Extensions of Projective Varieties and Deformations, II

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0. Introduction

0.1. In the author's paper [L1], an upper bound on the number of steps of a nontrivial extension of a projective variety V was obtained under the additional assumption that V is linearly normal and can be defined by quadratic equations (see the definitions that follow). In this paper we show that the main result of [L1] is valid (at least over $\mathbf{C}$) without this assumption.

0.2. DEFINITIONS AND STATEMENT OF RESULTS. Throughout the paper the base field will be the field $\mathbf{C}$ of complex numbers. We say that a smooth projective variety $V \subseteq \mathbf{P}^n$ *can be extended k steps* to a projective variety $W \subseteq \mathbf{P}^{n+k}$ (or that W is a *k-step extension* of V) if $V = W \cap \mathbf{P}^n$, W is smooth along V and transversal to $\mathbf{P}^n$, and $\mathbf{P}^n$ is imbedded in $\mathbf{P}^{n+k}$ as a linear subspace. If W is not a cone, such an extension is called *nontrivial.*

If $N_{\mathbf{P}^n|V}$ is the normal bundle of a smooth projective variety $V \subseteq \mathbf{P}^n$, set $\alpha(V) = h^0(N_{\mathbf{P}^n|V}(-1)) - n - 1$ (cf. [L1]). The main result of this paper is the following theorem.

THEOREM 0.1. *If $\alpha(V) < n$, $V \neq \mathbf{P}^n$, and V is not a quadric and is not contained in a hyperplane in $\mathbf{P}^n$, then V cannot be nontrivially extended more than $\alpha(V)$ steps.*

If T_V is the tangent bundle of V, set $\Gamma_V = (P^1(O_V(1)))^*$, where $P^1(L)$ denotes the sheaf of principal parts of the first order of the sheaf L [K, IVA]. The following corollary can be easily derived from our theorem in the same way as in [L1].

COROLLARY 0.2.

(a) *If* $\dim V \geq 2$, *then the theorem holds with $\alpha(V)$ replaced by $h^1(\Gamma_V)$.*
(b) *If* $\dim V \geq 3$, *then the theorem holds with $\alpha(V)$ replaced by $h^1(T_V(-1))$.*

In [L1] it was shown by examples that the bound in our theorem is in a sense sharp for varieties of dimension ≥ 2: For each pair of numbers $\alpha \geq 0$ and $d \geq 2$, there exists a variety V for which the hypotheses of Theorem 0.1 hold

Received May 14, 1990. Revision received December 30, 1990.
Michigan Math. J. 39 (1992).

and such that $\alpha(V) = \alpha$, $\deg V = d$, and V can be extended α steps (to a smooth variety).

0.3. Our argument is a refinement of that contained in [L2], the first part of this paper, where a new interpretation of the invariant $\alpha(V)$ in terms of deformation theory was obtained.

0.4. NOTATION AND CONVENTIONS. We will say that a projective variety in $\mathbf{P}^n$ is *nondegenerate* if it is not contained in any hyperplane in $\mathbf{P}^n$. By $\mathrm{Gr}(k, \mathbf{P}^n)$ we mean the Grassmannian of k-dimensional projective subspaces (k-*planes*) in $\mathbf{P}^n$. By $\mathbf{P}(E)$ we mean the space of *lines* in the vector space E. If p is a smooth point of a projective variety $V \subseteq \mathbf{P}^n$, then $T_p V$ is the tangent space to V at p, and if $p \in V$ is any point, then $T_p V \subseteq \mathbf{P}^n$ is the imbedded Zariski tangent space (projective tangent space) to V at p.

If $X, Y \subseteq \mathbf{P}^n$ are projective varieties, then $X * Y$ is their *join* (closure of the union of lines joining points of X with points of Y). If $S \subseteq \mathbf{P}^n$, then $\langle S \rangle$ is the linear span of S (intersection of the linear subspaces containing S).

0.5. ACKNOWLEDGMENTS. I am grateful to F. L. Zak for his concern and for many helpful discussions. I would like to thank the Mathematics Department of the University of Arkansas and particularly D. Khavinson for providing a pleasant atmosphere during the writing of the first draft of this paper and for help in its distribution.

