Bounds for the Fundamental Frequencies of an Elastic Medium

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I. Introduction

Let Ω be an open bounded region in $\mathbf{R}^n$. Let Δ be the Laplace operator,

$$\Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_n^2},$$

on real-valued functions on Ω. Let $\boldsymbol{\Delta}$ be the Laplace operator applied to components of $\mathbf{R}^n$-valued functions on Ω. Let $\partial\Omega$, $\partial/\partial n$, grad, and div be (respectively) the boundary set of Ω, the outward-pointing normal derivative on $\partial\Omega$, the gradient, and the divergence. Let $L^2(\Omega)$ be the Hilbert space of square-integrable real-valued functions on Ω and let $\mathbf{L}^2(\Omega)$ be the Hilbert space of $\mathbf{R}^n$-valued functions $\mathbf{u}$ on Ω, so that the pointwise $\mathbf{R}^n$ norm $\|\mathbf{u}(x)\|$ is in $L^2(\Omega)$.

Consider the three eigenvalue problems:

$$\boldsymbol{\Delta}\mathbf{u}+\alpha\,\mathrm{grad}(\mathrm{div}\,\mathbf{u})+\Lambda^{(\alpha)}\mathbf{u}=\mathbf{0} \text{ in } \Omega, \qquad \mathbf{u}=\mathbf{0} \text{ on } \partial\Omega; \tag{1.1}$$

$$\Delta v+\lambda v=0 \text{ in } \Omega, \qquad v=0 \text{ on } \partial\Omega; \qquad \text{and} \tag{1.2}$$

$$\Delta^2\phi+\nu\Delta\phi=0 \text{ in } \Omega, \qquad \phi=\frac{\partial\phi}{\partial n}=0 \text{ on } \partial\Omega. \tag{1.3}$$

The number α is a nonnegative constant, and in this paper α always refers to the constant in problem (1.1). We consider the collection of eigenvalue problems in (1.1) for all nonnegative values of α. Problem (1.1) governs the behavior of an elastic medium and thus appears often in the theory of elasticity. The problems (1.2) and (1.3) are often referred to as the Dirichlet and buckling eigenvalue problems, respectively.

In this paper we extend some inequalities obtained recently by Kawohl and Sweers [3] for the smallest eigenvalue of problem (1.1) to all of its eigenvalues. We assume that the eigenvalues of (1.1), (1.2), and (1.3) are ordered $\Lambda_1^{(\alpha)}\le\Lambda_2^{(\alpha)}\le\cdots$, $\lambda_1\le\lambda_2\le\cdots$, and $\nu_1\le\nu_2\le\cdots$, respectively.

We establish that the following inequalities hold:

$$\lambda_{[(n+k-1)/n]}\le\Lambda_k^{(\alpha)} \quad \text{for all } \alpha\ge 0, \text{ all } k, n, \tag{1.4}$$

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where $[x]$ is the largest integer less than or equal to x;

$$\Lambda_k^{(\alpha)} \leq \left(1+\frac{m\alpha}{n}\right)\lambda_{l+1}+\alpha\sum_{i=1}^{l}\lambda_i \quad \text{for all } \alpha\geq 0, \text{ all } k, n, \tag{1.5}$$

where $l=[k/n]$ and $m=k \pmod n$; and

$$\Lambda_k^{(\alpha)} \leq \nu_k \quad \text{for all } \alpha\geq 0, \text{ all } k \text{ and all even } n. \tag{1.6}$$

Inequalities (1.5) and (1.6) provide upper bounds for $\Lambda_k^{(\alpha)}$. Since (1.5) depends on α and (1.6) does not, for large α, (1.6) is a sharper upper bound for $\Lambda_k^{(\alpha)}$ than (1.5). For small $\alpha\geq 0$ the right-hand side of (1.5) approaches λ_{l+1} and since (as is well known) $\lambda_{l+1}<\nu_{l+1}$ for all l, inequality (1.5) provides the sharper upper bound for small α. Inequalities (1.4) and (1.5) can be applied to earlier results of Hile and Protter [1] and Levine and Protter [4] to establish:

