Semicocycles and Weighted Composition Semigroups on H^p

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1. Introduction

We consider semigroups $(T_t)_{t\geq 0}$ on the Hardy space H^p of the unit disc $\mathbf{D}$, which are of the form

$$(1.1) \quad T_t: H^p \to H^p, \quad T_t f(z) = h_t(z) f(\Phi_t(z)) \quad (t \geq 0, f \in H^p, z \in \mathbf{D})$$

with suitable analytic functions $\Phi_t: \mathbf{D} \to \mathbf{D}$ and $h_t: \mathbf{D} \to \mathbf{C}$. We suppose that $(\Phi_t)_{t\geq 0}$ is a semiflow (sometimes called semigroup) of analytic functions; that is, the mapping $t \mapsto \Phi_t(z)$ is continuous for every $z \in \mathbf{D}$, $\Phi_0(z) \equiv z$ and $\Phi_{t+s}(z) = \Phi_t(\Phi_s(z))$ for all $z \in \mathbf{D}$ and $t, s \in [0, \infty)$. An application of Vitali's theorem shows the joint continuity of the mapping $(z, t) \mapsto \Phi_t(z)$. We often write Φ instead of $(\Phi_t)_{t\geq 0}$. Semiflows are studied very comprehensively by Berkson and Porta [1].

In this paper we discuss the manner in which properties of semigroups $(T_t)_{t\geq 0}$ of the form (1.1) are related to the properties of the functions h_t.

DEFINITION 1. Let Φ be a semiflow. A family $(h_t)_{t\geq 0}$ of analytic functions $h_t: \mathbf{D} \to \mathbf{C}$ is called a *semicocycle for* Φ if

(i) the mapping $t \mapsto h_t(z)$ is continuous for every $z \in \mathbf{D}$,
(ii) $h_{t+s} = h_t \cdot (h_s \circ \Phi_t)$ for $t, s \geq 0$, and
(iii) $h_0 \equiv 1$.

$(h_t)_{t\geq 0}$ is said to be

continuous, if the mapping $(t, z) \mapsto h_t(z)$ is continuous,
differentiable, if for every $z \in \mathbf{D}$ the mapping $t \mapsto h_t(z)$ is differentiable, and
bounded, if every h_t is bounded $(t \geq 0)$.

By using Vitali's theorem one can show that a bounded semicocycle is continuous. If Φ is a semiflow and $(h_t)_{t\geq 0}$ a bounded semicocycle for Φ, then the family $(T_t)_{t\geq 0}$, given by (1.1), is a semigroup of bounded linear operators on H^p.

Let $\omega: \mathbf{D} \to \mathbf{C}$ be an analytic function satisfying $\omega \not\equiv 0$. If all zeros of ω are in the set $\{z \in \mathbf{D}: \Phi_t(z) = z$ for all $t \in [0, \infty)\}$ of fixed points of Φ, then

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(1.2) $$h_t(z)=\frac{\omega(\Phi_t(z))}{\omega(z)} \quad \text{for } t\geq 0 \text{ and } z\in \mathbf{D}$$

defines a semicocycle for Φ. It is differentiable with

$$\frac{\partial}{\partial t}h_t(z)\bigg|_{t=0}=\frac{\omega'(z)G(z)}{\omega(z)} \quad \text{when } z\in\mathbf{D},$$

where G is the so-called infinitesimal generator of Φ (see below). Semicocycles arising in this way are discussed by Siskakis [6].

Let $g:\mathbf{D}\to\mathbf{C}$ be an analytic function and define $(h_t)_{t\geq 0}$ by

(1.3) $$h_t(z)=\exp\left(\int_0^t g(\Phi_s(z))\,ds\right) \quad \text{for } t\geq 0 \text{ and } z\in\mathbf{D}.$$

Then $(h_t)_{t\geq 0}$ is a differentiable semicocycle for Φ also. Furthermore, we have

$$\frac{\partial}{\partial t}h_t(z)\bigg|_{t=0}=g(z) \quad \text{for } z\in\mathbf{D}.$$

In Lemma 2.2 we prove that every semicocycle given by (1.2) has a representation of the form (1.3), and we state conditions for which the converse holds.

Our main result shows that for a strongly continuous semigroup $(T_t)_{t\geq 0}$ of the form (1.1), the functions h_t are given by (1.3).