1. Preliminaries

Let $V \subseteq \mathbf{P}^n$ be a projective variety and $H \subseteq \mathbf{P}^n$ a hyperplane transversal to V; set $X = V \cap H$. Denote the Hilbert scheme of closed subschemes of $\mathbf{P}^n$ containing X by Z_X and the group of projective automorphisms of $\mathbf{P}^n$ fixing all points of H by G_H (see [L2] for details). If the homogeneous coordinates $(x_0 : \cdots : x_n)$ in $\mathbf{P}^n$ are chosen so that H is defined by the equation $x_n = 0$, then one may consider G_H as the set of $(n+1)$-tuples $(a_0, \ldots, a_n)$ of complex numbers such that $a_n \neq 0$. The image of a point $(x_0 : \cdots : x_n) \in \mathbf{P}^n$ under the action of $(a_0, \ldots, a_n) \in G_H$ is $(x_0 + a_0 x_n : \cdots : x_{n-1} + a_{n-1} x_n : a_n x_n) \in \mathbf{P}^n$. G_H acs on Z_X: informally speaking, an element $g \in G$ sends a subscheme $Y \supseteq X$ to $g^{-1} Y$. If $\phi : G_H \to Z_X$ is the morphism informally defined by $g \mapsto g^{-1} V$ (cf. [L2]), let $d\phi$ be its derivative at the identity of G_H.

PROPOSITION 1.1 [L2]. *The dimension of the cokernel of $d\phi$ is $\alpha(V)$.*

Since G_H is reduced as a group scheme, its action on Z_X induces the action of G_H on $(Z_X)_{\mathrm{red}}$. What we will need from the deformation theory is contained in the following evident corollary to Proposition 1.1.

COROLLARY 1.2. *If ξ is the point of $(Z_X)_{\mathrm{red}}$ corresponding to V, then the codimension of the G_H-orbit of ξ in the component of $(Z_X)_{\mathrm{red}}$ that contains ξ is less than or equal to $\alpha(V)$.*

2. An Elementary Lemma

LEMMA 2.1. *Let* $S \subset \mathrm{Gr}(l, \mathbf{P}^m)$ *be a closed family of l-planes,* $0 < l < m$.

(a) *If* $\dim S > \dim(\mathrm{Gr}(l, \mathbf{P}^{m-1}))$, *then the union of all planes in S is the whole of* $\mathbf{P}^m$.

(b) *If* $\dim S > 1 + \dim(\mathrm{Gr}(l, \mathbf{P}^{m-2}))$, *then the union of all planes in S is either the whole of* $\mathbf{P}^m$ *or a hyperplane in* $\mathbf{P}^m$.

REMARK. The bound in the lemma is sharp. Indeed, if $l \leq m-2$ and $Z \subseteq \mathbf{P}^m$ is a hypersurface ruled by $\mathbf{P}^{m-2}$'s, then Z contains a $1 + \dim(\mathrm{Gr}(l, \mathbf{P}^{m-2}))$-dimensional family of $\mathbf{P}^l$'s.

Proof. Without loss of generality, we may assume that S is irreducible. Denote the assertion of the lemma by $A_{l,m}$ and let us prove it by induction on l and m.

It suffices to prove assertion (b), since (a) follows from it immediately. Suppose that the union of all l-planes in S is a variety $Z \subseteq \mathbf{P}^m$, $Z \neq \mathbf{P}^m$, and $\dim Z = t$. We must prove that Z is a hyperplane. Indeed, if M is the family of l-planes of S passing through a generic smooth point $p \in Z$, then

$$\dim(M) \geq \dim(S) + l - t \geq \dim(\mathrm{Gr}(l, \mathbf{P}^{m-2})) + l - t + 2.$$

The projectivizations of the tangent spaces at p to the planes from the family M form a family of $(l-1)$-planes in $\mathbf{P}(T_p Z) = \mathbf{P}^{t-1}$, and the dimension of this family equals $\dim M$. Since $t \leq m-1$,

$$\dim(\mathrm{Gr}(l, \mathbf{P}^{m-2})) + l - t + 2 > \dim(\mathrm{Gr}(l-1, \mathbf{P}^{t-1})),$$

and we see by $A_{l-1,t-1}$ that the union of the planes of M is $T_p Z$. Hence $T_p Z \subseteq Z$, so $Z = T_p Z$ and Z is a linear subspace of $\mathbf{P}^m$. Because $\dim(S) > \dim(\mathrm{Gr}(l, \mathbf{P}^{m-2}))$, Z must be a hyperplane. ❑

