{r}_2$	$\breve{b}_1$	$\breve{b}_2$	$\breve{b}_0$
$\breve{g}$	$\breve{k}$	$1'$	s	$\breve{s}$	$\breve{h}$	g	k	h	r_0	r_1	r_2	b_1	b_2	b_0	$\breve{r}_1$	$\breve{r}_2$	$\breve{r}_0$	$\breve{b}_1$	$\breve{b}_2$	$\breve{b}_0$
$\breve{h}$	$\breve{g}$	$\breve{s}$	$1'$	s	$\breve{k}$	k	h	g	r_2	r_0	r_1	b_0	b_1	b_2	$\breve{r}_1$	$\breve{r}_2$	$\breve{r}_0$	$\breve{b}_2$	$\breve{b}_0$	$\breve{b}_1$
$\breve{k}$	$\breve{h}$	s	$\breve{s}$	$1'$	$\breve{g}$	h	g	k	r_1	r_2	r_0	b_2	b_0	b_1	$\breve{r}_1$	$\breve{r}_2$	$\breve{r}_0$	$\breve{b}_0$	$\breve{b}_1$	$\breve{b}_2$
r_0	r_0	r_1	r_1	r_1	r_0	r_2	r_2	r_2	b	b	b	$\breve{b}$	$\breve{b}$	$\breve{b}$	$g\breve{g}1'$	$\breve{s}\breve{h}k$	$sh\breve{k}$	$\breve{r}$	$\breve{r}$	$\breve{r}$
r_1	r_1	r_2	r_2	r_2	r_1	r_0	r_0	r_0	b	b	b	$\breve{b}$	$\breve{b}$	$\breve{b}$	$sh\breve{k}$	$g\breve{g}1'$	$\breve{s}\breve{h}k$	$\breve{r}$	$\breve{r}$	$\breve{r}$
r_2	r_2	r_0	r_0	r_0	r_2	r_1	r_1	r_1	b	b	b	$\breve{b}$	$\breve{b}$	$\breve{b}$	$\breve{s}\breve{h}k$	$sh\breve{k}$	$g\breve{g}1'$	$\breve{r}$	$\breve{r}$	$\breve{r}$
b_0	b_1	b_1	b_2	b_0	b_2	b_2	b_1	b_0	$\breve{b}$	$\breve{b}$	$\breve{b}$	$\breve{r}$	$\breve{r}$	$\breve{r}$	r	r	r	$h\breve{h}1'$	$\breve{s}g\breve{k}$	$s\breve{g}k$
b_1	b_2	b_2	b_0	b_1	b_0	b_0	b_2	b_1	$\breve{b}$	$\breve{b}$	$\breve{b}$	$\breve{r}$	$\breve{r}$	$\breve{r}$	r	r	r	$s\breve{g}k$	$h\breve{h}1'$	$\breve{s}g\breve{k}$
b_2	b_0	b_0	b_1	b_2	b_1	b_1	b_0	b_2	$\breve{b}$	$\breve{b}$	$\breve{b}$	$\breve{r}$	$\breve{r}$	$\breve{r}$	r	r	r	$\breve{s}g\breve{k}$	$s\breve{g}k$	$h\breve{h}1'$
$\breve{r}_0$	$\breve{r}_2$	$\breve{r}_0$	$\breve{r}_2$	$\breve{r}_1$	$\breve{r}_1$	$\breve{r}_0$	$\breve{r}_1$	$\breve{r}_2$	$s\breve{s}1'$	ghk	$\breve{g}\breve{h}\breve{k}$	r	r	r	$\breve{b}$	$\breve{b}$	$\breve{b}$	b	b	b
$\breve{r}_1$	$\breve{r}_0$	$\breve{r}_1$	$\breve{r}_0$	$\breve{r}_2$	$\breve{r}_2$	$\breve{r}_1$	$\breve{r}_2$	$\breve{r}_0$	$\breve{g}\breve{h}\breve{k}$	$s\breve{s}1'$	ghk	r	r	r	$\breve{b}$	$\breve{b}$	$\breve{b}$	b	b	b
$\breve{r}_2$	$\breve{r}_1$	$\breve{r}_2$	$\breve{r}_1$	$\breve{r}_0$	$\breve{r}_0$	$\breve{r}_2$	$\breve{r}_0$	$\breve{r}_1$	ghk	$\breve{g}\breve{h}\breve{k}$	$s\breve{s}1'$	r	r	r	$\breve{b}$	$\breve{b}$	$\breve{b}$	b	b	b
$\breve{b}_0$	$\breve{b}_2$	$\breve{b}_1$	$\breve{b}_0$	$\breve{b}_2$	$\breve{b}_1$	$\breve{b}_2$	$\breve{b}_0$	$\breve{b}_1$	$\breve{r}$	$\breve{r}$	$\breve{r}$	$k\breve{k}1'$	$sg\breve{h}$	$\breve{s}\breve{g}h$	b	b	b	r	r	r
$\breve{b}_1$	$\breve{b}_0$	$\breve{b}_2$	$\breve{b}_1$	$\breve{b}_0$	$\breve{b}_2$	$\breve{b}_0$	$\breve{b}_1$	$\breve{b}_2$	$\breve{r}$	$\breve{r}$	$\breve{r}$	$\breve{s}\breve{g}h$	$k\breve{k}1'$	$sg\breve{h}$	b	b	b	r	r	r
$\breve{b}_2$	$\breve{b}_1$	$\breve{b}_0$	$\breve{b}_2$	$\breve{b}_1$	$\breve{b}_0$	$\breve{b}_1$	$\breve{b}_2$	$\breve{b}_0$	$\breve{r}$	$\breve{r}$	$\breve{r}$	$sg\breve{h}$	$\breve{s}\breve{g}h$	$k\breve{k}1'$	b	b	b	r	r	r

