# Symmetric Properties of Eigenfunctions of the Laplace Operator on Compact Riemannian Manifolds

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## Introduction

In this paper, we deal with amplitude symmetric properties of eigenfunctions of the Laplace operator on compact Riemannian manifolds. We prove that all the eigenfunctions of the Laplace operator on compact Riemannian manifolds with nonnegative Ricci curvature have a certain amplitude symmetry, and that appropriate conditions on the Ricci curvature and the volume of the manifolds yield strong amplitude symmetries of the first eigenfunctions.

Let $M$ be a compact Riemannian manifold, $\Delta$ the Laplace operator acting on smooth functions on $M$, and $\lambda_1 > 0$ the first eigenvalue of $\Delta$. It is well known that

$$\sup_M u > 0 \quad \text{and} \quad \inf_M u < 0$$

for every eigenfunction $u$ corresponding to $\lambda_1$. When the equality

$$\sup_M u = -\inf_M u$$

holds for an eigenfunction $u$ corresponding to $\lambda_1$, it is usually easier to estimate $\lambda_1$ in terms of geometric quantities of $M$ (see [4], [6], and [9]). In [9], Yang and Zhong asked if the above equality holds for all eigenfunctions $u$ corresponding to $\lambda_1$ when $M$ has nonnegative Ricci curvature. The following example gives a negative answer to this question.

Consider the real projective space $\mathbf{RP}^2$ with the standard unit sphere

$$\mathbf{S}^2 = \{(x,y,z) \in \mathbf{R}^3; x^2+y^2+z^2=1\}$$

as its Riemannian covering space. Clearly, $\mathbf{RP}^2$ has positive Ricci curvature. The function $f$ on $\mathbf{S}^2$ defined by

$$f(x,y,z) = z^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 = \tfrac{3}{2}z^2 - \tfrac{1}{2}$$

induces a function $u$ on $\mathbf{RP}^2$, and $u$ is an eigenfunction corresponding to $\lambda_1$ of $\mathbf{RP}^2$ (see, e.g., [2] for the proof of this claim). Direct calculation gives

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$$\sup_{\mathbf{RP}^2} u = 1 \quad \text{and} \quad \inf_{\mathbf{RP}^2} u = -\frac{1}{2}.$$

We study the relations between the ratio

$$\frac{-\inf_M u}{\sup_M u} \quad \text{for eigenfunctions } u \text{ of } \Delta$$

and the geometric properties of $M$. Before stating our results, we introduce some notation.

For each positive integer $k$, let $\mathfrak{F}_k$ denote the eigenspace of $\Delta$ corresponding to the $k$th eigenvalue $\lambda_k$. We say that $u \in \mathfrak{F}_k$ is normalized if

$$\sup_M u = 1 \quad \text{and} \quad \inf_M u \geq -1.$$

For every nonzero $u \in \mathfrak{F}_k$ there exists a unique constant $c$ such that $cu$ is normalized. Set

$$\alpha_k(M) = \inf\{-\inf_M u;\ u \in \mathfrak{F}_k \text{ is normalized}\}.$$

Since $\mathfrak{F}_k$ is finite-dimensional, we have $0 < \alpha_k(M) \leq 1$. For example, it is easy to show that

$$\alpha_1(\mathbf{S}^n) = 1 \quad \text{and} \quad \alpha_1(\mathbf{RP}^n) = \alpha_2(\mathbf{S}^n) = \frac{1}{n} \quad \text{for } n > 1.$$

When $\alpha_k(M) = 1$, $\sup_M u = -\inf_M u$ for all $u \in \mathfrak{F}_k$. So, in this case, we say that the $k$th eigenfunctions of $\Delta$ are *amplitude symmetric*.

In general, to estimate

$$\frac{-\inf_M u}{\sup_M u} \quad \text{when } u \in \mathfrak{F}_k \text{ and } u \neq 0,$$

it is enough to estimate $\alpha_k(M)$ from below. We first find a lower bound of $\alpha_k(M)$ for manifolds $M$ with nonnegative Ricci curvature. The bound depends only on the dimension of $M$.