$$\Lambda_k^{(\alpha)} \leq \left(1+\frac{k\alpha}{n}\right)\left(\Lambda_{(m-1)n+1}^{(\alpha)}+\frac{4}{nm}\sum_{i=1}^{m}\Lambda_{(i-1)n+1}^{(\alpha)}\right), \tag{1.7}$$

where $m=[k/n]$;

$$\Lambda_k^{(\alpha)} \geq \frac{4\pi^2 n}{n+2}\left(\frac{1}{B_n V}\right)^{2/n}\left(1+\left[\frac{k-1}{n}\right]\right)^{2/n}; \quad \text{and} \tag{1.8}$$

$$\frac{1}{k}\sum_{j=1}^{k}\Lambda_j^{(\alpha)} \geq \frac{4\pi^2 n}{n+2}\left(\frac{1}{B_n V}\right)^{2/n}\left(\frac{nl}{k}l^{2/n}+\frac{m}{k}(l+1)^{2/n}\right), \tag{1.9}$$

where $m=k \pmod n$ and $l=[k/n]$. In (1.8) and (1.9), B_n is the volume of the unit ball in $\mathbf{R}^n$ and V is the volume of Ω.

II. Relation to the Dirichlet Problem

It is well known (see, e.g., [3]) that the eigenvalues of equation (1.1) satisfy

$$\Lambda_k^{(\alpha)} = \min R_\alpha(\mathbf{u}), \tag{2.1}$$

where $\mathbf{u}\in \mathbf{L}^2(\Omega)$ with $\mathbf{u}=\mathbf{0}$ on $\partial\Omega$ is $\mathbf{L}^2(\Omega)$-orthogonal to the first $k-1$ eigenvectors of (1.1), and where

$$R_\alpha(\mathbf{u}) = \frac{\int_\Omega(-\Delta\mathbf{u}\cdot\mathbf{u}+\alpha(\operatorname{div}\mathbf{u})^2)}{\int_\Omega\|\mathbf{u}\|^2}. \tag{2.2}$$

We will use (2.1) and (2.2) often below. We begin by establishing a relationship between the eigenvalues of (1.2) and those of (1.1) when $\alpha=0$.

LEMMA 1. *If* $\lambda_1\leq\lambda_2\leq\cdots$ *and* $\Lambda_1^{(0)}\leq\Lambda_2^{(0)}\leq\cdots$ *then* $\lambda_{[(n+k-1)/n]}=\Lambda_k^{(0)}$ *for all* k.

Proof. When $\alpha=0$, equation (1.1) becomes

$$\Delta\mathbf{u}+\Lambda^{(0)}\mathbf{u}=0 \text{ in } \Omega; \qquad \mathbf{u}=\mathbf{0} \text{ on } \partial\Omega. \tag{2.3}$$

Equation (2.3) is simply an uncoupled system of n copies of the Dirichlet problem (1.2). Thus the eigenvalues of (2.3) are exactly the eigenvalues of (1.2) with n-fold multiplicity (multiplied by the multiplicity of the corresponding eigenvalue of (1.2)). Thus

$$\lambda_1 = \Lambda_1^{(0)} = \Lambda_2^{(0)} = \cdots = \Lambda_n^{(0)};$$
$$\lambda_2 = \Lambda_{n+1}^{(0)} = \Lambda_{n+2}^{(0)} = \cdots = \Lambda_{2n}^{(0)}; \ldots ;$$
$$\lambda_k = \Lambda_{(k-1)n+1}^{(0)} = \Lambda_{(k-1)n+2}^{(0)} = \cdots = \Lambda_{kn}^{(0)}; \ldots .$$

In other words, if $mn+1 \le k \le (m+1)n$ then $\lambda_{m+1} = \Lambda_k^{(0)}$. Evidently we must have $m = [(k-1)/n]$ or $m+1 = [(n+k-1)/n]$. □

LEMMA 2. *If $\alpha \le \beta$ then $\Lambda_k^{(\alpha)} \le \Lambda_k^{(\beta)}$ for all k.*

Proof. For $\alpha \le \beta$, $R_\alpha(\mathbf{u}) \le R_\beta(\mathbf{u})$. It is then easy to show, using (2.2), that $\Lambda_k^{(\alpha)} \le \Lambda_k^{(\beta)}$ for all $k = 1, 2, \ldots$. □

We can summarize the above two lemmas in the following theorem.

