Extensions of Projective Varieties and Deformations, I

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0. Introduction

0.1. In this paper we deal with the following question. Let $V \subset \mathbf{P}^n$ be a projective variety; what are the obstructions for V being a hyperplane section of a projective variety $W \subset \mathbf{P}^{n+1}$? We will say that such a variety W is an *extension* of V.

Of course, if W is a cone with the base V and a vertex in $\mathbf{P}^{n+1} \setminus \mathbf{P}^n$, then V will be its hyperplane section. The point is whether we can find such a W that is not a cone. We will call such a W a *nontrivial extension* of V.

The question we have just asked has quite a long history. As early as 1909, G. Scorza proved that if V is a Veronese variety of dimension greater than 1 or a Segre variety other than $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$, then V admits no nontrivial extension. Nowadays, extensions of projective varieties have been studied by many authors (see Section 0.4 and the references in [L2]). The goal of this paper is twofold: first, to prove a result on non-extendibility of a smooth projective variety; and second, to give an interpretation of the obstruction to extendibility we use in terms of deformation theory.

0.2. PRELIMINARIES. The base field will be the field $\mathbf{C}$ of complex numbers. Let V be a smooth projective variety in $\mathbf{P}^n = \mathbf{P}(E)$, where E is an $(n+1)$-dimensional vector space. Throughout the paper $\mathbf{P}(E)$ denotes Proj Sym(E^*), so closed points of $\mathbf{P}(E)$ are lines in E. From now on, we assume that V is not contained in a hyperplane of $\mathbf{P}^n$, unless stated otherwise.

Let us state some facts known as the folklore. Consider the vector bundle (locally free sheaf) $\Gamma_V = (P^1(O_V(1)))^*$, where P^1 denotes the sheaf of principal parts of the first order. The bundle Γ_V can be included in the following exact sequences:

$$0 \to O_V(-1) \to \Gamma_V \to T_V(-1) \to 0; \tag{0.1}$$

$$0 \to \Gamma_V \to E \otimes O_V \to N_{\mathbf{P}(E)|V}(-1) \to 0. \tag{0.2}$$

Here, T_V denotes the tangent bundle of V and $N_{\mathbf{P}(E)|V}$ denotes the normal bundle of the imbedding $V \subset \mathbf{P}(E)$. The rank of Γ_V equals $\dim V + 1$; if $(\Gamma_V)_p \subset E$ is the fiber of Γ_V at the point $p \in V$ imbedded in E by the injection

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in the sequence (0.2), then $\mathbf{P}((\Gamma_V)_p) \subset \mathbf{P}(E)$ is the projective tangent space to V at p, that is, the union of all the lines passing through p and tangent to V. Consider the Gaussian map γ from V to the Grassmanian of $(\dim V+1)$-dimensional vector subspaces of E sending a point $p \in V$ to the subspace of E whose projectivization is the projective tangent space to V at p; then Γ_V is the pullback via γ of the "universal subbundle" on the Grassmanian.

Presumably all these notions were introduced in 1957 by M. Atiyah [A] (in a slightly modified form). The reader may consult the beginning of [L1] for more details.

0.3. STATEMENT OF RESULTS. Consider the homomorphism

$$\alpha_V : E \to H^0(V, N_{\mathbf{P}(E)|V}(-1))$$

obtained by taking H^0 of the sequence (0.2). Set

$$\alpha(V) = \dim \operatorname{coker}(E \to H^0(V, N_{\mathbf{P}(E)|V}(-1))). \tag{0.3}$$

Now we can state the main results of the paper.

THEOREM 0.1. *If $V \neq \mathbf{P}^n$, V is not a quadric, and $\alpha(V)=0$, then V is not a hyperplane section of a projective variety other than a cone.*

The first proof of Theorem 0.1 was obtained by F. L. Zak in 1984 (unpublished). Zak's proof was based on entirely different ideas; he made use of the theory of projective duality.