THEOREM 1. *If an $n$-dimensional compact Riemannian manifold $M$ has nonnegative Ricci curvature, then*

$$\alpha_k(M) \geq \frac{1-\beta_{k,n}}{1+\beta_{k,n}} \quad \textit{for all } k,$$

*where $\beta_{k,n} < 1$ is the constant determined by*

$$\int_{-1}^{1} \frac{dt}{\sqrt{(1+\beta_{k,n}t)(1-t^2)}} = kn\sqrt{2(n+4)}.$$

Denote the volume and Ricci curvature of $M$ by $\mathrm{vol}(M)$ and $\mathrm{Ric}_M$, respectively, and let $v_n$ be the volume of the $n$-dimensional standard unit sphere. We have the following theorem.

THEOREM 2. *For every positive integer $n$ and each positive number $c<1$, there exists a positive number $\epsilon$ such that, if an $n$-dimensional compact Riemannian manifold $M$ satisfies $\mathrm{Ric}_M \geq n-1$ and $\mathrm{vol}(M) > v_n - \epsilon$, then $\alpha_1(M) > c$.*

We give some technical lemmas in Section 1. Theorem 1 is proved in Section 2 and Theorem 2 is proved in Section 3.

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## 1. Gradient Estimates

In this section, we derive some results that will be used in the proofs of our theorems. The $n$-dimensional compact Riemannian manifold $M$ dealt with here is assumed to have nonnegative Ricci curvature. We first establish a lemma.

LEMMA 1.1. *If $u \in \mathfrak{F}_k$ is normalized so that*

$$\Delta u = -\lambda_k u, \quad \sup_M u = 1, \quad \textit{and} \quad \inf_M u \geq -1,$$

*then we have $|\nabla u|^2 \leq \lambda_k(1-u^2) \leq \lambda_k$ on $M$, where $\nabla u$ denotes the gradient of $u$ relative to the given Riemannian metric on $M$.*

When $k=1$, this is Lemma 2 of [9]. The proof given there works in the general case. We need a somewhat different result, which we establish in our main lemma (i.e., Lemma 1.11). Given a normalized eigenfunction $u \in \mathfrak{F}_k$, let

$$\beta = \frac{1+\inf_M u}{1-\inf_M u}, \qquad v = \frac{2}{1-\inf_M u} u - \beta. \tag{1.2}$$

It is straightforward to check that

$$0 \leq \beta < 1, \quad \Delta v = -\lambda_k(v+\beta), \quad \text{and} \quad \sup_M v = 1 = -\inf_M v. \tag{1.3}$$

Before stating Lemma 1.11, we prove a proposition which will imply that lemma.

PROPOSITION 1.4. *Let $\beta$ and $v$ be defined by* (1.2). *If $f$ is a nonnegative function with continuous second-order derivative $f''$ on $[-1,1]$ such that*

$$\frac{n}{2}(f')^2 + (n-1)f''f > 0 \quad \textit{on} \quad [-1,1],$$

*then $|\nabla v|^2 \leq B \cdot (f \circ v)$ on $M$, where $B$ is the constant given by*

$$B = \lambda_k \sup_{-1\le t\le 1} \frac{2(n-1)f(t)-(n+1)(t+\beta)f'(t)}{(n/2)f'(t)^2-(n-1)f''(t)f(t)}.$$

*Proof.* First we note that the real number $B$ is well-defined by the given conditions on $f$. Denote the covariant derivatives of $v$ under an orthonormal frame by $v_i$, $v_{ij}$, and $v_{ijp}$, with $i$, $j$, and $p$ ranging from 1 to $n$. Then

$$\Delta v = \sum_i v_{ii}, \quad v_{ij} = v_{ji}, \quad v_{ijp} = v_{jip}, \quad \text{and} \quad v_{ijp} = v_{ipj} + \sum_q v_q R_{qijp},$$

where $\{R_{ijpq}\}$ is the sectional curvature tensor. Hence $\{R_{ij}\} = \{\sum_p R_{ipjp}\}$ is the Ricci curvature tensor.