THEOREM 1. *If Ω is an open bounded region in $\mathbf{R}^n$ so that problems* (1.1) *and* (1.2) *are self-adjoint with discrete spectra, then*

$$\lambda_{[(n+k-1)/n]} = \Lambda_k^{(0)} \le \Lambda_k^{(\alpha)} \le \Lambda_k^{(\beta)} \quad \text{if } 0 \le \alpha \le \beta. \tag{2.4}$$

Now that we have established the lower bound for $\Lambda_k^{(\alpha)}$ in terms of the eigenvalues of the Dirichlet problem (1.2), we turn to the upper bound (1.5).

THEOREM 2. *The eigenvalues of* (1.1) *and* (1.2) *are related by*

$$\Lambda_k^{(\alpha)} \le \left(1 + \frac{m\alpha}{n}\right)\lambda_{l+1} + \alpha \sum_{i=1}^{l} \lambda_i \tag{2.5}$$

for $n \ge 2$, where $l = [k/n]$ and $m = k$ (mod n). Moreover, if $\alpha > 0$ then inequality (2.5) *is strict. (Note: If $l = 0$ then the sum in* (2.5) *should be interpreted to be zero.)*

Proof. Let $u_1, u_2, \ldots, u_i, \ldots$ be $L^2(\Omega)$-unit eigenvectors of (1.2). Let $l = [k/n]$ and $m = k$ (mod n). Suppose that

$$\int_\Omega \left(\frac{\partial u_{l+1}}{\partial x_1}\right)^2 \le \int_\Omega \left(\frac{\partial u_{l+1}}{\partial x_2}\right)^2 \le \cdots \le \int_\Omega \left(\frac{\partial u_{l+1}}{\partial x_n}\right)^2. \tag{2.6}$$

Define the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \\ 0 \end{bmatrix}; \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \\ 0 \end{bmatrix}; \ \ldots ; \ \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \\ \\ 1 \end{bmatrix} \tag{2.7}$$

in $\mathbf{R}^n$. We assume that the usual basis of $\mathbf{R}^n$ is used. If (2.6) does not hold we can arrange the integrals in ascending order and make appropriate modifications in the definitions of (2.7). So the assumption in (2.6) results in no loss of generality. □

Consider the collection of the k $\mathbf{L}^2(\Omega)$-unit vectors:

$$u_i\mathbf{e}_j \quad \text{for } i=1,2,\dots,l;\ j=1,2,\dots,n, \qquad \text{and}$$
$$u_{l+1}\mathbf{e}_j \quad \text{for } j=1,2,\dots,m. \tag{2.8}$$

There is a unit vector, $\mathbf{v}$, in the span of the vectors in (2.8) which is $\mathbf{L}^2(\Omega)$-orthogonal to the first $k-1$ eigenvectors of (1.1). Write $\mathbf{v}$ as

$$\mathbf{v}=\sum_{i,j}\mu_{ij}u_i\mathbf{e}_j, \tag{2.9}$$

where the μ_{ij} are constants which must satisfy

$$\sum_{i,j}\mu_{ij}^2=1, \tag{2.10}$$

since $\mathbf{v}$ is an $\mathbf{L}^2(\Omega)$-unit vector and the $u_i\mathbf{e}_j$ are mutually orthogonal in $\mathbf{L}^2(\Omega)$.