COROLLARY 1. *Suppose that $V \neq \mathbf{P}^n$ and V is not a quadric.*

(a) *If $H^1(V, \Gamma_V)=0$ then V is not a hyperplane section of a projective variety other than a cone.*
(b) *If $\dim V \geq 2$ and $H^1(V, T_V(-1))=0$, then V is not a hyperplane section of a projective variety other than a cone.*

This corollary follows immediately from Theorem 0.1, Kodaira's vanishing theorem, and the exact sequences (0.1) and (0.2).

To state the next corollary we must fix some notation (cf. [CHM], [CM]). If M and N are line bundles on a smooth curve C, let $R(M,N)$ be the kernel of the natural map $H^0(C,M)\otimes H^0(C,N) \to H^0(C, M\otimes N)$. Let us denote by $\phi_{M,N}: R(M,N) \to H^0(C, \omega_C \otimes M \otimes N)$ the map $f\otimes g \mapsto g\,df - f\,dg$.

COROLLARY 2. *Let C be a smooth curve imbedded in a projective space via a complete linear system. If the map $\phi_{\omega_C, \mathcal{O}_C(1)}$ is epimorphic, then C is not a hyperplane section of a projective surface other than a cone.*

This corollary follows from Theorem 0.1 and the fact that the cokernel of $\phi_{\omega_C, \mathcal{O}_C(1)}$ is dual to the cokernel of the map in the right-hand side of (0.3) ([W2]; cf. also [CM]).

Corollary 2 implies the following result.

COROLLARY 3. *If C is a canonical curve of genus g and if the Wahl map of C is epimorphic, then C cannot be a hyperplane section of a projective surface other than a cone.*

Here, the Wahl map is the homomorphism

$$W_L: \Lambda^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 3})$$

defined by $s\,dz \wedge t\,dz \mapsto (s(dt/dz) - t(ds/dz))\,dz^3$ (cf. [CHM]). This corollary generalizes slightly some results of Wahl [W1] and of Beauville and Mérindol [BM].

In Section 2 we prove the following technical result which may be of some independent interest.

PROPOSITION 2.1. *Let $V \subset \mathbf{P}^n$ be a smooth projective variety which is neither a projective space nor a quadric. Then, for a generic hyperplane $H \subset \mathbf{P}^n$, there is no nontrivial automorphism of $\mathbf{P}^n$ that fixes all points of H and maps V onto itself.*

0.4. The cokernel of α_V was studied in the 1970s by Schlessinger and Pinkham ([S], [P1], [P2]); if V is projectively normal, then this cokernel is just the weight -1 subspace of the tangent space T^1 to the formal moduli space of the singularity at the vertex of the cone over V. The main construction in our proof is similar to Pinkham's construction of sweeping out a hyperplane through a cone [P1, Remark 7.6(iii)]. The difference is that we sweep a hyperplane not through a cone but through a nontrivial extension of V.

As the referee pointed out to me, under some additional assumptions Theorem 0.1 can be derived from the results of Pinkham and Schlessinger (one should assume not only that α_V is epimorphic but that *all* negative weight subspaces of T^1 vanish, i.e., $H^0(V, N_{\mathbf{P}(E)|V}(-i)) = 0$ for all $i \geq 2$, and that V is projectively normal; cf. Badescu [B]). Hence, from this point of view it may seem necessary that the hypotheses of Theorem 0.1 contain the vanishing of $H^0(V, N_{\mathbf{P}(E)|V}(-i))$ for $i > 1$ as well. However, it turns out that the hypotheses of our Theorem 0.1 imply this vanishing. This result can be obtained by putting together a construction used by Zak in his proof of Theorem 0.1 with an idea of Badescu [B, Thm. 5]. The proof will be published elsewhere.

Corollary 1(b) is similar to some results of Fujita [F], who studied the varieties that cannot be ample divisors rather than hyperplane sections. Fujita's results imply that a smooth projective variety $V \subset \mathbf{P}^n$ of dimension ≥ 2 admits no nontrivial extension, provided that $H^1(V, T_V(-i)) = 0$ for all $i > 0$. Again, it may be shown that vanishing of $H^1(V, T_V(-1))$ implies vanishing of $H^1(V, T_V(-i))$ for all $i > 1$, provided that V is neither a quadric nor a projective space.