Given $\epsilon > 0$, let $B_\epsilon$ be the maximum value of $|\nabla v|^2/(f\circ v+\epsilon)$ and let $x_\epsilon$ be a point where this value is attained. Then, at $x_\epsilon$,

$$|\nabla v|^2 = B_\epsilon\cdot(f\circ v+\epsilon) > 0, \tag{1.5}$$

$$2\sum_i v_i v_{ij} = B_\epsilon\cdot(f'\circ v)\cdot v_j \quad \text{for all } j, \tag{1.6}$$

and

$$2\sum_{i,j} v_{ij}^2 + 2\sum_{i,j} v_i v_{ijj} \le B_\epsilon\cdot(f''\circ v)\cdot|\nabla v|^2 + B_\epsilon\cdot(f'\circ v)\cdot\Delta v. \tag{1.7}$$

By (1.3) and (1.5), from (1.7) we get

$$2\sum_{i,j}|v_{ij}|^2 + 2\sum_{i,j} v_i v_{ijj} \le B_\epsilon^2\cdot(f''\circ v)(f\circ v+\epsilon) - B_\epsilon\lambda_k\cdot(v+\epsilon)(f'\circ v). \tag{1.8}$$

We may assume the orthonormal frame is chosen so that $v_1(x_\epsilon) = |\nabla v(x_\epsilon)|$ and $v_i(x_\epsilon) = 0$ for all $i \ge 2$. Then (1.6) yields

$$v_{11}(x_\epsilon) = \tfrac{1}{2}B_\epsilon\cdot(f'\circ v)(x_\epsilon).$$

From this and (1.3), we have the following estimates at $x_\epsilon$:

$$\begin{aligned}\sum_{i,j} v_{ij}^2 &\ge \sum_i v_{ii}^2 \ge v_{11}^2 + \frac{1}{n-1}(v_{22}+\cdots+v_{nn})^2 = v_{11}^2 + \frac{1}{n-1}(\Delta v - v_{11})^2 \\ &= \frac{n}{4(n-1)}B_\epsilon^2\cdot(f'\circ v)^2 + \frac{1}{n-1}B_\epsilon\lambda_k\cdot(v+\beta)(f'\circ v) + \frac{\lambda_k^2}{n-1}(v+\beta)^2;\end{aligned} \tag{1.9}$$

$$\begin{aligned}\sum_{i,j} v_i v_{ijj} &= \sum_{i,j} v_i v_{jij} = \sum_{i,j} v_i\cdot\left(v_{jji} + \sum_p v_p R_{pjij}\right) \\ &= -\lambda_k|\nabla v|^2 + \sum_{i,j} v_i R_{ij} v_j \ge -\lambda_k|\nabla v|^2 = -B_\epsilon\lambda_k\cdot(f\circ v+\epsilon).\end{aligned} \tag{1.10}$$

Combining (1.8), (1.9), and (1.10), we obtain

$$B_\epsilon \le \lambda_k \left.\frac{2(n-1)(f\circ v+\epsilon)-(n+1)(v+\beta)(f'\circ v)}{(n/2)(f'\circ v)^2-(n-1)(f''\circ v)(f\circ v+\epsilon)}\right|_{x_\epsilon},$$

which in particular indicates that for all small enough $\epsilon$'s, the $B_\epsilon$'s have a common upper bound; hence we can assume that $B_0 = \lim_{\epsilon\to 0} B_\epsilon$ exists. Because $M$ is compact, we can assume further that $x_\epsilon \to x_0 \in M$ as $\epsilon \to 0$. Then

$$|\nabla v|^2 \le B_0 \cdot (f \circ v) \quad \text{on } M$$

and

$$B_0 \le \lambda_k \frac{2(n-1)(f \circ v) - (n-1)(v+\beta)(f' \circ v)}{(n/2)(f' \circ v)^2 - (n-1)(f'' \circ v)(f \circ v + \epsilon)}\bigg|_{x_0} \le B.$$

These immediately imply the claim and complete the proof of the proposition. □

Now we are ready to prove our main lemma.

LEMMA 1.11. *If $\beta$ and $v$ are defined by* (1.2), *then we have*

$$|\nabla v|^2 \le n\lambda_k (1+\beta v)(1-v^2) \quad \text{on } M.$$

*Proof.* Define $f$ by

$$f(t) = (1+\beta t)(1-t^2).$$

Then it is not difficult to show that $f$ satisfies the conditions of Proposition 1.4 and

$$n\left[\frac{n}{2}(f')^2 - (n+1)f''f\right] - [2(n-1)f - (n+1)(t+\beta)f'] \ge 0 \quad \text{on } [-1,1].$$

So, from Proposition 1.4, we obtain

$$|\nabla v|^2 \le n\lambda_k \cdot (f \circ v) = n\lambda_k(1+\beta v)(1-v^2) \quad \text{on } M.$$