Since $\mathbf{v}$ is orthogonal to each of the first $k-1$ eigenvectors of (1.1), we have

$$\begin{aligned}\Lambda_k^{(\alpha)}\le R_\alpha(\mathbf{v})&=\int_\Omega(-\Delta\mathbf{v}\cdot\mathbf{v}+\alpha(\operatorname{div}\mathbf{v})^2)\\&=\int_\Omega\sum_{i,j}(\mu_{ij})^2(-\Delta u_i)u_i+\alpha\int_\Omega\Big(\sum_{i,j}\mu_{ij}\frac{\partial u_i}{\partial x_j}\Big)^2.\end{aligned} \tag{2.11}$$

The right-hand equality in (2.11) follows from the definitions of $\mathbf{v}$ and Δ. Continuing, we see that

$$\begin{aligned}R_\alpha(\mathbf{v})&=\sum_{i,j}(\mu_{ij})^2\int_\Omega(-\Delta u_i)u_i+\alpha\int_\Omega\Big(\sum_{i,j}\mu_{ij}\frac{\partial u_i}{\partial x_j}\Big)^2\\&=\sum_{i,j}(\mu_{ij})^2\lambda_i+\alpha\int_\Omega\Big(\sum_{i,j}\mu_{ij}\frac{\partial u_i}{\partial x_j}\Big)^2.\end{aligned} \tag{2.12}$$

Equation (2.12) follows from the fact that $-\Delta u_i=\lambda_i u_i$. Next we apply the Schwarz inequality to the right-hand term of (2.12), and remember from (2.10) that $\sum_{i,j}\mu_{ij}^2=1$. That is,

$$\begin{aligned}\int_\Omega\Big(\sum_{i,j}\mu_{ij}\frac{\partial u_i}{\partial x_j}\Big)^2&\le\int_\Omega\sum_{i,j}(\mu_{ij})^2\sum_{i,j}\Big(\frac{\partial u_i}{\partial x_j}\Big)^2\\&=\int_\Omega\sum_{i,j}\Big(\frac{\partial u_i}{\partial x_j}\Big)^2.\end{aligned} \tag{2.13}$$

Thus from (2.12) and (2.13) we see that

$$R_\alpha(\mathbf{v}) \le \sum_{i,j} \lambda_i(\mu_{ij})^2 + \alpha \sum_{i,j} \int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)^2$$

$$(2.14) \qquad = \sum_{i,j} \lambda_i(\mu_{ij})^2 + \alpha \sum_{i=1}^{l} \int_\Omega \|\operatorname{grad} u_i\|^2 + \alpha \sum_{j=1}^{m} \int_\Omega \left(\frac{\partial u_{l+1}}{\partial x_j}\right)^2$$

$$= \sum_{i,j} \lambda_i(\mu_{ij})^2 + \alpha \sum_{i=1}^{l} \lambda_i + \alpha \sum_{j=1}^{m} \int_\Omega \left(\frac{\partial u_{l+1}}{\partial x_j}\right)^2.$$

Notice that if $l=0$ then $i=1$ and $j=1,2,\ldots,m<n$, so

$$\sum_{i,j} \int_\Omega \left(\frac{\partial u_i}{\partial x_j}\right)^2 = \sum_{j=1}^{m} \int_\Omega \left(\frac{\partial u_{l+1}}{\partial x_j}\right)^2$$

and the sum $\sum_{i=1}^{l} \int_\Omega \|\operatorname{grad} u_i\|^2 = 0$ in this case.

The assumption of (2.6) implies that

$$(2.15) \qquad \sum_{j=1}^{m} \int_\Omega \left(\frac{\partial u_{l+1}}{\partial x_j}\right)^2 \le \frac{m}{n} \int_\Omega \|\operatorname{grad} u_{l+1}\|^2 \le \frac{m}{n} \lambda_{l+1}.$$

Placing the result (2.15) into (2.14) and (2.11) yields

$$\Lambda_k^{(\alpha)} \le R_\alpha(\mathbf{v}) \le \sum_{i,j} \lambda_i(\mu_{ij})^2 + \alpha \sum_{i=1}^{l} \lambda_i + \frac{\alpha m}{n} \lambda_{l+1}$$

$$(2.16) \qquad \le \lambda_{l+1} + \alpha \sum_{i=1}^{l} \lambda_i + \frac{\alpha m}{n} \lambda_{l+1}$$

$$= \left(1 + \frac{m\alpha}{n}\right) \lambda_{l+1} + \alpha \sum_{i=1}^{l} \lambda_i.$$

This establishes the weak inequality. If the inequality in (2.5) is an equality with $\alpha > 0$ then we must have all of the following:

(i) $\mathbf{v}$ is an eigenvector of (1.1);
(ii) there is a function φ on Ω so that $\partial u_i/\partial x_j = \varphi(x)\mu_{ij}$;
(iii) $\int_\Omega (\partial u_{l+1}/\partial x_j)^2$ is independent of j; and
(iv) $\mu_{ij} = 0$ unless $i = l+1$.

