Roughness Properties of Norms on Non-Asplund Spaces

GILLES GODEFROY & VACLAV ZIZLER

I. Introduction

In this note we apply James' techniques on norm-attaining linear forms to roughness properties of *every* equivalent norm on a non-Asplund space. We answer in particular a question of Klee [10] which we recall: A Banach space X is said to have the *A-property* [1] if for every norm-compact subset K of X, the restriction of the duality mapping J to K has a selector σ_K such that $\sigma_K(K)$ is norm-relatively compact. Klee asked whether every space, or separable space, can be renormed to have the A-property; it follows from Theorem II.1 that only Asplund spaces may have such a norm. The proof relies heavily on an inequality of Simons [16].

NOTATION. We work in real Banach spaces, and keep notation which is standard in Banach space theory. In particular, $S_1(X)$ and X_1 denote the unit sphere and the unit ball of X, respectively. More generally, if $r>0$ and $x\in X$, then $B_r(x)$ denotes the closed ball of radius r centered at x. We denote by J the duality maping of X, that is, the multivalued map of X into X^* defined by

$$J(x)=\{y\in X^*\mid y(x)=\|y\|^2=\|x\|^2\}.$$

We refer to ([12], [9], [4]) for the construction of rough norms on non-Asplund spaces, and to ([3], [18]) for James' theorem and its applications.

ACKNOWLEDGMENT. This work was done during the visit of the first-named author to the University of Alberta in Edmonton in November 1989. It is his pleasure to thank the Faculty of Science and especially the Department of Mathematics of the University of Alberta for their generous hospitality and support. The authors also thank the referee for helpful suggestions which improved the final version of this note.

II. The Results

The following result asserts that, if X is not an Asplund space, then there are points at which the norm is very "not Frechét differentiable".

Received May 30, 1990. Revision received April 1, 1991.
Supported in part by NSERC (Canada).
Michigan Math. J. 38 (1991).

THEOREM II.1. *Let X be a Banach space which is not an Asplund space. Then for every $\epsilon > 0$, there is a norm-convergent sequence $\{x_n\}$ in $S_1(X)$ such that, for every $n \neq k$,*

$$\operatorname{dist}(J(x_n), J(x_k)) > 1 - \epsilon.$$

Proof. Let us assume first that X is separable. Then, by a result of Mazur [14], the set G of points of $S_1(X)$ at which the norm is Gâteaux-smooth is a norm-dense G_δ of $S_1(X)$. We recall that $x \in G$ if and only if $J(x)$ is reduced to a point.

We now fix a countable dense subset D of G. For every $x \in S_1(X)$, we pick a sequence $\{x_n\}$ in D which is norm-convergent to x, and we also pick a w^*-cluster point $\sigma(x)$ of $\{J(x_n)\}$ in (X_1^*, w^*); it is clear that $\sigma(x) \in J(x)$. We let

$$B = \{\sigma(x) \mid x \in S_1(X)\}.$$

The following lemma is the crucial point of the proof; it relies on Simons' inequality [16], which itself stems from James' characterization of weakly compact sets [7].

LEMMA II.2. *Given $\epsilon > 0$, there exists $x_0 \in S_1(X)$ such that for every $x \in D$, $\|\sigma(x_0) - J(x)\| > 1 - \epsilon$.*

Proof. If the result is false, we have

$$B \subset \bigcup \{B_{1-\epsilon}(J(x)) \mid x \in D\}. \tag{1}$$

Since X^* is not separable and D is countable, there is $z \in X^{**}$ with $\|z\| = 1$ and $z(J(x)) = 0$ for every $x \in D$. We pick $y_0 \in X_1^*$ such that $z(y_0) > 1 - \epsilon/2$.

Since X_1 is w^*-dense in X_1^{**}, z belongs to the closure of X_1 for the topology of pointwise convergence on $J(D) \cup \{y_0\}$, and thus we may find $\{x_n\}$ in X_1 such that

$$\forall n, \quad x_n(y_0) > 1 - \epsilon/2 \tag{2}$$

and

$$\forall x \in D, \quad \lim_{n \to \infty} x_n(J(x)) = 0. \tag{3}$$

Clearly, (1), (3), and $\|x_n\| \leq 1$ imply

$$\forall y \in B, \quad \overline{\lim_{n \to \infty}}\, x_n(y) \leq 1 - \epsilon. \tag{4}$$

Observe that every $x \in X$ attains its norm at some point of B. Therefore, by using the Simons' inequality [16, Thm. 3], (4), and (2), it follows that

$$1 - \epsilon \geq \sup_{y \in B}\Big[\overline{\lim_{n \to \infty}}\, x_n(y)\Big] \geq \overline{\lim_{n \to \infty}}\, x_n(y_0) \geq 1 - \frac{\epsilon}{2}.$$