0.5. AN OUTLINE OF THE PROOF OF THE THEOREM. Let $H \subset \mathbf{P}(E)$ be a generic hyperplane and $X = V \cap H$. We show that if $\alpha(V) = 0$ then almost all deformations of V within the family of subvarieties of $\mathbf{P}(E)$ containing X are induced by projective automorphisms of $\mathbf{P}(E)$ leaving every point of H fixed. Vaguely speaking, $\alpha(V)$ measure "the number of nontrivial deformations" of V within the family of subvarieties of $\mathbf{P}(E)$ containing X.

Now if $W \subset \mathbf{P}^{n+1}$ is an extension of V, we consider the pencil of hyperplanes in $\mathbf{P}^{n+1}$ passing through H. Projecting the sections of W by the hyperplanes of the pencil into $\mathbf{P}^n$, we obtain a family of subvarieties of $\mathbf{P}(E)$ containing X. Making use of the fact that almost all subvarieties in this family are images of V under the action of automorphisms of $\mathbf{P}(E)$, we conclude that W is a cone.

0.5. A preliminary version of this paper was deposited at VINITI (Soviet Institute of Scientific Information) in the beginning of 1987 [L2]. There the reader may find some straightforward but tedious proofs that we have omitted from Section 1 of this paper.

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1. The Interpretation of $\alpha(V)$ in Terms of Deformation Theory

Let X be a closed subscheme of $\mathbf{P}^n$. We will denote by Z_X the Hilbert scheme of the closed subschemes of $\mathbf{P}^n$ containing X, that is, the scheme representing the functor

$$Z_X \colon S \mapsto \{\text{closed subschemes of } S \times \mathbf{P}^n \text{ flat over } S \text{ and containing } S \times X\}.$$

If $I_X \subset O_{\mathbf{P}^n}$ is the sheaf of ideals of X, then this functor is isomorphic to Grothendieck's functor $\mathrm{Quot}_{I_X/\mathbf{P}^n/\mathrm{Spec}\,\mathbf{C}}$ (see [G]), so Z_X exists and each of its components is a projective scheme.

PROPOSITION 1.1. (a) *If $V \subset \mathbf{P}^n$ is a closed subscheme of $\mathbf{P}^n$ containing X and if $I_V \subset O_{\mathbf{P}^n}$ is the sheaf of ideals of V, then the Zariski tangent space to Z_X at the closed point corresponding to V is* $\mathrm{Hom}_{O_V}(I_V/I_V^2, I_{X,V})$, *where $I_{X,V} \subset O_V$ is the sheaf of ideals of X as a subscheme of V.*

(b) *If V is a nonsingular subvariety of $\mathbf{P}^n$ and X is a hyperplane section of V, then the same Zariski tangent space is isomorphic to $H^0(V, N_{\mathbf{P}^n|V}(-1))$.*

The proof is quite straightforward and will be omitted.

Now let H be a hyperplane in $\mathbf{P}^n = \mathbf{P}(E)$. Consider the group G_H of projective automorphisms of $\mathbf{P}^n$ leaving all points of H fixed. Choose the basis

$\langle e_0, e_1, \ldots, e_n \rangle$ of E so that H will be the projectivization of the linear span of vectors $e_0, e_1, \ldots, e_{n-1}$. Then G_H is isomorphic to the group of linear automorphisms of E with the matrix

$$\|c_{ij}\| = \begin{pmatrix} 1 & 0 & \cdots & a_0 \\ & 1 & \cdots & a_1 \\ & 0 & \ddots & a_{n-1} \\ & & & a_n \end{pmatrix},$$

where $a_n \neq 0$, $c_{ij} = 0$ if $i > j$ or $i < j < n$, and $c_{ij} = 1$ if $i = j < n$.

