Holomorphic Extension of Proper Meromorphic Mappings

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Introduction

Forstnerič [F1] has proved that every proper holomorphic mapping from a ball in C^n to a ball in C^N ($N \geq n \geq 2$) that is sufficiently smooth on the closure of the ball is in fact a rational mapping. He left open the possibility that such a mapping could be indeterminate at a point of the boundary sphere. Cima and Suffridge have shown recently that this does not occur, and hence every such mapping extends to be holomorphic in a neighborhood of the closed ball. They prove the following local result.

THEOREM [CS]. *Suppose that B is the open unit ball in C^n and that U is a neighborhood of a point $p \in bB$. Suppose that $F: U \to C^N$ is a meromorphic mapping whose restriction to B maps B holomorphically into the open unit ball in C^N ($N \geq n$), and that for each point $q \in U \cap bB$, $\|F(z)\| \to 1$ as $z \to q$ ($z \in B$). Then F extends to a holomorphic mapping in some neighborhood of p. As a consequence, F is rational.*

In this note we prove the following more general result.

THEOREM 1. *Suppose that M is a real analytic (nonsingular) real hypersurface in C^n, U is a neighborhood of a point $p \in M$, and Ω is the portion of U lying on one side of M. Suppose that $F: U \to C^N$ is a meromorphic mapping whose restriction to Ω maps Ω holomorphically into the open unit ball in C^N ($N \geq n$), and that for each point $q \in U \cap M$, $\|F(z)\| \to 1$ as $z \to q$ ($z \in \Omega$). Then F extends to a holomorphic mapping in some neighborhood of p.*

Note that we make no geometric assumptions about the hypersurface other than its real analyticity. Here is the idea of the proof. Since the ring of germs of holomorphic functions at a point $p \in C^n$ is a unique factorization domain, we may assume that F is of the form f/g where no factor of g divides all the components of f. Certainly F would extend holomorphically past M at p if $g(p)$ did not vanish. We prove that if $g(p) = 0$, then some factor of g would necessarily divide each component of f. First we prove that it is sufficient to

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pull back to an arbitrary parametrized holomorphic curve that lies in the zero set of g sufficiently near 0. We then polarize the defining equation and prove a lemma that relies on the fact that the defining function is real-valued.

It is natural to seek a larger class of hypersurfaces in the range C^N for which the conclusion of the theorem still holds. Our present argument, with modifications given below, also enables us to prove the following result.

THEOREM 2. *Suppose that M is a real analytic (nonsingular) real hypersurface in C^n, U is a neighborhood of a point $p \in M$, and Ω is the portion of U lying on one side of M. Suppose that $F: U \to C^N$ ($N \geq n$) is a meromorphic mapping whose restriction to Ω maps Ω holomorphically into the domain $E = \{w \in C^N: \sum |w_i|^{2m_i} < 1$, where each m_i is a positive integer$\}$. If for each point $q \in U \cap M$, $\operatorname{dist}(F(z), bE) \to 0$ as $z \to q$ $(z \in \Omega)$, then F extends to a holomorphic mapping in some neighborhood of p.*

If we replace the ball by an arbitrary domain, however, we need not arrive at the same conclusion. For example, suppose that Ω is the ball $\{(z, w) \in C^2: |z|^2 + |w|^2 < 1\}$, $p = (0, 1)$, and m is a positive integer. Then the mapping $(z, w) \to (z^m/(1-w), w)$ sends Ω properly to some domain in C^2 and has a point of indeterminacy at p. We mention also that there are proper mappings from a ball to a ball that do not extend continuously to the boundary, and that there are others that are continuous on the closed ball but are not rational. The article [F2] surveys some of these and other recent developments in the study of proper holomorphic mappings.

This work is part of the author's Ph.D. thesis [Ch], and he wishes to thank his adviser John D'Angelo for introducing him to the subject and for many helpful conversations.

Proofs of the Theorems

First we prove a simple result about real-valued functions of a single complex variable.