THEOREM 1. *Let p be in $[1,\infty)$, Φ be a semiflow, and h_t be analytic in* $\mathbf{D}$ *for $t\geq 0$ such that $(T_t)_{t\geq 0}$, defined by* (1.1), *is a strongly continuous semigroup on H^p. Then $(h_t)_{t\geq 0}$ is a differentiable semicocycle for Φ, the function $g:=(\partial/\partial t)h_t|_{t=0}$ is analytic in* $\mathbf{D}$, *and* (1.3) *holds.*

The proof characterizes, for every $z\in\mathbf{D}$, the mapping $t\mapsto h_t(z)$ as the unique solution of the differential equation on $\mathbf{R}_+$:

(1.4) $$\frac{d}{dt}w(t)=w(t)\cdot g(\Phi_t(z)) \quad \text{with } w(0)=1.$$

There are parallels to analogous properties of the semiflow Φ. Berkson and Porta [1] showed that for every semiflow Φ the limit

$$G(z)=\lim_{t\to 0+}\frac{\Phi_t(z)-z}{t}=\frac{\partial}{\partial t}\Phi_t(z)\bigg|_{t=0}$$

exists uniformly on compact subsets of $\mathbf{D}$ and, for every $z\in\mathbf{D}$, the mapping $t\mapsto\Phi_t(z)$ satisfies the differential equation

(1.5) $$\frac{d}{dt}w(t)=G(w(t)) \quad \text{with } w(0)=z.$$

The analytic function G is called the *infinitesimal generator* of Φ. We list some properties of a semiflow Φ and its generator G (see [1], [5]): If G is not identically zero, then G has the unique representation

$$G(z)=F(z)(z-b)(\bar{b}z-1)$$

with $|b|\le 1$ and analytic $F:\mathbf{D}\to\mathbf{C}$ with nonnegative real part. So the set of the zeros of G is equal to $\{b\}\cap\mathbf{D}$. The point b is called the *Denjoy-Wolff point* of Φ. If $|b|<1$, then it is a fixed point for every Φ_t. In this case there is a unique schlicht function $h:\mathbf{D}\to\mathbf{C}$ with $h(0)=0$ and $h'(0)=1$ such that $h(\gamma_b(\Phi_t(z))) = e^{G'(b)\cdot t}\cdot h(\gamma_b(z))$ for all $z\in\mathbf{D}$ and $t\ge 0$, where $\gamma_b(z) = (z-b)(1-\bar{b}z)^{-1}$. This function h is called the *univalent* (or schlicht) *function* associated with Φ.

On the other hand, the following question arises: Which conditions for an analytic function g imply the strong continuity of the semigroup $(T_t)_{t\ge 0}$, defined by (1.1) and (1.3)? We state the following theorem. (By $\mathbf{N}$ we denote the set $\{1,2,3,\dots\}$ and $\mathbf{N}_0=\mathbf{N}\cup\{0\}$.)

THEOREM 2. *Let p be in $[1,\infty)$ and Φ be a semiflow with generator G and (if $G\neq 0$) Denjoy-Wolff point b. Furthermore, let $g:\mathbf{D}\to\mathbf{C}$ be analytic.*

(a) *If*

$$M:=\sup_{z\in\mathbf{D}}\operatorname{Re} g(z)<\infty, \tag{1.6}$$

then the semigroup $(T_t)_{t\ge 0}$, defined by (1.1) *and* (1.3), *is strongly continuous on H^p.*

(b) *If the semigroup $(T_t)_{t\ge 0}$, defined by* (1.1) *and* (1.3), *is strongly continuous, then its generator $(A,D(A))$ is given by*

$$D(A)=\{f\in H^p\colon G\cdot f'+g\cdot f\in H^p\}$$

and

$$Af=G\cdot f'+g\cdot f \quad \textit{for } f\in D(A).$$

If $G\neq 0$ and $|b|<1$, then choose any $c\in\mathbf{D}\setminus\{b\}$ and define $\alpha=g(b)/G'(b)$, $\gamma_b(z)=(z-b)/(1-\bar{b}z)$, and

$$\omega(z)=\exp\left(\int_c^z \frac{g(\zeta)}{G(\zeta)}\,d\zeta\right) \quad \textit{for } z\in\mathbf{D}\setminus\{b\}.$$

Then the point spectrum $\pi(A)$ of A satisfies

$$\pi(A)=\left\{k\cdot G'(b)+g(b)\colon k\in\mathbf{N}_0 \textit{ and } \frac{(h\circ\gamma_b)^{k+\alpha}}{\omega}\in H^p\right\}.$$

REMARK. $(h\circ\gamma_b)^{k+\alpha}/\omega$ is always analytic at b, but ω is not.

So we see that there are at least as many strongly continuous weighted composition semigroups for a given semiflow as there are analytic functions on $\mathbf{D}$ whose real parts are bounded above. Theorem 2 is a generalization of Theorems 1, 2, and 3 of Siskakis [6].