PROPOSITION 1.2. *The tangent space to G_H at the identity is isomorphic to E.*

Now let V be a smooth projective subvariety of $\mathbf{P}^n = \mathbf{P}(E)$, H a hyperplane in $\mathbf{P}^n$, and $X = V \cap H$. Consider the morphism $\phi: G_H \to Z_X$ which acts on closed points by sending $g \in G_H$ to the point corresponding to the subscheme $g^{-1} V \subset \mathbf{P}^n$. The formal definition of ϕ is as follows: ϕ is the G_H-valued point of Z_X defined as the left-hand side of the Cartesian square

$$\begin{array}{ccc} \bullet & \longrightarrow & V \\ \downarrow & & \downarrow^{i} \\ G_H \times \mathbf{P}^n & \xrightarrow{a} & \mathbf{P}^n, \end{array} \tag{1.1}$$

where i is the inclusion of V into $\mathbf{P}^n$ and a is the morphism corresponding to the action of G_H on $\mathbf{P}^n$.

The derivative of ϕ at the identity of G_H is a homomorphism from the tangent space to G_H at the identity to the Zariski tangent space to Z_X at the point corresponding to V. Propositions 1.1(b) and 1.2 imply that the former of those spaces is E and the latter is $H^0(V, N_{\mathbf{P}^n|V}(-1))$.

PROPOSITION 1.3. *The derivative $d\phi: E \to H^0(V, N_{\mathbf{P}(E)|V}(-1))$ coincides with the homomorphism α_V defined in Section 0.3.*

The proof of this proposition is straightforward but tedious; we omit it and refer the reader to [L2].

Proposition 1.3 shows that, vaguely speaking, $\alpha(V)$ is the codimension in Z_X of the G_H-orbit of V, that is, the "number of nontrivial deformations of V with a fixed hyperplane section".

Now suppose that $V \subset \mathbf{P}(E) = \mathbf{P}^n$ is a smooth projective variety and that $H \subset \mathbf{P}^n$ is a hyperplane transversal to V. The following proposition summarizes what we will need in the sequel from the deformation theory.

PROPOSITION 1.4. *Suppose that the following hypotheses are satisfied:*

(a) *there is no non-identical automorphism of $\mathbf{P}^n$ leaving points of H fixed and mapping V into itself, and*

(b) $\alpha(V) = 0$;

let $\{V_s\}_{s\in S}$ *be a flat family of subvarieties of* $\mathbf{P}^n$ *with a quasi-projective base* S *such that all the* V_s*'s contain* X *and* $V_{s_0}=V$ *for some* $s_0\in S$*. Then there is a Zariski-open subset* $U\subset S$ *containing* s_0 *and a morphism* $f\colon U\to G_H$ *such that* $V_s=f(s)\cdot V$ *for all* $s\in U$*, and* $f(s_0)$ *is the identity in* G_H*.*

Proof. We begin with a simple lemma.

LEMMA 1.5. *If* $V\neq\mathbf{P}(E)$ *then* $H^0(V,\Gamma_V)=0$.

Proof of the lemma. Since $\Gamma_V\subset E\otimes O_V$, $H^0(\Gamma_V)\subset H^0(E\otimes O_V)=E$. Now if v is a nonzero global section of Γ_V and $z\in\mathbf{P}(E)$ is the point corresponding to $v\in E$, then $z\in T_pV$ for any closed point $p\in V$, where T_pV is the projective tangent space (see Section 0.2). Hence, the dual variety $V^*\subset(\mathbf{P}(E))^*$ lies in the hyperplane of $(\mathbf{P}(E))^*$ corresponding to z. Now by the projective duality theorem [La, Thm. 2.2], V is a cone with the vertex z, which is impossible since V is smooth and $V\neq\mathbf{P}(E)$. This contradiction proves the lemma. □

The lemma and the exact sequence (0.2) imply that α_V is injective. Now from Proposition 1.3 and the hypothesis (b) it follows that $d\phi$ is an isomorphism at the identity of G_H and hence, since ϕ is G_H-equivariant, everywhere on G_H. From (a) it follows that ϕ is a one-to-one map on the set of closed points. From these two observations it follows easily that $\phi\colon G_H\to Z_X$ is an open inclusion.