Table 1 $b = b_0 \cup b_1 \cup b_2$; $r = r_0 \cup r_1 \cup r_2$. Table entries consisting of three terms are to be interpreted as unions of atoms. For example, $k\breve{k}1'$ denotes $k \cup k^{\vee} \cup \mathrm{id}\, U$.

Let us collect these facts:

P8 (a) $ft = tf$, for all $f \in \mathrm{Aut}_0 \mathfrak{A}$, $t \in \mathfrak{A}$;
(b) $\mathrm{Aut}_0 \mathfrak{A} \leq Z(W)$ and $\mathrm{Aut}_0 \mathfrak{A} \cap W = \{\mathrm{id}\, U\}$;
(c) every block of W is a block of $\mathrm{Aut}_0 \mathfrak{A}$;
(d) the degree of $\mathrm{Aut}_0 \mathfrak{A}$ is $|U|$.

In the sequel, let $L_i = \{i\} \times \mathbf{9}$ for each $i \in \mathbf{5}$. Each L_i is a block of W (or, equivalently, of e), and every block has this form. Since $\mathrm{Aut}_0 \mathfrak{A} \leq Z(W)$, we infer from P7(d) that if $f \in \mathrm{Aut}_0 \mathfrak{A}$ agrees with some $t \in W$ on one point of L_i, then $f | L_i = t | L_i$. Furthermore, if $f[L_i] = L_i$ for some $i \in \mathbf{5}$ then $f[L_j] = L_j$ for all $j \in \mathbf{5}$. This follows from the fact that each $f \in \mathrm{Aut}_0 \mathfrak{A}$ commutes with r_0.

P9 Suppose that $f(00) = 0k$, and $f(10) = 1m$; then f is completely determined by these values.

Proof. Since $f \in \mathrm{Aut}_0 \mathfrak{A}$, $f(20) \in im_{b_0}(0k) \cap im_{r_0}(1m)$. Now B_0 takes elements of $\mathbf{9}$ into orbits of K, and R_0 takes elements of $\mathbf{9}$ into orbits of S. By P2, these intersect in exactly one element. It follows that $f(20)$ is determined, and therefore, by P7, f is determined on L_2. □

More generally, let us define $\varphi : \mathbf{9} \times \mathbf{9} \to \mathbf{9}$ by

$$\varphi(k, m) = \text{the unique } u \in \mathbf{9} \text{ with } (k, u) \in B_0,\ (m, u) \in R_0.$$

Thus, if $f(i0) = ik$ and $f((i+1)0) = (i+1)m$, then

$$f((i+2)0) = (i+2)\,\varphi(k, m).$$

Thus, given k and m, we can define a sequence $F_{k,m}$ by setting

$$F_{k,m}(0) = k, \quad F_{k,m}(1) = m, \quad F_{k,m}(i+2) = \varphi(F_{k,m}(i), F_{k,m}(i+1)).$$

P10 If $f \in \mathrm{Aut}_0 \mathfrak{A}$ and $f(00) = 0k$, $f(10) = 1m$, then for all $i \in \omega$, $F_{k,m}(i) = F_{k,m}(i \bmod 5)$, and if $i \in \mathbf{5}$ then $f(i0) = (iF_{k,m}(i))$.