2. Semicocycles and the Proof of Theorem 1

Throughout this section Φ denotes a semiflow with generator G and (if $G \neq 0$) Denjoy–Wolff point b. Let $(h_t)_{t \geq 0}$ be a semicocycle for Φ. We state some properties of $(h_t)_{t \geq 0}$.

LEMMA 2.1.

(a) $(h_t)_{t \geq 0}$ *is bounded if and only if* $\limsup_{t \to 0+} \|h_t\|_\infty < \infty$.

(b) h_t *has no zero* $(t \geq 0)$.

Proof. (a) Let $M_t := \sup_{z \in \mathbf{D}} |h_t(z)| \in [0, \infty]$ for $t \geq 0$.

"$\Rightarrow$": Note that $M_t < +\infty$ for every $t \geq 0$. Condition (ii) of Definition 1 shows the subadditivity of $t \mapsto \log M_t$. From [2, VIII, 1.4 and 1.5] we know that there are $M, w \in \mathbf{R}$ such that $M_t \leq Me^{wt}$ for all $t \geq 0$. The conclusion follows.

"$\Leftarrow$": There are $M, \delta > 0$ with $|h_t(z)| \leq M$ for $z \in \mathbf{D}$ and $t \in [0, \delta]$. For arbitrary $t > 0$ there exists $\tau \in [0, \delta]$ and $n \in \mathbf{N}$ such that $t = n\tau$. Then the equation

$$h_t(z) = \prod_{k=0}^{n-1} h_\tau(\Phi_{k\tau}(z)) \quad \text{for } z \in \mathbf{D}$$

shows that $h_t \in H^\infty$.

(b) Assume the existence of $z_0 \in \mathbf{D}$ and $t_0 \in [0, \infty)$ with $h_{t_0}(z_0) = 0$. Define $I = \{t \in [0, \infty) : h_t(z_0) = 0\}$ and $\tau = \inf I$. Part (i) of Definition 1 implies that $\tau \in I$. Part (iii) implies that $0 \notin I$. Since

$$h_{\tau+s}(z_0) = h_\tau(z_0) h_s(\Phi_\tau(z_0)) = 0 \quad \text{for all } s \in [0, \infty),$$

we have $I = [\tau, \infty)$. Choose $\epsilon \in (0, \tau)$. Then $h_{\tau+s-\epsilon}(z_0) = h_s(z_0) h_{\tau-\epsilon}(\Phi_s(z_0))$ implies $h_{\tau-\epsilon}(\Phi_s(z_0)) = 0$ for all $s \in [\epsilon, \tau)$.

If $\Phi_s(z_0) = z_0$ for all $s \in [\epsilon, \tau)$, then $\tau - \epsilon \in I$, which is a contradiction to $I = [\tau, \infty)$. Otherwise, the analytic function $h_{\tau-\epsilon}$ is zero on the nonconstant path $[\epsilon, \tau) \ni s \mapsto \Phi_s(z_0)$, hence $h_{\tau-\epsilon} \equiv 0$ and $\tau - \epsilon \in I$. This is again a contradiction to $I = [\tau, \infty)$. □

The next lemma shows the way in which the representations (1.2) and (1.3) are connected.

LEMMA 2.2. (a) *If* ω *is analytic in* $\mathbf{D}$ *without zeros in* $\mathbf{D} \setminus \{b\}$, *then the analytic function* $g : \mathbf{D} \to \mathbf{C}$, *given by* $g = G\omega'/\omega$, *satisfies*

$$\frac{\omega(\Phi_t(z))}{\omega(z)} = \exp\left(\int_0^t g(\Phi_s(z))\, ds\right) \quad \textit{for } t \geq 0 \textit{ and } z \in \mathbf{D}. \tag{2.1}$$

(b) *Assume* $G \neq 0$. *If* g *is analytic in* $\mathbf{D}$, *then there exists an analytic* ω: $\mathbf{D} \setminus \{b\} \to \mathbf{C}$ *such that* (2.1) *holds* (*even for* $z = b$ *in the case* $b \in \mathbf{D}$). *If* $|b| < 1$, *then* ω *is analytic at* b *if and only if* $\alpha := g(b)/G'(b) \in \mathbf{N}_0$. *In this case* α *is the order of the zero* b *at* ω.