This completes the proof. □

## 2. Proof of Theorem 1

Assume that $M$ is a compact Riemannian manifold with nonnegative Ricci curvature, and let $d(M)$ be its diameter. Fix a normalized $u \in \mathfrak{F}_k$, and define $\beta$ and $v$ by (1.2). By Lemma 1.11, we have

$$|\nabla v|^2 \le n\lambda_k(1+\beta v)(1-v^2) \quad \text{on } M.$$

Let $\gamma$ be a minimal path from $\{x \in M;\ v(x) = -1\}$ to $\{x \in M;\ v(x) = 1\}$, and denote by $s$ the arc length of $\gamma$. Then we obtain

$$d(M) \cdot \sqrt{\lambda_k} \ge \int_\gamma \frac{|\nabla v|\, ds}{\sqrt{n(1+\beta v)(1-v^2)}} \ge \int_{-1}^{1} \frac{dt}{\sqrt{(1+\beta t)(1-t^2)}}.$$

On the other hand, from [3, Cor. 2.2] we have

$$d(M) \cdot \sqrt{\lambda_k} \le k\sqrt{2n(n+4)}.$$

Thus,

$$\int_{-1}^{1} \frac{dt}{\sqrt{(1+\beta t)(1-t^2)}} \le kn\sqrt{2(n+4)}.$$

Define $g\colon [0,1) \to \mathbf{R}$ by

$$g(s)=\int_{-1}^{1}\frac{dt}{\sqrt{(1+st)(1-t^2)}}.$$

Then:

(1) $g(0)=\pi<n\sqrt{2k(n+4)}$.

(2) $g$ is strictly increasing; in fact, for $s\in(0,1)$,

$$g'(s)=\frac{1}{2}\int_0^1\left[\frac{1}{\sqrt{(1-st)^3}}-\frac{1}{\sqrt{(1+st)^3}}\right]\frac{t\,dt}{\sqrt{1-t^2}}>0.$$

(3) $\lim_{s\to 1}g(s)=+\infty$. This comes from the following estimates:

$$g(s)>\int_{-s}^{0}\frac{dt}{\sqrt{(1+st)(1-t^2)}}>\int_{-s}^{0}\frac{dt}{1-t^2}>\frac{1}{2}\int_{-s}^{0}\frac{dt}{1+t}=\ln\frac{1}{\sqrt{1-s}}.$$

Hence, for each positive integer $n$, the equation

$$\int_{-1}^{1}\frac{dt}{\sqrt{(1+xt)(1-t^2)}}=kn\sqrt{2(n+4)}$$

in $x$ has a unique root $x=\beta_{k,n}\in(0,1)$, and $\beta<\beta_{k,n}$. Therefore,

$$-\inf_M u=\frac{1-\beta}{1+\beta}\geq\frac{1-\beta_{k,n}}{1+\beta_{k,n}}.$$

This implies that $\alpha_k(M)\geq(1-\beta_{k,n})/(1+\beta_{k,n})$ and completes the proof of Theorem 1. □

## 3. Proof of Theorem 2

Let $M$ be an $n$-dimensional compact Riemannian manifold such that $\mathrm{Ric}_M\geq n-1$. Since the proof in the case $n=1$ is trivial, we assume $n>1$.

Fix a normalized $u\in\mathfrak{F}_1$ and choose $m\in M$ so that $u(m)$ is maximal; then $u(m)=1$. We denote by $N$ the unit sphere in the tangent space of $M$ at $m$. For each $\theta\in N$, let $l(\theta)$ be the distance from $m$ to its cut-locus along the direction $\theta$. Moreover, we assume that $f(\rho,\theta)\,d\rho\,d\theta$ is the volume element of $M$; that is,

$$\mathrm{vol}(M)=\int_N\int_0^{l(\theta)}f(\rho,\theta)\,d\rho\,d\theta.$$

In order to prove Theorem 2, it suffices to show that when $\mathrm{vol}(M)$ is close enough (but independent of $u$) to $v_n$ there exists a direction $\theta\in N$ such that $l(\theta)$ is close to $\pi$ and $u(\rho,\theta)$ is close to $\cos\rho$ for $\rho\in(0,l(\theta))$.