LEMMA 1. *Suppose that D is a disk centered at the origin in the complex plane, $r: D \to R$ and $\varphi: D \to C$ are smooth, $r(0) = 0$, and k is a positive integer. For $t \in D$ put $a(t, \bar{t}) = r(t, \bar{t})\varphi(t, \bar{t})/t^k$. If $\lim_{t\to 0} a(t, \bar{t})$ exists, then this limit must be zero.*

Proof. It suffices to show that the order of vanishing ν of the product $r\varphi$ exceeds k. Since $\lim_{t\to 0} a(t, \bar{t})$ exists, $\nu \geq k$. So we assume that $\nu = k$ and derive a contradiction. Write $r = r_j + O(|t|^{j+1})$ and $\varphi = \varphi_{k-j} + O(|t|^{k-j+1})$, where r_j and φ_{k-j} are homogeneous polynomial functions of $(t, \bar{t})$ of (total) degree j and $k-j$, respectively, and write $t = |t|e^{i\theta}$ for $\theta \in [0, 2\pi]$. Then $r_j(e^{i\theta}, e^{-i\theta}) = \lim_{t\to 0}(r(t, \bar{t})/|t|^j)$ is real for all θ, and so r_j is real-valued. Since $r_j\varphi_{k-j}$ is homogeneous in $(t, \bar{t})$ of degree k and $\lim_{t\to 0}(r_j\varphi_{k-j}/t^k)$ exists, there is a constant $c \neq 0$ so that $r_j\varphi_{k-j} = ct^k$. Since r_j is a polynomial, there is then a constant $c_1 \neq 0$ such that $r_j = c_1 t^j$. But since r_j is real-valued and $r(0)=0$, we obtain a contradiction. □

The following lemma is also crucial to our argument.

LEMMA 2. *Suppose that U is a neighborhood of the origin in C^n and that $H: U \to C^m$ and $g: U \to C$ are holomorphic with $g(0)=0$. Suppose further that, for every parametrized holomorphic curve $t \to z(t)$ from the open unit disk B_1 in C to $\{z \in U: g(z)=0\}$ with $z(0)=0$, if $g(z(t))=0$ then $H(z(t))=0$. Then there is a neighborhood U_1 of 0 contained in U such that $H(z)=0$ for all $z \in U_1$ such that $g(z)=0$.*

Proof. After making a generic linear change of coordinates, we may apply the Weierstraß preparation theorem (see [W], for example) to conclude that sufficiently near 0, the zero set of g coincides with the zero set of a Weierstraß polynomial of the form

$$z_n^k + \sum_{j=1}^{k} a_j(z') z_n^{k-j},$$

where each a_j is a holomorphic function of z' that vanishes at 0. Let us denote by A this portion of the zero set of g. Now consider the intersection of A with a complex 2-plane P containing the origin. For a generic choice of 2-plane P, $A \cap P$ is a 1-dimensional curve. According to [Ca, Prop. 3.1], we may assume that, for a generic set of such 2-planes P, the curve $A \cap P$ varies holomorphically with P. By that proposition, then, we obtain parametrized holomorphic curves $z_P: B_1 \to A \cap P$ with $z_P(0)=0$. By hypothesis, $H(z_P(t))=0$ for all such curves. So H vanishes on $A \cap P$ for a generic set of 2-planes P containing the origin. Now to conclude that H vanishes on A, it is sufficient to show that H vanishes on a dense open subset of A. So we choose a smooth point $q_0 = z_{P_0}(t_0)$ of A on $A \cap P_0$ for a generic P_0. Since P_0 is generic, we then obtain a dense open subset of A containing (t_0, P_0) by letting t vary near t_0 and P vary near P_0. The lemma is proved. □

We now begin to prove Theorems 1 and 2. We take $p=0$. Since the ring of germs of holomorphic functions at $0 \in C^n$ is a unique factorization domain (see [W], for example), we may assume that F has the form f/g, where $f: U \to C^N$ and $g: U \to C$ are holomorphic and have no (nonunit) common factor near 0. We prove that this implies that $g(0)$ must not equal zero, and the conclusion of the theorem then follows. In fact, we prove the equivalent statement that if $g(0)$ vanishes then f and g have a common factor near 0. So we suppose that $g(0)=0$, and we consider the following statement.