3. The Main Construction

LEMMA 3.1. *Let* $V \subseteq \mathbf{P}^n$ *be a nonsingular and nondegenerate projective variety, and suppose that* $W \subseteq \mathbf{P}^{n+k}$ *is its k-step extension. Assume that* $\alpha(V) < k$ *and that* $H \subseteq \mathbf{P}^n$ *is a hyperplane for which the following hypotheses hold:*

(a) *there is no nontrivial automorphism of* $\mathbf{P}^n$ *fixing all the points of H and mapping V into itself;*

(b) *H is transversal to V; and*

(c) *V is nondegenerate in H.*

Then there is a point $p \in W \setminus \mathbf{P}^n$ *such that* $W \supseteq p * (W \cap H)$.

Proof. Choose the homogeneous coordinates in $\mathbf{P}^n$ and $\mathbf{P}^{n+k}$ so that the equation of H in $\mathbf{P}^n$ is $x_n = 0$ and the equations of $\mathbf{P}^n$ in $\mathbf{P}^{n+k}$ are $x_{n+1} = \cdots = x_{n+k} = 0$. For each $u = (u_1; \ldots; u_k) \in \mathbf{C}^k$ define an n-plane $H_u \subseteq \mathbf{P}^{n+k}$ by the equations $x_{n+i} = u_i x_n$ for $1 \leq i \leq k$.

Define the projection $p: \mathbf{P}^{n+k} \to \mathbf{P}^n$ by the truncation of the last k coordinates, and consider the family over $\mathbf{C}^k$ of subvarieties of $\mathbf{P}^n$ in which the variety $V_u = p(V \cap H_u)$ lies over $u \in \mathbf{C}^k$. Over a Zariski open subset $U \subseteq \mathbf{C}^k$ this family is flat. Set $X = V \cap H$, and consider the morphism $a: U \to (Z_X)_{\text{red}}$ defined by the restriction of this family to U (Z_X, ϕ, and G_H mean the same as in Section 1). By Corollary 1.2, the codimension of the G_H-orbit of the point corresponding to V in $(Z_X)_{\text{red}}$ is less than or equal to $\alpha(V)$. Condition (a) of the hypothesis implies that the map $\phi: G_H \to (Z_X)_{\text{red}}$ is one-to-one. Since $\alpha(V) < k$, this implies that there is an affine curve $\Gamma \subseteq \mathbf{C}^k$ such that $V_u = a(u)V$ for each $u \in \Gamma \cap U$ (cf. [L2, Prop. 1.4]).

Let S be the normalization of the compactification of Γ. By restricting to Γ and lifting to S we may regard a as a rational map from S to G_H and the coordinate functions u_i on $\mathbf{C}^k$ as rational functions on S. Writing a in matrix form, one obtains rational functions $a_0, \ldots, a_n$ on S such that, for each point $(x_0 : \cdots : x_n) \in V$ and each $p \in S$ where all a_j's and u_j's are defined, the point

$$(x_0 + a_0(p)x_n : \cdots : x_{n-1} + a_{n-1}(p)x_n : a_n(p)x_n : u_{n+1}(p)a_n(p)x_n \\ : \cdots : u_{n+k}(p)a_n(p)x_n)$$

belongs to W. Denoting $u_{n+k}a_n$ by a_{n+k}, one may write this point as

$$(x_0 + a_0(p)x_n : \cdots : x_{n-1} + a_{n-1}(p)x_n : a_n(p)x_n : a_{n+1}(p)x_n : \cdots : a_{n+k}(p)x_n).$$

Note that not all a_j's are constants, since $a_{n+j} = u_j a_n$ and not all u_j's are constants. Hence some of the a_j's must have poles on S.

Let m be the maximal order of poles of a_j's, $0 \le j \le n+k$, and suppose that m is attained at a point $\xi \in S$. Choose an open imbedding $\lambda: \Delta \to S$ of the unit disk in the complex plane into S such that $\lambda(0) = \xi$, and set $\tilde{a}_j = a_j \circ \lambda$. Set $b_j = \lim_{t \to 0} \tilde{a}_j(t) \cdot t^m$; all the b_j's are finite and not all of them are 0.