The first 26 terms of $F_{0,1}$ are as follows:

$$0, 1, 5, 8, 8, 7, 5, 6, 1, 3, 1, 4, 3, 0, 2, 7, 4, 4, 5, 7, 3, 2, 6, 2, 8, 6,$$

and $F_{0,1}(26) = F_{0,1}(0)$, $F_{0,1}(27) = F_{0,1}(1)$. Thus, $F_{0,1}$ has a period of 26.

Assume that $f \in \mathrm{Aut}_0 \mathfrak{A}$ and that $f(00) = 0k$, $f(10) = 1m$. Then $(k, m) \in R_0$, since f commutes with r_0 and $(00, 10) \in r_0$. Note that for every pair $(k, m) \in R_0$ with $km \neq 00$, k and m occur in consecutive places in the 26-long period of $F_{0,1}$. This, together with P10, shows that no $f \in \mathrm{Aut}_0 \mathfrak{A}$ can move a point within L_i, since f cannot move $i0$ within L_i, and therefore neither can it move any other point within L_i. Consequently, $\mathrm{Aut}_0 \mathfrak{A}$ is not transitive.

2.4. $\mathfrak{A}$ IS NOT PERMUTATIONAL. Let $\mathfrak{B}$ be a proper RA on a set T, and let $p : \mathfrak{A} \to \mathfrak{B}$ be an isomorphism. Recall that $e = \bigcup W$; from Table 1 we infer that $p(e)$ is an equivalence relation on T with each block M containing exactly nine elements, and that the subgroup of Sym M which is generated by $\{p_M(s), p_M(g)\}$ is isomorphic to $\mathfrak{F}$. Here, $p_M(s)$ and $p_M(g)$ are the restrictions of $p(s), p(g)$ to M.

As before, set $r = r_0 \cup r_1 \cup r_2$ and $b = b_0 \cup b_1 \cup b_2$. From

$$ex = xe = x \quad \text{and} \quad x^{\vee}x = e$$

for each $x \in \{r, b\}$, it follows that $p(r)$ and $p(b)$ are functions on the blocks of $p(e)$ in the following sense: If μ is the set of blocks of $p(e)$, then there are functions $\rho, \beta: \mu \to \mu$ such that

$$p(r) = \bigcup\{M \times \rho(M): M \in \mu\},$$

$$p(b) = \bigcup\{M \times \beta(M): M \in \mu\}.$$

From

$$rr = b \quad \text{and} \quad bb = r^{\vee}$$

we infer that

$$\rho\rho = \beta \quad \text{and} \quad \beta\beta = \rho^{\vee},$$

whence it follows that both ρ and b have order five. Thus $p(e)$ consists of exactly five blocks $M_0, \dots, M_4$, and ρ and β are as shown in Figure 1. Hence,

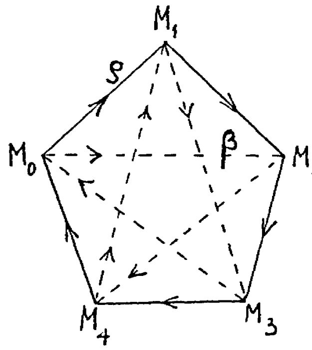

Figure 1

there is an isomorphism between $(T, p(s), p(g), p(r), p(b))$ and (U, s, g, r, b), and we may suppose that $T = U$, $p(s) = s$, $p(g) = g$, $p(r) = r$, and $p(b) = b$, and consequently that $p(e) = e$. Note that $\mathfrak{A}$ and $\mathfrak{B}$ differ only on how r and b are split into three parts.

For $i, j \in \mathbf{5}$, let $T_{ij} = L_i \times L_j$. We now want to show that

P11 $r_0 \cap T_{i(i+1)} = p(r_j) \cap T_{i(i+1)}$ for some $j \in \mathbf{3}$;
$b_0 \cap T_{i(i+2)} = p(b_j) \cap T_{i(i+2)}$ for some $j \in \mathbf{3}$.