(1) For all z in some neighborhood of 0, $f(z)=0$ whenever $g(z)=0$.

If we knew that (1) were true, then f and g would have a common factor near 0. To see this we write the germs of g and of the components of f as products of irreducible germs. Since the ring of germs of holomorphic functions at $0 \in C^n$ is a unique factorization domain, this factorization is unique up to associates and order of the factors. If h is an irreducible factor of g near 0, then by (1) each component of f must have h as a factor, since h is irreducible. So it suffices to prove that (1) holds if $g(0)=0$.

First we choose coordinates and a local defining function r for Ω in some neighborhood V of 0 such that, for $z \in V$, r is of the form

$$(2) \qquad r(z,\bar{z}) = 2\operatorname{Re}(z_n) + a(z,\bar{z}) \quad \text{with } a(0)=0 \text{ and } da(0)=0,$$

where $a: V \to R$ is real-analytic.

From this point forward the proof of Theorem 2 requires a bit more technical argument than that of Theorem 1. In order to make clear the main ideas, we first complete the proof of Theorem 1 and later make the appropriate changes necessary to establish Theorem 2.

We continue with the proof of Theorem 1. From the hypotheses (i) the image of Ω under f/g is bounded and (ii) $\|f(z)/g(z)\|^2 \to 1$ as $z \to M$ $(z \in \Omega)$, we then conclude that

$$(3) \qquad \|f(z)\|^2 = |g(z)|^2 \quad \text{for all } z \in V \text{ such that } r(z,\bar{z})=0.$$

We polarize, or complexify, the statement (3) with respect to z_n; that is, we allow z_n and $\overline{z_n}$ to be independent. Writing ζ for the former $\overline{z_n}$ and putting $z' = (z_1, \dots, z_{n-1})$, we have

$$(4) \quad \langle f(z), f(z', \bar{\zeta})\rangle = g(z)\overline{g(z', \bar{\zeta})} \quad \text{if } r(z, (\overline{z'}, \zeta)) = 0, \text{ sufficiently near } 0.$$

Next we transform (4) into an identity in z by eliminating ζ. From the form (2) of r we have

$$r(z, (\overline{z'}, \zeta)) - r(z,\bar{z}) = \zeta - \overline{z_n} + a(z, (\overline{z'}, \zeta)) - a(z,\bar{z})$$

sufficiently near 0. Since $a(0)=0$ and $da(0)=0$, there is a function b that is analytic in $(z, \overline{z'}, \zeta)$ near 0, vanishes at the origin, and satisfies the relation $a(z, (\overline{z'}, \zeta)) - a(z,\bar{z}) = (\zeta - \overline{z_n}) b(z, \overline{z'}, \zeta)$. We then conclude that

$$r(z, (\overline{z'}, \zeta)) - r(z,\bar{z}) = (\zeta - \overline{z_n})(1 + b(z, \overline{z'}, \zeta)) \quad \text{with } b(0)=0.$$

Because $(\partial r/\partial \zeta)(0) \neq 0$, the equation $r(z, (\overline{z'}, \zeta)) = 0$ near 0 defines ζ as a function of z and $\overline{z'}$ near 0, by the implicit function theorem. This function $(z, \overline{z'}) \to \zeta(z, \overline{z'})$ vanishes at 0, and so we get that sufficiently near 0, the conditions

$$r(z, (\overline{z'}, \zeta)) = 0 \quad \text{and} \quad -r(z,\bar{z}) = (\zeta - \overline{z_n})(1 + b(z, \overline{z'}, \zeta(z, \overline{z'})))$$

are equivalent. Writing $c(z, \overline{z'})$ for $-1/(1 + b(z, \overline{z'}, \zeta(z, \overline{z'})))$, we then have that sufficiently near 0, the statements

$$r(z, (\overline{z'}, \zeta)) = 0 \quad \text{and} \quad \zeta = \overline{z_n} + r(z,\bar{z}) c(z, \overline{z'})$$

are equivalent. Upon conjugating the latter statement and substituting the resulting expression for $\bar{\zeta}$ into (4), we obtain