Now, by Lemma 3.1 of [L2], for each point $(x_0 : \cdots : x_n : 0 : \cdots : 0) \in X$ and each complex number $c \neq 0$ there exists a map $h: \Delta \to V$, $h: t \mapsto (\tilde{x}_0(t) : \cdots : \tilde{x}_{n-1}(t) : \tilde{x}_n(t) : 0 \cdots 0)$ such that $\tilde{x}_i(0) = x_i$ for $0 \le i \le n-1$, $\tilde{x}_n(t)/ct^m \to 1$ as t tends to 0. For each $t \in \Delta$ the point

$$(\tilde{x}_0(t) + \tilde{a}_0(t)\tilde{x}_n(t) : \cdots : \tilde{x}_{n-1}(t) + \tilde{a}_{n-1}(t)\tilde{x}_n(t) : \tilde{a}_n(t)\tilde{x}_n(t) : \tilde{a}_{n+1}(t)\tilde{x}_n(t) \\ : \cdots : \tilde{a}_{n+k}(t)\tilde{x}_n(t))$$

is in W. As t goes to 0, this point tends to

$$(x_0 + cb_0x_n : \cdots : x_{n-1} + cb_{n-1}x_n : cb_nx_n : \cdots : cb_{n+k}x_n) \in W.$$

Because the choices of the point in X and of the number c were arbitrary, we conclude that W contains the cone over X with the vertex $z = (b_0 : \cdots : b_{n+k}) \in \mathbf{P}^{n+k}$. If $z \in \mathbf{P}^n$, then $W \cap \mathbf{P}^n = V$ would contain such a cone, contradicting the smoothness of V. Hence $z \notin \mathbf{P}^n$ and the lemma is proved. □

4. Proof of the Theorem

Under the hypotheses of Theorem 0.1, suppose that V is extended to a variety $W \subseteq \mathbf{P}^{n+k}$, $k > \alpha(V)$. We must prove that W is a cone.

To begin with, let us reduce the theorem to the case $k=\alpha(V)+1$. To that end, assume that in this case the theorem is true and consider a generic linear subspace $L\subseteq \mathbf{P}^{n+k}$ of dimension n. The variety $V'=W\cap L$ is smooth, and by the semicontinuity of cohomology we see that $\alpha'=\alpha(V')\leq\alpha(V)$. Now consider a generic linear space P, containing L and having dimension $n+\alpha'+1$. The variety $W'=W\cap P$ is an $(\alpha'+1)$-step extension of V'. From the case $k=\alpha+1$ of our theorem, which we have assumed to hold, it follows that W' is a cone. If we denote the vertex of this cone by p and if we let the linear subspaces L and $P\supseteq L\cup\{p\}$ vary, we shall see that W is also a cone with the vertex p.

Observe further that V is not a hypersurface. Indeed, if V were a hypersurface in $\mathbf{P}^n$, then $\alpha(V)=h^0(N_{\mathbf{P}^n|V})-n-1$ would be greater than n, since $\deg V>2$. Note also that, since V is not a hypersurface, its extension W is not a hypersurface either.

Assume now that $k=\alpha(V)+1$ and consider a generic n-plane $L\subseteq\mathbf{P}^{n+k}$. Then the hypotheses of Lemma 3.1 hold; the hypothesis (a) follows from Proposition 2.1 of [L2]. The variety $V_L=W\cap L$ is smooth and by semicontinuity $\alpha(V_L)\leq\alpha(V)$. Since W is a k-step extension of V_L, our main lemma implies that for a generic $(n-1)$-plane $H\subseteq L$ there is a point $p\in W$ such that $W\supseteq p*(V_L\cap H)=p*(W\cap H)$. In other words, for a generic $(n-1)$-plane $H\subseteq\mathbf{P}^{n+k}$ there is a point $p\in W$ such that $W\supseteq p*(W\cap H)$.