For $l \ge 1$, conditions (ii) and (iv) are inconsistent since then $\partial u_1/\partial x_j = 0$ in Ω for all j. Thus $u_1 = 0$ in Ω, which is not possible.

If $l=0$ and equality holds in (2.5), we must have $\mathbf{v}$ with components that are constant multiples of u, the first eigenfunction of (1.2). Say $\mathbf{v} = \mathbf{a}u$ where $\mathbf{a}$ has components $a_1, a_2, \ldots, a_n$. Writing out the system (1.1) assuming equality in (2.5), we have

$$(2.17) \qquad a_i \Delta u + \alpha \frac{\partial}{\partial x_i}(\operatorname{div} \mathbf{v}) + \left(1 + \frac{m\alpha}{n}\right) \lambda_1 a_i u = 0 \text{ in } \Omega$$

for $i = 1, 2, \ldots, n$, and where

$$(2.18) \qquad \Delta u + \lambda_1 u = 0 \text{ in } \Omega.$$

Application of condition (2.18) to (2.17) yields the simpler system

$$\frac{\partial}{\partial x_i}(\operatorname{div} \mathbf{v}) + \frac{m\lambda_1 a_i}{n} u = 0 \text{ in } \Omega. \tag{2.19}$$

An alternative form of (2.19) is

$$\operatorname{grad}(\mathbf{a}\cdot\operatorname{grad}(u)) + \frac{m\lambda_1 u\mathbf{a}}{n} = 0, \tag{2.20}$$

using the fact that $\operatorname{div}(\mathbf{v}) = \mathbf{a}\cdot\operatorname{grad}(u)$ by definition of $\mathbf{v}$.

The system in the form of (2.20) implies that $\mathbf{a}\cdot\operatorname{grad}(u)$ depends only on its coordinate in the direction of $\mathbf{a}$ by taking dot products with vectors orthogonal to $\mathbf{a}$. Since $\mathbf{a}\cdot\operatorname{grad}(u)$ depends only on one coordinate, so does its gradient, and thus, upon dotting (2.20) with $\mathbf{a}$, we see that u depends only on that one coordinate also. In this case we have $u = 0$ in Ω, since u is constant on all hyperplanes orthogonal to $\mathbf{a}$ and all such hyperplanes meet $\partial\Omega$ someplace where $u = 0$. As u, the first eigenfunction of (1.2), is never identically zero, inequality (2.5) must be strict in this case also.

III. Relation to the Buckling Problem

In this section we establish inequality (1.6) for even n. We begin with a few preliminaries.

PROPOSITION 1. *The eigenvalues of* (1.3) *are characterized by*

$$\nu_k = \min \frac{\int_\Omega (\Delta^2\phi)\phi}{\int_\Omega \|\operatorname{grad}\phi\|^2}, \tag{3.1}$$

where ϕ is chosen among all functions in $L^2(\Omega)$ with $\phi = \partial\phi/\partial n = 0$ on $\partial\Omega$ whose gradient is in $\mathbf{L}^2(\Omega)$ and $\operatorname{grad}\phi$ *and* $\operatorname{grad}\phi_i$ *are $\mathbf{L}^2(\Omega)$-orthogonal, where $i = 1, 2, \ldots, k-1$ and ϕ_i is an eigenfunction of* (1.3) *corresponding to ν_i.*

LEMMA 3. *Let n be an even positive integer. Let $\mathcal{U}: \mathbf{R}^n \to \mathbf{R}^n$ be given by*

$$\mathcal{U}(z_1, z_2, \ldots, z_n) = (z_2, -z_1, z_4, -z_3, \ldots, z_n, -z_{n-1}) \tag{3.2}$$

in standard coordinates. Extend $\mathcal{U}$ to $\mathbf{L}^2(\Omega)$ pointwise. Then