(5) $$\langle f(z), f(z', z_n + r(z,\bar{z})\overline{c(z,\overline{z'})})\rangle = g(z)\overline{g(z', z_n + r(z,\bar{z})\overline{c(z,\overline{z'})})}$$

for all z sufficiently near 0. We apply Taylor's theorem to f and rearrange the terms of (5) to get

(6) $$\|f(z)\|^2 + \langle f(z), r(z,\bar{z})e(z,\bar{z})\rangle = g(z)\overline{g(z', z_n + r(z,\bar{z})\overline{c(z,\overline{z'})})}$$

for all z sufficiently near 0, where e is a mapping whose components are bounded.

We wish to show that, sufficiently near 0, f vanishes on the zero set of g. According to Lemma 2, it suffices to prove that f vanishes along every parametrized holomorphic curve that lies in the zero set of g sufficiently near 0. So suppose that $t \to z(t)$ is a holomorphic mapping from the unit disk in the complex plane to the zero set of g sufficiently near 0, with $z(0) = 0$. When we pull back via this mapping, the identity (6) in z becomes the following identity in t:

(7) $$\|f(z(t))\|^2 + \langle f(z(t)), r(z(t), \overline{z(t)})e(z(t), \overline{z(t)})\rangle = 0$$

for all t sufficiently near 0.

If $f(z(t))$ were not identically zero near 0, then for some $\gamma > 0$ we would be able to write, for all t such that $|t| < \gamma$,

(8) $$f(z(t)) = Lt^k + O(t^{k+1}) \quad \text{for some } L \in C^N \setminus \{0\}$$

and some positive integer k. Upon substituting this expression for $f(z(t))$ into (7) and dividing by $|t|^{2k}$, we would then get the identity

$$\|L + O(t)\|^2 + \langle L + O(t), r(z(t), \overline{z(t)})e(z(t), \overline{z(t)})/t^k\rangle = 0, \quad |t| < \gamma.$$

Because $\|L + O(t)\|^2$ tends to a limit (namely, $\|L\|^2$) as $t \to 0$, the expression $\langle L + O(t), r(z(t), \overline{z(t)})e(z(t), \overline{z(t)})/t^k\rangle$ would necessarily tend to $-\|L\|^2$ as $t \to 0$. Since r is real-valued, we would have

$$\lim_{t \to 0}\{r(z(t), \overline{z(t)})\langle L + O(t), e(z(t), \overline{z(t)})\rangle / t^k\} = -\|L\|^2.$$

We could then conclude from Lemma 1 that $-\|L\|^2 = 0$, and hence L would equal zero.

Therefore it is not possible to fulfill the conditions in (8), and hence there is a $\delta > 0$ such that $f(z(t))$ vanishes identically for $|t| < \delta$. Thus f vanishes along every parametrized holomorphic curve that lies in the zero set of g sufficiently near 0. Lemma 2 then implies that sufficiently near 0, f vanishes on the zero set of g. Hence statement (1) is true, and so Theorem 1 is proved. □

Now we continue with the proof of Theorem 2. Recall that we may assume that $F = f/g$ and that it remains to prove that if $g(0) = 0$, then f must vanish along each parametrized holomorphic curve that lies in the zero set of g sufficiently near 0. As above, we choose coordinates and a local defining function r for Ω in some neighborhood V of 0 such that for $z \in V$, r is of the form

$$r(z,\bar{z}) = 2\,\mathrm{Re}(z_n) + a(z,\bar{z}) \quad \text{with } a(0)=0 \text{ and } da(0)=0,$$

where $a: V \to R$ is real-analytic.