Proof. By $p(r_0) \cup p(r_1) \cup p(r_2) = p(r) = r$, we have $(i0, (i+1)0) \in p(r_j)$ for some $j \in \mathbf{3}$. Since

$$p(r_j)s = s^{\vee}p(r_j)g = g^{\vee}p(r_j) = gp(r_j) = p(r_j),$$

from $(i0,(i+1)0)\in p(r_j)$ and from the fact that every element of $r_0\cap T_{i(i+1)}$ can be obtained from $(i0,(i+1)0)$ by successively applying g from the left and s from the right, we can infer

$$r_0\cap T_{i(i+1)}\subseteq p(r_j)\cap T_{i(i+1)}$$

and that

$$p(r_j)g\cap p(r_j)=p(r_j)g^{\vee}\cap p(r_j)=\emptyset$$

implies $p(r_j)\cap r_k\cap T_{i(i+1)}=\emptyset$ for $k\neq 0$. To see this, assume for example that $p(r_j)\cap r_1\cap T_{i(i+1)}\neq\emptyset$. Since $r_1g^{\vee}=r_0$, we then have $p(r_j)g^{\vee}\cap r_0\cap T_{i(i+1)}\neq\emptyset$ and therefore $p(r_j)\cap p(r_j)g^{\vee}\neq\emptyset$, a contradiction.

The proof for the second statement of P11 is analogous and is left to the reader. □

Now, assume that $f\in \mathrm{Aut}_0\mathfrak{B}$ such that $f(ij)=ik$ for some $i\in\mathbf{5}$ and $j,k\in\mathbf{9}$, $j\neq k$. Since $p(x)=x$ for $x\in\{s,g,r,b\}$, f preserves the latter relations and P7 implies that $f[L_n]=L_n$ for each $n\in\mathbf{5}$. By P11, f then preserves r_0 and b_0 as well; hence, it is also an automorphism of $\mathfrak{A}$. As shown in Section 2.3, this is not possible. This proves Theorem 1. □

The example of a nonpermutational integral RA just exhibited is not the smallest possible. We next outline the construction of such an RA on a base set of 32 elements with 17 atoms. Checking that this algebra is integral and nonpermutational is analogous to the previous example, and we leave this to the reader. We shall use the notational conventions introduced so far.

Let $U=\mathbf{8}\times\mathbf{4}$, and let $S,G:\mathbf{4}\to\mathbf{4}$ with $S=(01)(23)$ and $G=(02)(13)$. Also, let R,B,T be the relations on $\mathbf{4}$ defined by

$$R:\ 01\to 02,\ 23\to 13,$$
$$B:\ 02\to 03,\ 13\to 12,$$
$$T:\ 01\to 03,\ 23\to 12.$$

Next, define

$$s=\{(ij,iS(j)): i\in\mathbf{8}, j\in\mathbf{4}\},$$
$$g=\{(ij,iG(j)): i\in\mathbf{8}, j\in\mathbf{4}\},$$
$$r=\{(ij,i+1)k): i\in\mathbf{8}, jk\in R\},$$
$$b=\{(ij,(i+2)k): i\in\mathbf{8}, jk\in B\},$$
$$t=\{(ij,(i+3)k): i\in\mathbf{8}, jk\in T\}.$$

$\mathfrak{A}$ is defined to be the relation algebra on U generated by $\{s,g,r,b,t\}$.

It may be interesting to note that for $n<7$ the corresponding algebra on $U=\mathbf{n}\times\mathbf{4}$ is not integral, and that for $U=\mathbf{7}\times\mathbf{4}$ the algebra is integral and permutational. $\mathfrak{A}$ as above is thus the smallest example of a nonpermutational integral RRA which can be obtained by using this construction. It is unknown to us whether any smaller examples exist.

PROBLEM. What is the smallest cardinality that an integral nonpermutational RRA can have?

3. Many Integral and Nonpermutational RRAs

In this section we shall construct an infinite family of representable, integral, and nonpermutational RA's, and then show that a nonprincipal ultraproduct of these is permutational. As a corollary we obtain that $\mathcal{P}$ is not finitely axiomatizable relative to $\mathcal{I}$.