Consider the map $\psi\colon S\to Z_X$ induced by the family $\{V_s\}$ and set $U=\psi^{-1}(G_H)$, where we consider G_H as imbedded in Z_X by ϕ. Let us restrict ψ to U and consider ψ as a morphism from U to G_H. Since the pullback of the universal family over Z_X to G_H is given by the left-hand arrow of diagram (1.1), $\psi(s)^{-1}V=V_s$ for every $s\in U$. Set $f(s)=(\psi(s))^{-1}$ where -1 denotes the inverse element in G_H. The morphism $f\colon U\to G_H$ has the desired property. □

REMARK. In the next section we will show that if a smooth variety V is not a projective space or a quadric, then the hypothesis (a) of the proposition is true for almost all hyperplanes $H\subset\mathbf{P}^n$.

2. Smooth Varieties Cannot Have Too Many Symmetries

PROPOSITION 2.1. *Let* $V\subset\mathbf{P}^n$ *be a smooth projective variety that is neither a projective space nor a quadric. Then, for a generic hyperplane* $H\subset\mathbf{P}^n$*, there is no nontrivial automorphism of* $\mathbf{P}^n$ *that fixes all points of* H *and maps* V *onto itself.*

For the proof we will need a theorem of Mori–Sumihiro and the notion of Lefschetz pencil.

THEOREM OF MORI-SUMIHIRO [MS] (a weakened version). *If* $V\subset\mathbf{P}^n$ *is a smooth projective variety which is not a projective space or a plane conic, and if* T_V *is the tangent bundle of* V*, then* $H^0(V,T_V(-1))=0$.

Proof. If $\dim V \geq 2$, then from the sequence (0.1) and the Kodaira vanishing theorem it follows that $H^0(V, T_V(-1)) \neq 0 \Rightarrow H^0(V, \Gamma_V) \neq 0$. The latter inequality is impossible by Lemma 1.5. □

If $\dim V = 1$ then the proof is even simpler, and is left to the reader.

LEFSCHETZ PENCILS. If $\mathbf{P}^n$ is a projective space and $L \subset \mathbf{P}^n$ is a linear subspace of codimension 2, then the pencil of hyperplanes with the axis L consists of all hyperplanes containing L. The main result on Lefschetz pencils in characteristic 0 is in the following theorem.

THEOREM. *If $V \subset \mathbf{P}^n$ is a smooth projective variety not contained in a hyperplane, then for almost all linear subspaces $L \subset \mathbf{P}^n$ of codimension 2 the pencil of hyperplanes with the axis L has the following properties:*

(a) *L is transversal to V;*
(b) *for all but a finite number of hyperplanes $H \supset L$, the intersection $H \cap V$ is a smooth variety;*
(c) *for those $H \supseteq L$ whose intersections with V are not smooth, this intersection $H \cap V$ has only one singular point, and the singularity of $H \cap V$ at this point is analytically isomorphic to the singularity at the vertex of a cone over a nonsingular projective quadric.*

A pencil of hyperplanes satisfying conditions (a)–(c) is called a *Lefschetz pencil.* The proof of this theorem is contained in [SGA7, Exposé 17].

Proof of Proposition 2.1. Suppose that $V \subset \mathbf{P}^n$ is a smooth projective variety such that for each hyperplane $H \subset \mathbf{P}^n$ there exists a nontrivial projective automorphism $g_H : \mathbf{P}^n \to \mathbf{P}^n$ such that $g_H|_H = \mathrm{id}$ and $g_H(V) = V$. To prove the proposition it suffices to show that such V must be a projective space or a quadric.

First note that we may assume that all the automorphisms g_H are of finite order. Indeed, if the order of some g_H were not finite, then g_H would generate a subgroup $G \subset \mathrm{Aut}(\mathbf{P}^n)$ of a positive dimension, all of whose elements map V into itself and fix points of H. The corresponding action of Lie(G) on $H^0(V, T_V)$ would give rise to a nonzero section of T_V zero on $V \cap H$, that is, a section of $T_V(-1)$. According to the theorem of Mori–Sumihiro, this is only possible if V is a projective space or a plane conic (i.e., a quadric).