Given $\theta\in N$, fix a small $\rho_0\in(0,l(\theta))$. Let

$$\zeta_\theta(\rho)=\frac{\partial^2}{\partial\rho^2}u(\rho,\theta)+\frac{\lambda_1}{n}u(\rho,\theta)\quad\text{for }\rho\in(0,l(\theta)).$$

Then $u(\rho,\theta)$ satisfies the linear differential equation in $x$:

$$x''(t)+L^2x(t)=\zeta_\theta(t)\quad\text{for }t\in[\rho_0,l(\theta)]$$

with initial conditions

$$x(\rho_0)=u(\rho_0,\theta) \quad \text{and} \quad x'(\rho_0)=\frac{\partial u}{\partial \rho}(\rho_0,\theta),$$

where $L=\sqrt{\lambda_1/n}$. Since $\mathrm{Ric}_M \geq n-1$, $L\geq 1$ by Lichnerowiez's theorem (see, e.g., [7]). So, when $\rho\in[\rho_0, l(\theta))$,

$$\begin{aligned} u(\rho,\theta)=\xi_\theta(\rho)&+\left(u(\rho_0,\theta)\cdot\sin L\rho_0+\frac{\partial u}{\partial \rho}(\rho_0,\theta)\cdot\frac{\cos L\rho_0}{L}\right)\sin L\rho \\ &+\left(u(\rho_0,\theta)\cdot\cos L\rho_0-\frac{\partial u}{\partial \rho}(\rho_0,\theta)\cdot\frac{\sin L\rho_0}{L}\right)\cos L\rho, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \xi_\theta(\rho)=\sin L(\rho-\rho_0)\cdot\int_{\rho_0}^{\rho}\frac{\zeta_\theta(t)}{L}\cos L(t-\rho_0)\,dt \\ -\cos L(\rho-\rho_0)\cdot\int_{\rho_0}^{\rho}\frac{\zeta_\theta(t)}{L}\sin L(t-\rho_0)\,dt. \end{aligned}$$

Therefore, we only need to estimate $L$ and $\xi_\theta$. In order to do so, we need more lemmas.

For $r>0$, denote by $B(r)$ a geodesic ball with radius $r$ on the standard unit $n$-sphere, and by $\mu_1(r)$ the first eigenvalue of the Dirichlet problem of the Laplacian on $B(r)$. It is well known that $\mu_1$ is continuous on $(0,+\infty)$ and strictly decreasing on $(0,\pi)$, that $\lim_{r\to 0}\mu_1(r)=+\infty$, and that $\mu_1(r)=n$ for $r\geq\pi$.

LEMMA 3.2. *Let* $\epsilon>0$. *If* $\mathrm{vol}(M)\geq\mathrm{vol}(B(\mu_1^{-1}(n+\epsilon)))$, *then* $\lambda_1\leq n+\epsilon$.

*Proof.* From $\mathrm{Ric}_M\geq n-1$ and [3, Thm. 2.1] we obtain

$$\lambda_1\leq\mu_1\left(\frac{d(M)}{2}\right). \tag{3.3}$$

So, if $d(M)\geq 2\mu_1^{-1}(n+\epsilon)$ then one has

$$\lambda_1\leq n+\epsilon. \tag{3.4}$$

From $\mathrm{Ric}_M\geq n-1$ and the volume comparison theorem, we obtain

$$\mathrm{vol}(M)\leq\mathrm{vol}\left(B\left(\frac{2}{d(M)}\right)\right).$$

Therefore, when $v(M)\geq\mathrm{vol}(B(\mu_1^{-1}(n+\epsilon)))$, we must have

$$\frac{\pi}{2}\geq\frac{d(M)}{2}\geq\mu_1^{-1}(n+\epsilon)$$

and hence (3.4) holds. This completes the proof. □

Suppose that $g$ is the metric tensor of $M$, and that $\nabla^2 u$ is the second-order covariant derivative tensor of $u$ relative to $g$. Define $h\colon(0,+\infty)\to(0,+\infty)$ by

(3.5) $$h(\epsilon)=\min\left\{1, \frac{\epsilon}{(n+1)v_n}\right\} \quad \text{for } \epsilon\in(0,+\infty).$$

Then the following lemma will be established.