THEOREM 2. *There are infinitely many nonpermutational integral RRAs such that a nonprincipal ultraproduct of these is permutational.*

Proof. Suppose $\mathbf{n}=\{0,1,\dots,n-1\}$, $n>3$, and let S,G,R_i,B_i be the same relations on $\mathbf{9}\times\mathbf{9}$ as in Section 2.1; also, recall the definitions of W, $\mathfrak{F}$, φ, and $F_{k,m}$. In the sequel, we shall use $+$ and $-$ mod n. Let $U_n=\mathbf{n}\times\mathbf{9}$, and define

$$s=\{(ij,iS(j))\colon i\in\mathbf{n},j\in\mathbf{9}\},$$
$$g=\{(ij,iG(j))\colon i\in\mathbf{n},j\in\mathbf{9}\},$$
$$h=\{(ij,iH(j))\colon i\in\mathbf{n},j\in\mathbf{9}\},$$
$$k=\{(ij,iK(j))\colon i\in\mathbf{n},j\in\mathbf{9}\}.$$

Also, for each $m\in\mathbf{3}$,

$$r_m=\{(ij,(i+1)k)\colon i\in\mathbf{n},jk\in R_m\},$$
$$b_m=\{(ij,(i+2)k)\colon i\in\mathbf{n},jk\in B_m\}.$$

Let $\mathfrak{A}_n$ be generated by these relations (or, equivalently, by $\{s,g,r_0,b_0\}$). The proof that $\mathfrak{A}_n$ is integral is analogous to Section 2.2: For any $m<n$, let

$$t_m=\{(ij,(i+m)k)\colon i\in\mathbf{n},j,k\in\mathbf{9}\},$$
$$W=\{s,g,h,k,s^\vee,g^\vee,h^\vee,k^\vee,\mathrm{id}\,U_n\},$$
$$\mathrm{At}=W\cup\{r_m,b_m,r_m^\vee,b_m^\vee\colon m<3\}\cup\{t_i\colon 3\le i<9\}.$$

The multiplication table of At can be obtained as follows: The products of elements from $W\cup\{r_m,b_m,r_m^\vee,b_m^\vee\colon m<3\}$ are as on Table 1, except for the following, where $i,j<3$:

$$r_ib_j=b_jr_i=t_3;$$
$$b_ib_j=t_4;$$
$$r_i^\vee b_j^\vee=t_{-3};$$
$$b_i^\vee b_j^\vee=t_{-4}.$$

The rest is determined by the following:

$$t_iw=wt_i=t_i \text{ if } w\in W,\ i<n;$$
$$t_ir_m=r_mt_i=t_{i+1},\ t_ir_m^\vee=r_m^\vee t_i=t_{i-1};$$
$$t_ib_m=b_mt_i=t_{i+2},\ t_ib_m^\vee=b_m^\vee t_i=t_{i-2} \text{ if } i<9,\ m<3;$$
$$t_it_j=t_{i+j} \text{ if } i,j<9.$$

This shows that At is the set of atoms of $\mathfrak{A}_n$, and therefore $\mathfrak{A}_n$ is integral.

Since P7 and P8 depended only on W and not on n, they are still valid in the more general case. Furthermore, P10 also holds if we replace $\mathbf{5}$ by $\mathbf{n}$. Since the period of $F_{0,1}$ is 26, we have the following result:

if 26 does not divide n, then $\mathrm{Aut}_0 \mathfrak{A}_n$ is not transitive.

The proof that $\mathfrak{A}_n$ is not permutational is also analogous to Section 2.4. The only difference is the following: Let e, r, b, p be as in 2.4. In 2.4, the part of Table 1 concerning $\{e, r, b\}$ implied that $p(e)$ contained exactly five blocks $M_0, \ldots, M_4$. In the present case $n > 5$, the part of the multiplication table concerning $\{t_0, \ldots, t_{n-1}\}$ implies that $p(e)$ contains exactly n blocks $M_0, \ldots, M_{n-1}$. The rest of the proof is completely analogous to that of 2.4 and is therefore left to the reader.

Let $I = \{n \in \omega : n \text{ is prime and } n > 3\}$, let T be a nonprincipal ultrafilter on I, and let $U = \prod U_n/T$ and $\mathfrak{A} = \prod \mathfrak{A}_n/T$. We first define a representation of $\mathfrak{A}$: Let $a = \langle a_n : n \in I\rangle/T \in \mathfrak{A}$, and $u = \langle u_n : n \in I\rangle/T \in U$, $v = \langle v_n : n \in I\rangle/T \in U$. Define

$$(u, v) \in \mathrm{rep}(a) \quad \text{if and only if} \quad \{n \in I : (u_n, v_n) \in a_n\} \in T.$$

It follows from 3.1.90 of [HMT] that $(U, \mathrm{rep}(a))_{a \in \mathfrak{A}}$ is a representation for $\mathfrak{A}$. We will show that this representation is c-permutational. More precisely, let $\mathfrak{B}$ be the relation set algebra on U with universe $B = \{\mathrm{rep}(a) : a \in \mathfrak{A}\}$; then $\mathfrak{B} \cong \mathfrak{A}$ and $\mathfrak{B}$ is c-permutational.