(i) *$\mathcal{U}$ is unitary in $\mathbf{L}^2(\Omega)$;*
(ii) *for any $\mathbf{v} \in \mathbf{L}^2(\Omega)$ for which $\Delta\mathbf{v}$ exists,* $\Delta(\mathcal{U}(\mathbf{v})) = \mathcal{U}(\Delta\mathbf{v})$; *and*
(iii) *for all $\phi \in C^2(\Omega)$,* $\operatorname{div}(\mathcal{U}(\operatorname{grad}\phi)) = 0$.

Proposition 1 is well known (see, e.g., [3]). Lemma 3 is easily shown from the definitions.

THEOREM 3. *The eigenvalues of* (1.1) *and* (1.3) *are related by*

$$\Lambda_k^{(\alpha)} \le \nu_k, \quad k = 1, 2, \ldots, \tag{3.3}$$

for all $\alpha \ge 0$ if n is even.

Proof. Let $\phi_1, \phi_2, \ldots, \phi_k$ be eigenvectors of (1.3) corresponding to $\nu_1, \nu_2, \ldots, \nu_k$. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$ be $\mathbf{L}^2(\Omega)$-unit eigenvectors of (1.1) for some fixed α. Proposition 1 states that we may assume that $\phi_1, \ldots, \phi_k$ satisfy $\int_\Omega \operatorname{grad}\phi_i \cdot \operatorname{grad}\phi_j = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Lemma 3(i) implies that $\int_\Omega \mathcal{U}(\operatorname{grad}\phi_i)\cdot\mathcal{U}(\operatorname{grad}\phi_j) = \delta_{ij}$ also. Choose $\mathbf{v}$ in the span of $\mathcal{U}(\operatorname{grad}\phi_1), \mathcal{U}(\operatorname{grad}\phi_2), \ldots, \mathcal{U}(\operatorname{grad}\phi_k)$ so that $\mathbf{v}$ is an $\mathbf{L}^2(\Omega)$-unit vector and $\mathbf{v}$ is $\mathbf{L}^2(\Omega)$-orthogonal to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$.

Say $\mathbf{v} = \sum_{i=1}^k \beta_i \mathcal{U}(\operatorname{grad}\phi_i)$. Then, using Lemma 3, we have

$$\Lambda_k^{(\alpha)} \le R_\alpha(\mathbf{v}) = \int_\Omega \left(-\Delta \sum_{i=1}^k \beta_i \mathcal{U}(\operatorname{grad}\phi_i)\right)\cdot\left(\sum_{j=1}^k \beta_j \mathcal{U}(\operatorname{grad}\phi_j)\right). \tag{3.4}$$

Using Proposition 1 and Lemma 3, it can be shown that the right-hand side of (3.4) is equal to $\sum_{i=1}^k \beta_i^2 \nu_i$, which does not exceed ν_k. □

REMARK. It is not clear whether inequality (3.3) holds in $\mathbf{R}^n$ for n odd and $n \ge 3$. However in $\mathbf{R}^1$ the eigenvalue problem (1.1) reduces to

$$(\Delta + \alpha\Delta)u + \Lambda^{(\alpha)}u = 0 \text{ in } \Omega; \qquad u = 0 \text{ on } \partial\Omega. \tag{3.5}$$

Equation (3.5) is equivalent to

$$\Delta u + \frac{\Lambda^{(\alpha)}}{1+\alpha}u = 0 \text{ in } \Omega; \qquad u = 0 \text{ on } \partial\Omega. \tag{3.6}$$

Clearly (3.6) is the Dirichlet problem (1.2) with eigenvalues $\Lambda_k^{(\alpha)}/(1+\alpha) = \lambda_k$. Thus in $\mathbf{R}^1$ we have $\Lambda_k^{(\alpha)} = (1+\alpha)\lambda_k$. Because $0 < \lambda_k \le \nu_k$ for all k, inequality (3.3) cannot hold for all α since for $\alpha = 2\nu_k/\lambda_k - 1$ we have $\Lambda_k^{(\alpha)} = 2\nu_k > \nu_k$.