LEMMA 3.6. *Let* $\epsilon>0$. *If* $\lambda_1\le n+h(\epsilon)$ *then*

$$\int_M\left|\nabla^2u+\frac{\lambda_1}{n}ug\right|^2\le\epsilon.$$

*Proof.* From

$$\lambda_1\int_M|\nabla u|^2-\int_M|\nabla^2u|^2=\int_M R(\nabla u,\nabla u),$$

where $R$ is the Ricci curvature tensor of $M$, we get

$$\int_M|\nabla^2u|^2\le[\lambda_1-(n-1)]\int_M|\nabla u|^2.$$

So,

$$\int_M\left|\nabla^2u+\frac{\lambda_1}{n}ug\right|^2=\int_M|\nabla^2u|^2-\frac{\lambda_1}{n}\int_M|\nabla u|^2\le(n-1)\left(\frac{\lambda_1}{n}-1\right)\int_M|\nabla u|^2.$$

By Lemma 1.1,

(3.7) $$\int_M\left|\nabla^2u+\frac{\lambda_1}{n}ug\right|^2\le n\left(\frac{\lambda_1}{n}-1\right)\lambda_1\operatorname{vol}(M)\le(\lambda_1-n)\lambda_1v_n.$$

Now assume that $\lambda_1\le n+h(\epsilon)$. From (3.5) and (3.7), we see that

$$\int_M\left|\nabla^2u+\frac{\lambda_1}{n}ug\right|^2\le(\lambda_1-n)(n+1)v_n\le\epsilon.$$

This completes the proof. □

Define $\delta_2\colon(0,+\infty)\times(0,\frac{\pi}{2})\to(0,+\infty)$ by

$$\delta_2(\epsilon,\rho)=\min\left\{\frac{1}{4}v_{n-1}\int_{\pi-\rho/2}^{\pi}\sin^{n-1}x\,dx,\right.$$
$$v_n-\operatorname{vol}\left(B\left(\mu_1^{-1}\left(n+h\left(\frac{\epsilon^2v_{n-1}\sin^{n-1}\rho}{8\pi}\right)\right)\right)\right),$$
$$\left.\frac{1}{8}v_{n-1}\int_{\rho/2}^{\rho}\sin^{n-1}x\,dx\right\}$$

for all $(\epsilon,\rho)\in(0,+\infty)\times(0,\frac{\pi}{2})$. Then we obtain the next lemma.

LEMMA 3.8. *Let* $\epsilon>0$ *and* $\rho_0\in(0,\frac{\pi}{2})$. *If* $\operatorname{vol}(M)\ge v_n-\delta_2(\epsilon,\rho_0)$, *then there exists* $\theta\in N$ *such that*

(3.9) $$l(\theta)>\pi-\frac{1}{2}\rho_0 \quad \textit{and} \quad \max_{\rho_0\le\rho\le\pi-\rho_0}|\xi_\theta(\rho)|\le2\epsilon.$$

*Proof.* Given $\epsilon > 0$ and $\rho_0 \in (0, \frac{\pi}{2})$, let

$$l_0 = \pi - \rho_0 \quad \text{and} \quad l_1 = \pi - \tfrac{1}{2}\rho_0.$$

From $\mathrm{Ric}_M \geq n-1$ and the Bishop-Gromov inequality, one knows that $0 \leq f(\rho,\theta) \leq \sin^{n-1}\rho$ and

$$\begin{aligned}
v_n - \mathrm{vol}(M) &= v_{n-1}\int_0^{\pi} \sin^{n-1}\rho\, d\rho - \int_N \int_0^{l(\theta)} f(\rho,\theta)\, d\rho\, d\theta \\
&\geq v_{n-1}\int_0^{\pi} \sin^{n-1}\rho\, d\rho - \int_N \int_0^{l(\theta)} \sin^{n-1}\rho\, d\rho \\
&\geq v_{n-1}\int_0^{\pi} \sin^{n-1}\rho\, d\rho - \mathrm{vol}(W_1)\int_0^{\pi} \sin^{n-1}\rho\, d\rho \\
&\quad - \mathrm{vol}(N - W_1)\int_0^{l_1} \sin^{n-1}\rho\, d\rho \\
&= (v_{n-1} - \mathrm{vol}(W_1))\int_{l_1}^{\pi} \sin^{n-1}\rho\, d\rho,
\end{aligned}$$