This contradiction completes the proof of Lemma II.2. □

We now come back to the proof of Theorem II.1. If $x_0 \in S_1(X)$ is the point provided by Lemma II.2, then (by construction of σ) there exists a sequence $\{x_n\}$ in D such that $\lim\|x_0 - x_n\| = 0$ and $\sigma(x_0)$ is a w^*-cluster point of $J(x_n)$. By taking a subsequence, we may and do assume that

$$(5) \qquad \sigma(x_0) = w^* - \lim_{n\to\infty} J(x_n)$$

in (X^*, w^*). Now, by Lemma II.2, we have

$$\|\sigma(x_0) - J(x_n)\| > 1 - \epsilon$$

for every n; this implies, by (5) and the w^*-lower semicontinuity of the norm, that

$$\forall n \geq 1, \quad \varliminf_{k\to\infty} \|J(x_k) - J(x_n)\| > 1 - \epsilon.$$

It is then easy to construct, by induction, a subsequence $\{x_n'\}$ of $\{x_n\}$ such that $\|J(x_n') - J(x_k')\| > 1 - \epsilon$ for every $n \neq k$. This concludes the proof if X is separable.

The general case will now follow from the separable one. If Y is any non-Asplund space, then Y contains a separable subspace X such that X^* is non-separable. By the above, there is a norm-convergent sequence $\{x_n\} \subset S_1(X)$ of points of Gâteaux-smoothness of the norm of X such that

$$\|J(x_n) - J(x_k)\| > 1 - \epsilon \quad \text{for} \quad n \neq k.$$

If $Q: Y^* \to X^*$ is the canonical quotient map and $\tilde{J}: S_1(Y) \to S_1(Y^*)$ is the duality mapping of Y, the restriction of $(Q \circ \tilde{J})$ to X is equal to J; because $\|Q\| = 1$, it follows that

$$\operatorname{dist}(\tilde{J}(x_n), \tilde{J}(x_k)) > 1 - \epsilon$$

for every $n \neq k$; this concludes the proof. □

It is not clear to us whether or not it is possible to improve Theorem II.1 by replacing in its statement the norm-convergent sequence by a norm-compact set without isolated points. However, it is possible to do so if we replace the mapping J by one of its "reasonable" selectors. Indeed, we have the next proposition.

PROPOSITION II.3. *Let X be a Banach space that is not an Asplund space. Then, for every $\epsilon > 0$, there exists a subset K of $S_1(X)$ which is norm homeomorphic to the Cantor set $\{0,1\}^\omega$, and a selector σ of the duality mapping J such that*

$$\|\sigma(x) - \sigma(x')\| > 1 - \epsilon$$

for every $x \neq x'$ in K.

Proof. Again, we first assume that X is separable. It follows from [8, Thm. 3] that then there exists a selector $\sigma: X \to X^*$ of the duality mapping which is (norm-w^*) of the first Borel class. If we let $B=\sigma(S_1(X))$, then B is a w^*-analytic subset of $S_1(X^*)$ (cf., e.g., [11, 38.III.5]) on which every element of X attains its norm. This latter condition allows us to apply the proof of Lemma II.2 which shows that B is not contained in a countable union of balls of radius $(1-\epsilon)$. Since B is w^*-analytic, this implies that there is a subset K_0 of B, w^*-homeomorphic to $\{0,1\}^\omega$, such that $\|y-y'\|>1-\epsilon$ for every $y\neq y'$ in K_0. This follows from a more general result ([5, Lemma 2.2]; [13]). For the reader's convenience, we include a few hints on how to prove it directly.

If B is a continuous image of a Polish space P and d is the semimetric on P obtained by lifting the norm metric on B, then the following statement holds true: Given $\delta>0$, if P is not δ-separable in d, then there are $x_0, x_1\in P$ such that $d(x_0,x_1)>\delta$ and no neighborhood of x_0 or x_1 in P is δ-separable in d. To see that this statement holds true, put $D=\{x\in P$, no neighborhood of x in P is δ-separable in $d\}$. From the Lindelöf property of P it follows that $P\backslash D$ is δ-separable in d. Because P is not δ-separable in d, it follows that D is not δ-separable in d and the statement follows. From the statement and the lower semicontinuity of d on P it follows that there are points x_0 and x_1 in P and closed neighborhoods V_0 and V_1 in P (of x_0 and x_1, respectively) such that $d(V_0,V_1)>\delta$. Because both V_0 and V_1 are not δ-separable in d, we can apply the above argument to both of them. It is then clear how to complete the construction of a Cantor-like set K_0 in B.

If we now let $\Omega=\sigma^{-1}(K_0)$, then Ω is a G_δ set since σ is (norm-w^*) of the first Borel class. Since Ω is obviously uncountable, it contains a Cantor subset K which clearly works since $\sigma(K)\subseteq K_0$ (cf. [17, Thm. 119]).