IV. Upper and Lower Bounds for the Frequencies of the Elastic Problem

We can now establish a lower bound for the eigenvalues $\Lambda_k^{(\alpha)}$ which is dependent on the volume of the domain Ω, as well as an upper bound which is independent of the domain but dependent on some of the lower eigenvalues. The lower bound follows from some work of Levine and Protter [4]. The upper bound follows from a paper of Hile and Protter [1]. The result we use from [4] is as follows.

PROPOSITION 2. *The eigenvalues of* (1.2) *satisfy*

$$\lambda_k \ge \frac{1}{k}\sum_{j=1}^k \lambda_j \ge \frac{4\pi^2 n}{n+2}\left(\frac{k}{B_n V}\right)^{2/n}, \tag{4.1}$$

where B_n is the volume of the unit ball in $\mathbf{R}^n$ and V is the volume enclosed by Ω.

We use Proposition 2 and Theorem 1 to establish the next theorem.

THEOREM 4. *The eigenvalues of* (1.1) *satisfy*

$$\frac{1}{k}\sum_{i=1}^{k}\Lambda_i^{(\alpha)} \geq \frac{4n\pi^2}{n+2}\left(\frac{1}{B_nV}\right)^{2/n}\left(\frac{nj}{k}j^{2/n}+\frac{m}{k}(j+1)^{2/n}\right), \tag{4.2}$$

where $m=k \pmod{n}$ *and* $j=[k/n]$.

Proof. Using Theorem 1 we have

$$\frac{1}{k}\sum_{i=1}^{k}\Lambda_i^{(\alpha)} \geq \frac{1}{k}\sum_{i=1}^{k}\lambda_{[(n+i-1)/n]}. \tag{4.3}$$

Using the fact that $k=nj+m$ and the definition of $[\cdot]$, we can rewrite the right-hand side of (4.3) as

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_{[(n+i-1)/n]} = \frac{nj}{k}\left(\frac{1}{j}\sum_{i=1}^{j}\lambda_i\right)+\frac{m}{k}\lambda_{j+1}. \tag{4.4}$$

According to Proposition 2 we have

$$\begin{aligned}\frac{nj}{k}\left(\frac{1}{j}\sum_{i=1}^{j}\lambda_i\right)+\frac{m}{k}\lambda_{j+1} \geq \frac{nj}{k}\,\frac{4\pi^2 n}{n+2}\left(\frac{j}{B_nV}\right)^{2/n}\\ +\frac{m}{k}\cdot\frac{4\pi^2 n}{n+2}\left(\frac{j+1}{B_nV}\right)^{2/n}.\end{aligned} \tag{4.5}$$

Combining (4.3), (4.4), and (4.5) yields the result. □

In [4] Levine and Protter established that

$$\Lambda_k^{(\alpha)} \geq \frac{1}{k}\sum_{j=1}^{k}\Lambda_j^{(\alpha)} \geq \frac{4\pi^2 n}{n+2}\left(\frac{1}{B_nV}\right)^{2/n}\left(\frac{k}{n}\right)^{2/n}. \tag{4.6}$$

In some cases inequality (4.2) is sharper than (4.6) and in others inequality (4.6) is sharper than (4.2). For example, if $n=3$ and $k=1$ then (4.2) yields a larger lower bound for $\Lambda_1^{(\alpha)}$ than (4.6) does. In fact, this is generally the case when $k<n$. On the other hand, if $n=3$ and $k=4$ then (4.6) yields a larger lower bound than (4.2).

Applying the result of Theorem 1 to inequality (4.1) for λ_k yields another interesting result.