Now choose a generic linear subspace $L \subset \mathbf{P}^n$ of codimension 2 which is the axis of a Lefschetz pencil. A nontrivial g_H cannot correspond to almost all hyperplanes H of that pencil (otherwise such a g_H would be the identity on almost all $H \supset L$, hence on the whole of $\mathbf{P}^n$). So the nontrivial g_H's for $H \supset L$ generate a subgroup $G \subset \mathrm{Aut}(\mathbf{P}^n)$ of positive dimension.

Since the g_H's are of finite order and the characteristic of the base field is 0, the g_H's are semi-simple. Since G contains nontrivial semi-simple elements, it contains a subgroup isomorphic to the multiplicative group G_m.

Since G and its subgroup G_m act trivially on L and the representations of G_m are completely reducible, we can choose a system of homogeneous coordinates $(x_0 : \cdots : x_{n-1} : x_n)$ in $\mathbf{P}^n$ such that:

(i) $L \subset \mathbf{P}^n$ is defined by the equations $x_{n-1} = x_n = 0$; and
(ii) there exist integers α and β such that $\alpha \neq 0$ and the action of $G_m \subset G$ on $\mathbf{P}^n$ is given by

$$(t, (x_0 : \cdots : x_{n-1} : x_n)) \mapsto (x_0 : \cdots : x_{n-2} : t^\alpha x_{n-1} : t^\beta x_n),$$

where t is an element of G_m (i.e., a nonzero complex number).

Now let H be the hyperplane defined by the equation $x_n = 0$, and set $X = V \cap H$ and $Y = V \cap L$. Since L is transversal to V, X is not contained in L. Now take any point $(x_0 : \cdots : x_{n-2} : x_{n-1} : 0)$ in $X \setminus L$ such that not all the x_j for $0 \le j \le n-2$ are zeros. The closure of the G-orbit of the point $(x_0 : \cdots : x_{n-2} : x_{n-1} : 0) \in V$ is the set of all points with coordinates $(x_0 : \cdots : x_{n-2} : c : 0)$ with all possible c. We have proved that if x contains a point $p \notin Y$, then it contains the line joining it with the point $(0 : \cdots : 0 : 1 : 0) \in H$. Hence X is the cone over Y with the vertex $(0 : \cdots : 0 : 1 : 0)$. Since the pencil of hyperplanes with the axis L is a Lefschetz pencil, the cone X must be either smooth or have an ordinary quadratic singularity at the vertex. In the former case, X must be a linear subspace of H, hence V is a projective space; in the latter case Y must be a quadric, hence V must also be a quadric. This completes the proof. □

3. Proof of Theorem 0.1

We begin with two simple lemmas.

LEMMA 3.1. *Let $V \subset \mathbf{P}^n$ be a smooth projective variety over $\mathbf{C}$, and suppose that the homogeneous coordinates $(x_0 : \cdots : x_n)$ in $\mathbf{P}^n$ are chosen in such a way that the hyperplane H defined by the equation $x_n = 0$ is transversal to V. Suppose $p = (x_0 : \cdots : x_{n-1} : 0)$ is a point in $V \cap H$. Then:*

(a) *For each $\xi \in \mathbf{C}$, complex number $c \neq 0$, and positive integer m, there exist an open disk Δ in the complex plane such that $\xi \in \Delta$ and a holomorphic mapping $h : \Delta \to V$, written in homogeneous coordinates as $h(t) = (h_0(t) : \cdots : h_{n-1}(t) : h_n(t))$, such that $h_i(\xi) = x_i$ for $0 \le i \le n-1$ and*

$$\lim_{t \to \xi} \frac{h_n(t)}{c(t-\xi)^m} = 1.$$

(b) *For each complex number $c \neq 0$ and positive integer m there exists a holomorphic mapping h from the exterior of a disk in $\mathbf{C}$ into V such that, when written in homogeneous coordinates in the form $h(t) = (h_0(t) : \cdots : h_{n-1}(t) : h_n(t))$, h satisfies the following conditions:*

$$\lim_{|t| \to \infty} h_i(t) = x_i \quad \text{for } 0 \le i \le n-1; \qquad \lim_{|t| \to \infty} t^m h_n(t) = c.$$