After reordering the integers m_i if necessary, we may assume that they satisfy $m_1 \leq m_2 \leq \cdots \leq m_N$. We first consider the case in which all these inequalities are strict. From the hypotheses (i) the image of Ω under f/g is bounded and (ii) $\sum |f_i(z)/g(z)|^{2m_i} \to 1$ as $z \to M$ $(z \in \Omega)$, we then conclude that, for all $z \in V$ such that $r(z,\bar{z}) = 0$,

$$\sum |g(z)|^{2(m_N - m_i)} |f_i(z)|^{2m_i} = |g(z)|^{2m_N}.$$

In order to suppress terms irrelevant to the argument, we write this statement in the form

(9) $$|f_N(z)|^{2m_N} = g(z)\Phi(z,\bar{z}) \quad \text{for all } z \in V \text{ such that } r(z,\bar{z}) = 0,$$

where Φ is analytic in $(z,\bar{z})$ near 0. We polarize, or complexify, the statement (9) with respect to z_n; that is, we allow z_n and $\overline{z_n}$ to be independent. Writing ζ for the former $\overline{z_n}$ and putting $z' = (z_1, \ldots, z_{n-1})$, we have

(10) $$f_N(z)^{m_N} \overline{f_N(z', \bar{\zeta})}^{m_N} = g(z)\Phi(z, (\overline{z'}, \zeta)) \quad \text{if } r(z, (\overline{z'}, \zeta)) = 0,$$

sufficiently near 0.

Next we transform (10) into an identity in z by eliminating ζ. Just as in the proof of Theorem 1, we know that there is a function c that is analytic in $(z, \overline{z'})$ near 0 such that sufficiently near 0, the statements

$$r(z, (\overline{z'}, \zeta)) = 0 \quad \text{and} \quad \zeta = \overline{z_n} + r(z,\bar{z})c(z, \overline{z'})$$

are equivalent. Upon conjugating the latter statement and substituting the resulting expression for $\bar{\zeta}$ into (10), we obtain

(11) $$\begin{aligned} &f_N(z)^{m_N} \overline{f_N(z', z_n + r(z,\bar{z})\overline{c(z,\overline{z'})})}^{m_N} \\ &\qquad = g(z)\Phi(z, (\overline{z'}, z_n + r(z,\bar{z})\overline{c(z,\overline{z'})})) \end{aligned}$$

for all z sufficiently near 0. We apply Taylor's theorem to f_N and rearrange the terms of (11) to get

(12) $$\begin{aligned} &|f_N(z)|^{2m_N} + f_N(z)^{m_N} r(z,\bar{z}) e(z,\bar{z}) \\ &\qquad = g(z)\Phi(z, (\overline{z'}, z_n + r(z,\bar{z})\overline{c(z,\overline{z'})})) \end{aligned}$$

for all z sufficiently near 0, where e is a bounded function.

We are now ready to show that f_N vanishes along each parametrized holomorphic curve that lies in the zero set of g sufficiently near 0. So suppose that $t \to z(t)$ is a holomorphic mapping from the unit disk in the complex plane to the zero set of g sufficiently near 0, with $z(0) = 0$. When we pull back via this mapping, the identity (12) in z becomes the following identity in t:

(13) $$|f_N(z(t))|^{2m_N} + f_N(z(t))^{m_N} r(z(t), \overline{z(t)}) e(z(t), \overline{z(t)}) = 0$$

for all t sufficiently near 0.

If $f_N(z(t))$ were not identically zero near 0, then for some $\gamma > 0$ we would be able to write, for all t such that $|t| < \gamma$,

(14) $$f_N(z(t))^{m_N} = Lt^k + O(t^{k+1}) \quad \text{for some } L \in C \setminus \{0\}$$

and some positive integer k. Upon substituting this expression for $f_N(z(t))^{m_N}$ into (13) and dividing by $|t|^{2k}$, we would then have the identity

$$|L+O(t)|^2 + (L+O(t))r(z(t),\overline{z(t)})e(z(t),\overline{z(t)})/t^k = 0, \quad |t| < \gamma.$$

Because $|L+O(t)|^2$ tends to a limit (namely, $|L|^2$) as $t \to 0$, the expression $(L+O(t))r(z(t),\overline{z(t)})e(z(t),\overline{z(t)})/t^k$ would necessarily tend to $-|L|^2$ as $t \to 0$. We could then conclude from Lemma 1 that $-|L|^2 = 0$, and hence L would equal zero.