To facilitate notation, we shall introduce the following conventions: Set $N = \prod\{\mathbf{n} : n \in I\}/T$. Observe that $(N, +)$ is a group under addition induced by the addition mod n on $\mathbf{n}$; thus, for any $\bar{k}, \bar{l} \in N$ there is an $\bar{m} \in N$ such that $\bar{l} = \bar{k} + \bar{m}$.

There is a natural bijection between U and $N \times \mathbf{9}$: Each $u \in U$ is of the form $\langle (k_n, j) : n \in I\rangle/T$ for some $j \in \mathbf{9}$ and some sequence $\langle k_n : n \in I\rangle$ with $k_n \in \mathbf{n}$ for all $n \in I$. Now,

$$u \mapsto (\langle k_n : n \in I\rangle/T, j)$$

establishes the desired bijection. In the sequel, we shall identify U and $N \times \mathbf{9}$.

Let $\mathbf{Z}$ denote the set of all integers, and recall that $(N, +)$ is a group. For any $n \in \omega$, let

$$\underline{n} = \langle n/\mathrm{mod}\, i : i \in I\rangle \quad \text{and} \quad \underline{-n} = -\underline{n}.$$

Since $n < i$ for all but finitely many $i \in I$, almost all terms of $\bar{n}$ are equal to n.

To extend the definition of $F_{k,m}$ over $\mathbf{Z}$, for all $k, m \in \mathbf{9}$ let $\psi(k, m)$ be the unique $u \in \mathbf{9}$ for which $(u, k) \in R_0$ and $(u, m) \in B_0$. Now define, for all $i \leq 1$ and $k, m \in \mathbf{9}$,

$$F_{k,m}(i-2) = \psi(F_{k,m}(i-1), F_{k,m}(i)).$$

It is not difficult to check that, for all $i \in \mathbf{Z}$, if $(k, m) \in R_0$ then $(F_{k,m}(i), F_{k,m}(i+1)) \in R_0$ and $(F_{k,m}(i), F_{k,m}(i+2)) \in B_0$.

The following facts follow from P5, and are easy to check:

A1 For all $i, j \in \mathbf{9}$ there is a unique base automorphism η of $\mathfrak{F}$ such that $\eta(i) = j$.

A2 Let δ, γ be base automorphisms of $\mathfrak{F}$, and let $X \in \{R_0, B_0\}$. Then, $(\delta(0), \gamma(0)) \in X$ if and only if

$$(\forall i, j \in \mathbf{9})[(i, j) \in X \Leftrightarrow (\delta(i), \gamma(j)) \in X].$$

To show that $\mathfrak{B}$ is c-permutational, let $(\bar{k}, i), (\bar{l}, j) \in U$. We shall exhibit a base automorphism f of $\mathfrak{B}$ with $f((\bar{k}, i)) = (\bar{l}, j)$. Let η be the unique base automorphism of $\mathfrak{F}$ with $\eta(i) = j$, and set $j' = \eta(0)$; also, let $(j', j'') \in R_0$ and let $\bar{m} \in N$, $t \in \mathbf{9}$ be arbitrary.

Define

$$f(\bar{k}+\bar{m}, t) = \begin{cases} (\bar{l}+\bar{m}, \gamma(t)) & \text{if } \bar{m} = \underline{n} \text{ for some } n \in \mathbf{Z}, \text{ and } \gamma \text{ is the base automorphism of } \mathfrak{F} \text{ taking } 0 \text{ to } F_{j',j''}(n), \\ (\bar{l}+\bar{m}, t) & \text{if } \bar{m} \notin \{\underline{n} : n \in \mathbf{Z}\}. \end{cases}$$

Choosing $\bar{m} = \underline{0}$, and noticing that $F_{j',j''}(0) = j' = \eta(0)$, we can infer that $f(\bar{k}, i) = (\bar{l}, j)$.