Proof. It suffices to prove part (a) for $\xi = 0$; the rest can be reduced to that case by a simple change of parameter. In this case, since V is transversal to the hyperplane H, it is obvious that there exists a neighbourhood of zero Δ and a holomorphic map $g : \Delta \to V$ which is transversal to H and such that

$g(0) = (x_0 : \cdots : x_{n-1} : 0)$. Hence, if we write $h(z) = (g_0(z) : \cdots : g_n(z))$, we have (after dividing by the appropriate power of z) $g_i(0) = x_i$ for $0 \le i \le n-1$, and $g_n(z)/\lambda z \to 1$ as z tends to zero, where λ is a nonzero constant. Now if we make a change of parameter $z = at^m$ with the appropriate nonzero constant a, we obtain the desired map. □

LEMMA 3.2. *If $V \subset \mathbf{P}^n$ is a smooth projective variety such that $\alpha(V) = 0$ and V is not a quadric or a projective space, then V is not a hypersurface.*

Proof. If V is a hypersurface of degree d, the exact sequence (0.2) becomes

$$0 \to \Gamma_V \to O_V^{n+1} \to O_V(d-1) \to 0.$$

Hence, $\alpha(V) \ge h^0(O_V(d-1)) - n - 1$; if $d > 2$, the right-hand side of this inequality is positive, contrary to the assumption $\alpha(V) = 0$. Hence, V cannot be a hypersurface unless V is a quadric or a projective space. □

Now we turn to the crucial step in the proof of Theorem 0.1. Suppose that a smooth projective variety $V \subset \mathbf{P}^n$ is a hyperplane section of a projective variety $W \subset \mathbf{P}^{n+1}$. Assuming that $\alpha(V) = 0$ and that V is not a quadric or a projective space, we must show that W is a cone over V.

LEMMA 3.3. *Suppose that $H \subset \mathbf{P}^n$ is a hyperplane satisfying the following hypotheses:*

(a) *$X = V \cap H$ is smooth and its linear span is H, and*

(b) *there is no nontrivial automorphism of $\mathbf{P}^n$ mapping V into itself and fixing all the points of H.*

Then there is a singular point $p \in W$ such that the intersection of W with the hyperplane spanned by p and H is the cone over $V \cap H$ with the vertex p.

Proof. Choose the homogeneous coordinates $(x_0 : \cdots : x_{n+1})$ in $\mathbf{P}^{n+1}$ so that $\mathbf{P}^n$ and H are defined by $x_{n+1} = 0$ and $x_n = x_{n+1} = 0$, respectively. Set $X = V \cap H$.

Every hyperplane in $\mathbf{P}^{n+1}$ containing H can be defined either by equation $x_{n+1} = tx_n$ (we will call such a hyperplane H_t) or by equation $x_n = 0$ (we will call this hyperplane H_∞). Let us denote by $\pi : \mathbf{P}^{n+1} \to \mathbf{P}^n$ the projection $(x_0 : \cdots : x_n : x_{n+1}) \to (x_0 : \cdots : x_n : 0)$.

Now, for every $t \in \mathbf{C}$, consider the variety $V_t = \pi(W \cap H_t) \subset \mathbf{P}^n$. The varieties $\{V_t\}$ form a family of subvarieties of $\mathbf{P}^n$ containing X, and it is clear that this family is flat. By Proposition 1.4 there exists a rational map $f: \mathbf{A}^1 \to G_H$, where G_H is the group of automorphisms of $\mathbf{P}^n$ which fix the points of H, so that $f(0)$ is the identity of G_H and $V_t = f(t) \cdot V$ for all $t \in \mathbf{A}^1$ for which f is defined. Writing f in matrix form we find that there are rational functions $a_0(t), \ldots, a_n(t)$ such that $a_j(0) = 0$ for $0 \le j \le n-1$, $a_n(0) = 1$, and, for each point $(x_0 : \cdots : x_n : 0)$ in V, the point

$$(x_0 + a_0(t)x_n : \cdots : x_{n-1} + a_{n-1}(t)x_n : a_n(t)x_n : 0)$$

belongs to V_t provided that all the a_j's are defined at t. Recalling the definition of V_t, we obtain that for each point $(x_0 : \cdots : x_n : 0) \in V$ and for each t

for which all a_j's are defined, the point

$$(x_0+a_0(t)x_n:\cdots:x_{n-1}+a_{n-1}(t)x_n:a_n(t)x_n:ta_n(t)x_n)$$

lies in W.