Therefore it is not possible to fulfill the conditions in (14), and hence there is a $\delta > 0$ such that $f_N(z(t))$ vanishes identically for $|t| < \delta$. Thus f_N vanishes along every parametrized holomorphic curve that lies in the zero set of g sufficiently near 0. Lemma 2 then implies that sufficiently near 0, f vanishes on the zero set of g. Thus f_N and g have a common factor near 0, by the reasoning preceding statement (1). Hence $F_N = f_N/g$ is holomorphic near the origin, and the proof is finished if $N = n = 1$. So we suppose that $N > 1$.

Now $|F_N(z)|^{2m_N} + \sum_{1 \le i \le N-1} |f_i(z)/g(z)|^{2m_i} \to 1$ as $z \to M$ $(z \in \Omega)$, where F_N is holomorphic near 0. From this statement and the hypothesis that the image of Ω under f/g is bounded, we then conclude that for all $z \in V$ such that $r(z,\bar{z}) = 0$,

$$|g(z)|^{2m_{N-1}}|F_N(z)|^{2m_N} + \sum_{i=1}^{N-1} |g(z)|^{2(m_{N-1}-m_i)}|f_i(z)|^{2m_i} = |g(z)|^{2m_{N-1}}.$$

In order to suppress terms irrelevant to the argument, we write this statement in the form

$$|f_{N-1}(z)|^{2m_{N-1}} = g(z)\Psi(z,\bar{z}) \quad \text{for all } z \in V \text{ such that } r(z,\bar{z}) = 0,$$

where Ψ is analytic in $(z,\bar{z})$ near 0. Note that this statement has precisely the form of statement (9), and so the argument given above proves in this case that f_{N-1} and g have a common factor near 0. Hence $F_{N-1} = f_{N-1}/g$ is holomorphic near the origin. We continue in this way to prove that each component of F is holomorphic near the origin.

In general the integers m_i are not distinct. In this case we first consider the set $A = \{i : 1 \le i \le N \text{ and } m_i = m_N\}$, and we write $f = (f', f_A)$, where f_A consists of those components f_i of f for which $i \in A$ and f' consists of any other components of f. Of course f' may be void (for example, in Theorem 1, in which each $m_i = 1$). The hypotheses of the theorem then ensure that if V and r are as above, then

(15) $$\|f_A(z)\|^{2m_N} = g(z)\Theta(z,\bar{z}) \quad \text{for all } z \in V \text{ such that } r(z,\bar{z}) = 0,$$

where Θ is analytic in $(z,\bar{z})$ near 0. Note the similarities between statement (15) and statements (3) and (9). The arguments given previously thus apply to prove that f_A vanishes along each parametrized holomorphic curve that lies in the zero set of g sufficiently near 0, and hence that the mapping $F_A = f_A/g$ is holomorphic at the origin. What we have shown is that the restriction of F to a certain subspace of C^N is holomorphic at the origin. As above, we write C^N as an orthogonal direct sum of such subspaces, and we show that the restriction of F to each of the summands is holomorphic at the origin, and Theorem 2 is proved. □

Finally, we remark that Theorem 2 remains valid if the domain E is replaced by the domain $E' = \{w \in C^N : \sum |w_i|^{2m_i} + \|Q(w)\|^2 < 1\}$, where $Q: C^N \to C^T$ ($N \geq n$, $T \geq 1$) is a polynomial mapping of degree d and $m_1, m_2, \ldots, m_N$ are positive integers such that $m_1 \leq m_2 \leq \cdots \leq m_N$ and $d \leq m_N - 1$. See [Ch] for further details.

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