The general case can be done as before: If Y is not an Asplund space, then it contains a separable subspace X with a nonseparable dual; by the above argument there is a Cantor subset K of $S_1(X)$ and a selector σ satisfying the above conditions; if now $j: X^*\to Y^*$ is a map such that, for $x^*\in X^*$, $\|j(x^*)\|=\|x^*\|$, $Qj=\mathrm{Id}_{X^*}$, $\tilde{\sigma}=j\sigma$, and $\hat{\sigma}$ is a selector: $Y\to Y^*$ which extends $\tilde{\sigma}$, it follows from $\|Q\|=1$ that K and $\hat{\sigma}$ work. □

REMARKS II.4. (1) When the above statements are compared with Šmulyan's lemma (see [2, p. 29]), they naturally appear as "roughness" assertions. For instance, Proposition II.3 means that *any* norm on a non-Asplund space is "uniformly rough" when restricted to an appropriate Cantor set; note that conversely, such a norm cannot exist on a separable Asplund space X, since the conditions of Proposition II.3 clearly imply that X^* is nonseparable. The same techniques provide several statements which stress the dichotomy between Asplund and non-Asplund spaces.

(2) A connection between these results and James' constructions of trees in non-superreflexive spaces is provided by the notion of flat Banach space; recall that X is said to be *flat* if there exist $x\in S_1(X)$ and a 2-Lipschitz map g from $[0,1]$ into $S_1(X)$ such that $g(0)=x$ and $g(1)=-x$; for this notion and related ones we refer to [15]. If X is flat then $K=g([0,1])$ is a norm-compact

subset of $S_1(X)$ such that $\text{dist}(J(x'), J(x'')) = 2$ for every $x' \neq x''$ in K (see [6]). However a non-Asplund Banach space is not necessarily isomorphic to a flat space, since (for example) a space with the Radon–Nikodym property cannot be flat; a proof of this latter fact is provided by the observation that the 2-Lipschitz map $g: [0,1] \to S_1(X)$ which joins two antipodal points is nowhere differentiable.

(3) We do not know whether, when X is separable and X^* is not, every subset B of $S_1(X^*)$ on which every $x \in X$ attains its norm contains an uncountable biorthogonal system. Note that non-norm-separable and w^*-analytic subsets of dual spaces contain such systems [19]; under a determinacy axiom, this is also true in the w^*-projective hierarchy (see [5]).

References

1. P. M. Anselone, *Compactness properties of sets of operators and their adjoints,* Math. Z. 113 (1970), 233–236.
2. J. Diestel, *Geometry of Banach spaces, selected topics,* Lecture Notes in Math., 485, Springer, Berlin, 1975.
3. K. Floret, *Weakly compact sets,* Lecture Notes in Math., 801, Springer, Berlin, 1980.
4. G. Godefroy, *Metric characterization of first Baire class linear forms and octahedral norms,* Studia Math. 95 (1989), 1–15.
5. G. Godefroy and A. Louveau, *Axioms of determinacy and biorthogonal systems,* Israel J. Math. 67 (1989), 109–116.
6. R. E. Harrell and L. A. Karlovitz, *On tree structures in Banach spaces,* Pacific J. Math. 59 (1975), 85–91.
7. R. C. James, *Weakly compact sets,* Trans. Amer. Math. Soc. 113 (1964), 129–140.
8. J. E. Jayne and C. A. Rogers, *Borel selectors for upper semicontinuous set-valued maps,* Acta Math. 155 (1985), 41–79.
9. K. John and V. Zizler, *On rough norms on Banach spaces,* Comment. Math. Univ. Carolin. 19 (1978), 335–349.
10. V. Klee, *Two renorming constructions related to a question of Anselone,* Studia Math. 33 (1969), 231–242.
11. K. Kuratowski, *Topology,* v. I, Academic Press, New York, 1966.
12. E. B. Leach and J. H. M. Whitfield, *Differentiable functions and rough norms on Banach spaces,* Proc. Amer. Math. Soc. 33 (1972), 120–126.
13. A. Louveau, *σ-idéaux engendrés par des ensembles fermés et théorèmes d'approximation,* Trans. Amer. Math. Soc. 257 (1980), 143–170.
14. S. Mazur, *Über konvexe Mengen in linearen normierten Räumen,* Studia Math. 4 (1933), 70–84.
15. J. J. Schäffer, *Geometry of spheres in normed spaces,* Lecture Notes in Pure and Appl. Math., 20, Dekker, New York, 1976.
16. S. Simons, *A convergence theorem with boundary,* Pacific J. Math. 40 (1972), 703–708.
17. W. Sierpinski, *General topology,* Univ. of Toronto Press, Toronto, 1952.
18. C. Stegall, *Applications of descriptive topology in functional analysis,* Universität Linz, 1985.

19. ———, *The Radon-Nikodym property in conjugate Banach spaces,* Trans. Amer. Math. Soc. 206 (1975), 213–223.

Gilles Godefroy
Equipe d'Analyse
Universite Paris VI
75272 Paris Cedex 05
France

Vaclav Zizler
Department of Mathematics
University of Alberta
Edmonton, Alberta T6G 2G1
Canada