THEOREM 5. *The eigenvalues of* (1.1) *satisfy*

$$\Lambda_k^{(\alpha)} \geq \frac{4\pi^2 n}{n+2}\left(\frac{1}{B_nV}\right)^{2/n}\left[\frac{n+k-1}{n}\right]^{2/n}. \tag{4.7}$$

Proof. By Theorem 1 we have

$$\Lambda_k^{(\alpha)} \geq \lambda_{[(n+k-1)/n]}. \tag{4.8}$$

Application of inequality (4.1) to (4.8) yields (4.7). □

REMARK. The result of Theorem 5 is at least as sharp as the inequality in (4.6) for all k and n, since for $k, n \geq 1$ we always have $k/n \leq [(n+k-1)/n]$.

The last application of our results is to a domain-independent upper bound for $\Lambda_k^{(\alpha)}$ in terms of some of the $\Lambda_j^{(\alpha)}$ for $j<k$. Results of this type have been considered by several authors, the most recent by this author [2]. However, we will use the earlier result of Hile and Protter [1].

PROPOSITION 3. *The eigenvalues of* (1.2) *satisfy*

$$\lambda_{m+1} \leq \lambda_m + \frac{4}{mn} \sum_{i=1}^{m} \lambda_i \tag{4.9}$$

for $\Omega \subseteq \mathbf{R}^n$.

Our next result follows from (4.9).

THEOREM 6. *The eigenvalues of* (1.1) *satisfy*

$$\begin{aligned} \Lambda_k^{(\alpha)} &\leq \left(\frac{n+\alpha k}{n}\right)\left(\Lambda_{(m-1)n+1}^{(\alpha)} + \frac{4}{nm} \sum_{i=1}^{m} \Lambda_{(i-1)n+1}^{(\alpha)}\right) \\ &\leq \frac{(n+\alpha k)(n+4)}{n^2} \Lambda_{(m-1)n+1}^{(\alpha)}, \end{aligned} \tag{4.10}$$

where $m=[k/n]$. *Inequality* (4.10) *is strict for* $\alpha>0$.

Proof. Choose $k=1+(j-1)n$ in (2.4). Then (2.4) becomes

$$\lambda_{[(n+1+(j-1)n-1)/n]} = \lambda_j \leq \Lambda_{(j-1)n+1}^{(\alpha)} \quad \text{for all } j=1,2,\dots. \tag{4.11}$$

We also need a lower bound for λ_{m+1} in terms of some of the $\Lambda^{(\alpha)}$. We use Theorem 2 for such a bound. Theorem 2 asserts that

$$\Lambda_k^{(\alpha)} \leq \left(1+\frac{\alpha j}{n}\right)\lambda_{m+1} + \alpha \sum_{i=1}^{m} \lambda_i, \tag{4.12}$$

where $k=mn+j$ and $0 \leq j < n$. The summation in (4.12) is bounded above by $m\alpha\lambda_{m+1}$. Thus (4.12) can be weakened to

$$\Lambda_k^{(\alpha)} \leq \left(1+\frac{\alpha j}{n}\right)\lambda_{m+1} + m\alpha\lambda_{m+1} = \left(1+\frac{k\alpha}{n}\right)\lambda_{m+1}. \tag{4.13}$$

Application of (4.11) and (4.13) to (4.9) yields

$$\begin{aligned} \Lambda_k^{(\alpha)} &\leq \left(1+\frac{\alpha k}{n}\right)\lambda_{m+1} \\ &\leq \left(1+\frac{\alpha k}{n}\right)\left(\Lambda_{(m-1)n+1}^{(\alpha)} + \frac{4}{mn} \sum_{i=1}^{m} \Lambda_{(i-1)n+1}^{(\alpha)}\right). \end{aligned} \tag{4.14}$$

Inequality (4.14) is the desired result, inequality (4.10). The second inequality of (4.10) is obtained from the fact that $\Lambda_{(i-1)n+1}^{(\alpha)} \leq \Lambda_{(m-1)n+1}^{(\alpha)}$ for all $i=1,2,\dots,m$. Finally, inequalities (4.12), (4.13) and thus (4.14) are strict for $\alpha>0$ by Theorem 2. □

A stronger implicit bound for the $\Lambda^{(\alpha)}$ may be obtained from the implicit inequality of Hile and Protter [1], but the result is not particularly pleasing.

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