To show that f is indeed a base automorphism of $\mathfrak{B}$, let us first look at At $\mathfrak{B}$, the set of atoms of $\mathfrak{B}$. Let $\bar{m} \in N$ and $X \subseteq \mathbf{9} \times \mathbf{9}$ be arbitrary. Define

$$a(\bar{m}, X) = \{\langle(\bar{n}, i), (\bar{n}+\bar{m}, j)\rangle : \bar{n} \in N, (i, j) \in X\}.$$

It is not hard to see that the representations of the atoms of $\mathfrak{A}$—that is, the atoms of $\mathfrak{B}$—are the following:

$a(\underline{0}, X)$ where $X \in \mathfrak{F}$,
$a(\underline{1}, X)$ where $X \in \{R_0, R_1, R_2\}$,
$a(\underline{2}, X)$ where $X \in \{B_0, B_1, B_2\}$,
$a(\underline{-1}, X)$ where $X \in \{R_0^{\vee}, R_1^{\vee}, R_2^{\vee}\}$,
$a(\underline{-2}, X)$ where $X \in \{B_0^{\vee}, B_1^{\vee}, B_2^{\vee}\}$,
$a(\bar{m}, \mathbf{9} \times \mathbf{9})$ if $\bar{m} \in N \setminus \{\underline{0}, \underline{1}, \underline{2}, \underline{-1}, \underline{-2}\}$.

Since $\bigcup \text{At}\, \mathfrak{B} = U \times U$, it is enough to show that f preserves all atoms of $\mathfrak{B}$: Let $\bar{x} \in N$ such that $\bar{k}+\bar{x} = \bar{l}$. Then the following are not difficult to check for all $\bar{p} \in N$:

B1 For all $i \in \mathbf{9}$ there is some $i' \in \mathbf{9}$ such that

$$f(\bar{p}, i) = (\bar{p}+\bar{x}, i').$$

B2 There is an automorphism of $\mathfrak{F}$ such that

$$f(\bar{p}, i) = (\bar{p}+\bar{x}, g(i)).$$

B3 $f(\bar{p}, 0) = (\bar{p}+\bar{x}, 0)$ if $\bar{p} \notin \{\bar{k}+\bar{n} : n \in \mathbf{Z}\}$, and $f(\bar{p}, 0) = (\bar{p}+\bar{x}, F_{j',j''}(n))$ if $\bar{p} = \bar{k}+\bar{n}$ for some $n \in \mathbf{Z}$.

Now we are ready to show that f preserves $a(\bar{m}, X)$ if $a(\bar{m}, X) \in \text{At}\, \mathfrak{B}$. It follows from B1 that f preserves $a(\bar{m}, \mathbf{9} \times \mathbf{9})$ for all $\bar{m} \in N$. From B2 it follows that f preserves $a(\underline{0}, X)$ for all $X \in \mathfrak{F}$. Next, we show that f preserves $a(\underline{1}, R_0)$: By A2, it is enough to prove that for all $\bar{p} \in N$ we have

$$(*) \qquad (f(\bar{p},0), f(\bar{p}+\underline{1},0)) \in a(\underline{1}, R_0).$$

If $\bar{p} \notin \{\bar{k}+\bar{n}: n \in \mathbf{Z}\}$ then $\bar{p}+\underline{1} \notin \{\bar{k}+\bar{n}: n \in \mathbf{Z}\}$, and thus (*) holds by B3 and $(0,0) \in R_0$. Suppose that $\bar{p} = \bar{k}+\bar{n}$ for some $n \in \mathbf{Z}$; then (*) holds by B3 and $(F_{j',j''}(n), F_{j',j''}(n+1)) \in R_0$.

Since $a(\underline{1}, R_1) = a(\underline{1}, SR_0) = a(\underline{0}, S)a(\underline{1}, R_0)$, f also preserves $a(\underline{1}, R_1)$; similarly, f preserves $a(\underline{1}, R_2)$ as well. Since $a(\underline{-1}, R_i^{\vee}) = a(\underline{1}, R_i)^{\vee}$, f preserves $a(\underline{-1}, R_i^{\vee})$ for all $i \in \mathbf{3}$. The proof for the remaining cases is analogous, and can safely be omitted. □

COROLLARY 3. *The class of all permutational relation algebras is not finitely axiomatizable over the class of all integral representable relation algebras.*

Proof. Assume that Δ is a finite set of sentences in the language of relation algebras expressing the fact that an integral representable relation algebra is not permutational. Let φ be the conjunction of the elements of Δ. By Łos' theorem, φ is preserved under ultraproducts, contradicting Theorem 2. □

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