Consider now the set of pairs $(H,p)\in \mathrm{Gr}(n-1,\mathbf{P}^{n+k})\times W$ satisfying the condition $W\supseteq p*(W\cap H)$. It is clear this set is closed; as we have just proved, it contains a component that projects epimorphically onto $\mathrm{Gr}(n-1,\mathbf{P}^{n+k})$ and hence has dimension no less than $\dim \mathrm{Gr}(n-1,\mathbf{P}^{n+k})$. Denote this component by Π and the projection map $(H,p)\mapsto p$ by $\pi:\Pi\to W$. Setting $\dim V=d$, we consider two cases.

Case 1: $\pi(\Pi)$ contains a smooth point of W.

Let p be a generic smooth point of W in $\pi(\Pi)$. I claim that

$$(*)\qquad \dim(\pi^{-1}(p))-\dim(\mathrm{Gr}(n-1,\mathbf{P}^{n+k-2}))-1>0.$$

Indeed, since

$$\dim(\pi^{-1}(p))=\dim\Pi-\dim W \quad\text{and}\quad \dim\Pi\geq\dim(\mathrm{Gr}(n-1,\mathbf{P}^{n+k})),$$

the left-hand side of $(*)$ is no less than

$$\dim(\mathrm{Gr}(n-1,\mathbf{P}^{n+k}))-\dim(\mathrm{Gr}(n-1,\mathbf{P}^{n+k-2}))-d-k-1=2n-d-k-1.$$

Since $k=\alpha(V)+1\leq n$ and $d<n-1$ (V is not a hypersurface), the inequality $2n-d-k-1>0$ holds and $(*)$ is proved.

In view of Lemma 2.1, inequality $(*)$ implies that the union of such n-planes H for which $W\supseteq p*(H\cap W)$ is either a hyperplane in $\mathbf{P}^{n+k}$ or the whole of $\mathbf{P}^{n+k}$. Observe that if $W\supseteq p*(H\cap W)$ then $T_pW\supseteq p*\langle H\cap W\rangle$; because p is a generic point of $\pi(\Pi)$, the variety $H\cap W$ is nondegenerate in H for almost all $H\in\pi^{-1}(p)$. Hence $\langle H\cap W\rangle=H$ for such H and $T_pW\supseteq p*H$ for almost all $H\in\pi^{-1}(p)$. Since the closure of the union of such H's is either the whole of $\mathbf{P}^{n+k}$ or a hyperplane in $\mathbf{P}^{n+k}$, $\dim T_pW\geq n+k-1$. Since

p is smooth on W, this means that W is either a hypersurface or the whole of $\mathbf{P}^{n+k}$, which is impossible. Hence the hypothesis of Case 1 cannot hold.

Case 2: $\pi(\Pi)$ is contained in the set of singular points of W.

Denote the set of singular points of W by Σ. Let p be a generic point of $\pi(\Pi)$. I claim that the union of the $(n-1)$-planes of $\pi^{-1}(p)$ is $\mathbf{P}^{n+k}$. Indeed, since $W \cap \mathbf{P}^n$ is smooth, $\Sigma \cap \mathbf{P}^n = \emptyset$. Hence $\dim \Sigma < k$ and

$$\dim(\pi^{-1}(p)) > \dim \Pi - k \geq \dim(\mathrm{Gr}(n-1, \mathbf{P}^{n+k})) - k.$$

The right-hand side of this inequality is no less than $\dim(\mathrm{Gr}(n-1, \mathbf{P}^{n+k-1}))$, because $\dim(\mathrm{Gr}(n-1, \mathbf{P}^{n+k})) - k - \dim(\mathrm{Gr}(n-1, \mathbf{P}^{n+k-1})) = n-k$ and $k = \alpha(V)+1 \leq n$. Hence $\dim(\pi^{-1}(p)) > \dim(\mathrm{Gr}(n-1, \mathbf{P}^{n+k-1}))$ and our claim follows from Lemma 2.1. Since the union of all the H's of $\pi^{-1}(p)$ is $\mathbf{P}^{n+k}$, it is clear that $W = p * W$; that is, W is a cone with the vertex p. The theorem is proved. □

References

[K] S. L. Kleiman, *The enumerative theory of singularities,* Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 297–396, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[L1] S. L'vovsky, *On the extension of varieties defined by quadratic equations,* Mat. Sb. (N.S.) 135 (1988), 312–324; translation in Math. USSR-Sb. 63 (1989), 305–317.

[L2] ———, *Extensions of projective varieties and deformations, I,* Michigan Math. J. 39 (1992), 41–51.

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