where $W_1 = \{\theta \in N;\, l(\theta) > l_1\}$ is an open subset of $N$. If

$$\mathrm{vol}(M) \geq v_n - \frac{1}{4} v_{n-1}\int_{l_1}^{\pi} \sin^{n-1}\rho\, d\rho,$$

then $\mathrm{vol}(W_1) \geq \frac{3}{4} v_{n-1}$. Set $\hat{\epsilon} = (\epsilon^2 v_{n-1} \sin^{n-1}\rho_0)/8\pi$. By Lemma 3.2, when $\mathrm{vol}(M) \geq \mathrm{vol}(B(\mu_1^{-1}(n + h(\hat{\epsilon}))))$ one has $\lambda_1 \leq n + h(\hat{\epsilon})$ and hence, by Lemma 3.6,

$$\int_N \int_0^{l(\theta)} \left|\nabla^2 u + \frac{\lambda}{n} u g\right|^2 f(\rho,\theta)\, d\rho\, d\theta \leq \hat{\epsilon}.$$

So, if $\mathrm{vol}(M) \geq \mathrm{vol}(B(\mu_1^{-1}(n + h(\hat{\epsilon}))))$, then there exists an open subset $W_2 \subset N$ such that

$$\text{(3.10)} \qquad \int_0^{l(\theta)} \left|\nabla^2 u + \frac{\lambda_1}{n} u g\right|^2 f(\rho,\theta)\, d\rho < \frac{4\hat{\epsilon}}{v_{n-1}} = \frac{\epsilon^2 \sin^{n-1}\rho_0}{2\pi} \quad \text{for } \theta \in W_2,$$

and $\mathrm{vol}(W_2) \geq \frac{3}{4} v_{n-1}$. Furthermore, if

$$\mathrm{vol}(M) \geq v_n - \frac{v_{n-1}}{8}\int_{\rho_0/2}^{\rho_0} \sin^{n-1}\rho\, d\rho,$$

then from

$$\int_N \int_0^{l(\theta)} (\sin^{n-1}\rho - f(\rho,\theta))\, d\rho \leq v_n - \mathrm{vol}(M)$$

we know that there also exists an open subset $W_3 \subset N$ such that

$$\text{(3.11)} \qquad \int_0^{l(\theta)} (\sin^{n-1}\rho - f(\rho,\theta))\, d\rho < \frac{1}{2}\int_{\rho_0/2}^{\rho_0} \sin^{n-1}\rho\, d\rho \quad \text{for } \theta \in W_3,$$

and $\mathrm{vol}(W_3) \geq \frac{3}{4} v_{n-1}$. If $\mathrm{vol}(M) \geq v_n - \delta_2(\epsilon, \rho_0)$, then $\mathrm{vol}(W_1 \cap W_2 \cap W_3) > 0$, and (3.10) and (3.11) hold for all $\theta \in W_1 \cap W_2 \cap W_3$. From now on, we fix $\theta \in W_1 \cap W_2 \cap W_3$. Since $f(\rho,\theta)/\sin^{n-1}\rho$ is a decreasing function of $\rho$, we have

$$\int_0^{l_1}(\sin^{n-1}\rho - f(\rho,\theta))\,d\rho \ge \int_{l_0}^{l_1}(\sin^{n-1}\rho - f(\rho,\theta))\,d\rho$$
$$\ge \left(1-\frac{f(l_0,\theta)}{\sin^{n-1} l_0}\right)\int_{l_0}^{l_1}\sin^{n-1}\rho\,d\rho.$$

Hence (3.11) implies that

$$f(l_0,\theta) \ge \tfrac{1}{2}\sin^{n-1} l_0$$

and

$$f(\rho,\theta) = \frac{f(\rho,\theta)}{\sin^{n-1}\rho}\sin^{n-1}\rho \ge \frac{1}{2}\sin^{n-1}\rho \ge \frac{1}{2}\sin^{n-1}\rho_0 \quad \text{for } \rho\in[\rho_0, l_0].$$