Now consider two cases.

Case 1: Not all a_j's are polynomials.

Let m be the maximal order of poles of a_j's. Suppose it is attained at the point ξ ($\xi\neq 0$ since all a_j's are finite at 0). If $b_j=\lim_{t\to\xi}(t-\xi)^m a_j(t)$, then all b_j's are finite and not all of them are zero.

Now for each point $(x_0:\cdots:x_{n-1}:0:0)\in V\cap H$ and each $c\neq 0$, consider a holomorphic map $h:\Delta\to V$ from a disk Δ containing ξ into V, given by the formula $h(t)=(h_0(t):\cdots:h_{n-1}(t):h_n(t):0)$, such that $h_i(\xi)=x_i$ for $0\le i\le n-1$ and $h_n(t)/c(t-\xi)^m\to 1$ as t tends to ξ (such mapping exists in view of Lemma 3.1(a)).

For each $t\in\Delta$, the point

$$p(t) \\ =(h_0(t)+a_0(t)h_n(t):\cdots:h_{n-1}(t)+a_{n-1}(t)h_{n-1}(t):a_n(t)h_n(t):ta_n(t)h_n(t))$$

belongs to W. As t goes to ξ, $p(t)$ tends to the point

$$(x_0+cb_0:\cdots:x_{n-1}+cb_{n-1}:cb_n:\xi cb_n)\in W.$$

If $b_n=0$, then this point lies in $W\cap H=X$, whence X is a cone with the vertex $(b_0:\cdots:b_{n-1}:0:0)\in H$. But this is impossible since X is smooth and not a projective space. Hence $b_n\neq 0$; then $W\cap H_\xi$ is a cone over X with the vertex $p=(b_0:\cdots:b_n:\xi b_n)$. Since X spans H, the dimension of the Zariski tangent space to this cone at its vertex is n. In view of Lemma 3.2, $\dim W<n$, so the point $p\in W$ must be singular.

Case 2: All a_j's are polynomials.

Let us denote ta_n by a_{n+1}, and let m be the maximal degree of a_j, $0\le j\le n+1$. Set $b_j=\lim_{|t|\to\infty}a_j/t^m$ for $0\le j\le n+1$. Because $\deg a_{n+1}>\deg a_n$, $b_n=0$.

Lemma 3.1(b) and an argument similar to that given for Case 1 yield that, for each point $(x_0:\cdots:x_{n-1}:0:0)\in X$ and each $c\neq 0$, the point $(x_0+cb_0:\cdots:x_{n-1}+cb_{n-1}:0:cb_{n+1})$ belongs to W. If $b_{n+1}=0$ then X is a cone, which is impossible. If $b_{n+1}\neq 0$ then $W\cap H_\infty$ is a cone over X with the vertex $(b_0:\cdots:b_{n-1}:0:b_{n+1})$, and by the same logic as used in Case 1 we conclude that this point is singular on W. □

Completion of the proof of the theorem. Since $W\cap\mathbf{P}^n$ is a smooth variety, W has only a finite number of singular points. On the other hand, it follows from Proposition 2.1 that the hypotheses of Lemma 3.3 are satisfied for almost all of the hyperplanes $H\subset\mathbf{P}^n$. Since W has only a finite number of singular points, it follows from Lemma 3.3 that there is a singular point $p\in W$ such that, for almost all hyperplanes $H\subset\mathbf{P}^n$, the intersection of W

with the hyperplane in $\mathbf{P}^{n+1}$ spanned by H and p is a cone with the vertex p. This is possible only if W is a cone with the vertex p. The theorem is proved. □

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