This, together with (3.10), yields

$$\int_{\rho_0}^{l_0}\zeta_\theta^2\,d\rho \le \int_{\rho_0}^{l_0}\left|\nabla^2 u + \frac{\lambda_1}{n}ug\right|^2 d\rho \le \frac{2}{\sin^{n-1}\rho_0}\int_0^{l_0}\left|\nabla^2 u+\frac{\lambda_1}{n}ug\right|^2 f(\rho,\theta)\,d\rho \le \frac{\epsilon^2}{\pi}$$

and

$$\int_{\rho_0}^{l_0}|\zeta_\theta(\rho)|\,d\rho \le \sqrt{(l_0-\rho_0)\int_{\rho_0}^{l_0}\zeta_\theta^2(\rho)\,d\rho} \le \epsilon.$$

Therefore

$$\max_{\rho_0\le\rho\le l_0}|\xi_\theta(\rho)| \le \frac{2}{L}\int_{\rho_0}^{l_0}|\zeta_\theta(\rho)|\,d\rho \le 2\epsilon.$$

Thus (3.9) holds for this arbitrarily fixed $\theta\in W_1\cap W_2\cap W_3$. This completes the proof. □

Now we are ready to prove Theorem 2. Fix a number $\epsilon\in(0,1)$. By Lemma 3.2, there exists $\delta_1(\epsilon)>0$ such that if $\mathrm{vol}(M)\ge v_n-\delta_1(\epsilon)$ then

$$L \le 2-\frac{1}{\pi}\arccos\sqrt[3]{-1+\epsilon} < 2$$

and therefore

$$\left|\frac{\partial u}{\partial\rho}(\rho,\theta)\right|^2 \le |\nabla u|^2_{(\rho,\theta)} \le \lambda_1(1-u^2(\rho,\theta)) \le 4n(1-u^2(\rho,\theta)).$$

Because $u$ attains its maximum value at $m$, we have

$$u(\rho,\theta)|_{\rho=0}=1 \quad\text{and}\quad \left.\frac{\partial}{\partial\rho}u(\rho,\theta)\right|_{\rho=0}=0.$$

So, if we let $\rho_0=\rho_0(\epsilon)=\min\{(\epsilon/8n)^2, \frac{1}{2}\arcsin\epsilon, \frac{1}{2}\arccos\sqrt[3]{1-\epsilon}\}$, then

$$u(\rho_0,\theta)\ge\sqrt[3]{1-\epsilon}, \qquad \left|\frac{\partial u}{\partial\rho}(\rho_0,\theta)\right|\le\frac{\epsilon}{2},$$
$$\sin L\rho_0\le\sin 2\rho_0\le\epsilon, \qquad \cos L\rho_0\ge\cos 2\rho_0\ge\sqrt[3]{1-\epsilon},$$
$$|L(\pi-\rho_0)-\pi|\le\arccos\sqrt[3]{1-\epsilon}, \qquad \cos L(\pi-\rho_0)\le\sqrt[3]{-1+\epsilon},$$

when $\mathrm{vol}(M)\ge v_n-\delta_1(\epsilon)$. Finally, when $\mathrm{vol}(M)\ge v_n-\delta_2(\frac{\epsilon}{2}, \rho_0(\epsilon))$, by Lemma 3.8 there exists a direction $\theta_0\in N$ such that

$$l(\theta_0) > \pi - \frac{\rho_0}{2} \quad \text{and} \quad \max_{\rho_0 \le \rho \le \pi - \rho_0} |\xi_{\theta_0}(\rho)| \le \epsilon.$$

Let $\delta(\epsilon) = \min\{\delta_1(\epsilon), \delta_2(\frac{\epsilon}{2}, \rho_0(\epsilon))\}$. If $\mathrm{vol}(M) \ge v_n - \delta(\epsilon)$ then, for the $\theta_0$ given above,

$$\begin{aligned}
\inf_M u &\le u(\pi - \rho_0, \theta_0) \\
&\le |\xi_{\theta_0}(\pi - \rho_0)| + \sin L\rho_0 + 2\left|\frac{\partial u}{\partial \rho}(\rho_0, \theta_0)\right| \\
&\quad + u(\rho_0, \theta_0) \cdot \cos L\rho_0 \cdot \cos L(\pi - \rho_0) \\
&\le \epsilon + \epsilon + \epsilon + \sqrt[3]{1-\epsilon} \cdot \sqrt[3]{1-\epsilon} \cdot \sqrt[3]{-1+\epsilon} \\
&= -1 + 4\epsilon
\end{aligned}$$

by (3.1). This implies our claim and completes the proof of Theorem 2. □

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