Proof. (a) Note that for every $z \in \mathbf{D}$ and $t \geq 0$ the mapping

$$\gamma : [0, t] \to \mathbf{D} \quad \text{with } \gamma(s) := \Phi_s(z)$$

is a continuously differentiable path in $\mathbf{D}$ from z to $\Phi_t(z)$ which satisfies (1.5). We have (first $z \neq b$)

$$\exp\left(\int_0^t g(\Phi_s(z))\,ds\right) = \exp\left(\int_0^t \frac{\omega'(\Phi_s(z))}{\omega(\Phi_s(z))}\gamma'(s)\,ds\right) = \exp\left(\int_z^{\Phi_t(z)} \frac{\omega'(\zeta)}{\omega(\zeta)}\,d\zeta\right)$$

$$= \exp\left(\log \omega(\zeta)\Big|_z^{\Phi_t(z)}\right) = \frac{\omega(\Phi_t(z))}{\omega(z)}.$$

(b) Fix any $c \in \mathbf{D}\backslash\{b\}$ and put

$$\omega(z) = \exp\left(\int_c^z \frac{g(\zeta)}{G(\zeta)}\,d\zeta\right) \quad \text{for } z \in \mathbf{D}\backslash\{b\}.$$

(We choose any integration path in $\mathbf{D}\backslash\{b\}$ between c and z; the definition is independent of this choice.) Without loss of generality we assume $b \in \mathbf{D}\backslash\{0\}$ and $c = 0$. A computation gives

$$\omega(z) = (z-b)^\alpha \psi(z) \quad \text{for } z \in \mathbf{D}\backslash\{b\},$$

where

$$\psi(z) = (-b)^{-\alpha} \exp\left(\int_0^z \frac{g(\zeta)(\zeta-b) - \alpha G(\zeta)}{G(\zeta)(\zeta-b)}\,d\zeta\right) \quad \text{for } z \in \mathbf{D}\backslash\{b\}.$$

The integrand $\zeta \mapsto [g(\zeta)(\zeta-b) - \alpha G(\zeta)]/[G(\zeta)(\zeta-b)]$ is analytic in $\mathbf{D}$. Hence, ψ is analytic in $\mathbf{D}$ without zeros. Therefore, ω is analytic at b if and only if $z \mapsto (z-b)^\alpha$ is, that is, if and only if $\alpha \in \mathbf{N}_0$. In this case α is obviously the order of the zero b of ω. If $\alpha \in \{-1, -2, -3, \ldots\}$, then ω has a pole of order $-\alpha$ at b, in all other cases an essential singularity. To show the validity of (2.1), use (1.5) and γ (see above) to parametrize a path in $\mathbf{D}$ from z to $\Phi_t(z)$. □

REMARK. Let b be in $\mathbf{D}$ and ω be analytic in $\mathbf{D}\backslash\{b\}$. If there exists an $\alpha \in \mathbf{C}$ such that $\omega(z)(z-b)^\alpha$ is analytic at b, then the representations (1.2) and (1.3) are equivalent.

We now show that the form (1.3) can always be achieved if $(T_t)_{t \geq 0}$, defined by (1.1), is strongly continuous.

PROOF OF THEOREM 1.

(1) *Analyticity of g.* Let $(A, D(A))$ denote the infinitesimal generator of $(T_t)_{t \geq 0}$. Fix $z \in \mathbf{D}$, $t > 0$ and choose a closed neighbourhood $U \subset \mathbf{D}$ of z. Note that $D(A)$ is dense in H^p with respect to the topology of uniform convergence on compact subsets. Hence, there exists a function $f \in D(A)$ such that

$$f(\Phi_s(\zeta)) \neq 0 \quad \text{for } s \in [0, t] \text{ and } \zeta \in U.$$

Now we see that

$$\lim_{s\to 0+}\frac{h_s(\zeta)-1}{s}=\lim_{s\to 0+}\frac{1}{f(\Phi_s(\zeta))}\left[\frac{T_sf(\zeta)-f(\zeta)}{s}-\frac{f(\Phi_s(\zeta))-f(\zeta)}{s}\right]$$
$$=\frac{1}{f(\zeta)}\left[Af(\zeta)-f'(\zeta)G(\zeta)\right]$$

uniformly for $\zeta\in U$. It follows that g is analytic in $\mathbf{D}$.

(2) $(h_t)_{t\geq 0}$ *is a differentiable semicocycle.* The semicocycle properties of $(h_t)_{t\geq 0}=(T_t\,\mathbf{1})_{t\geq 0}$ (where $\mathbf{1}\colon \mathbf{D}\to\{1\}$ denotes the constant function) are evident. The proof of the following fact is elementary: If $f\colon[0,\infty)\to\mathbf{C}$ is continuous and has a continuous right-hand derivative, then f is differentiable. Since

$$\lim_{s\to 0+}\frac{h_{t+s}(z)-h_t(z)}{s}=h_t(z)\lim_{s\to 0+}\frac{h_s(\Phi_t(z))-1}{s}=h_t(z)g(\Phi_t(z))$$

holds for every $t\geq 0$ and $z\in\mathbf{D}$, this fact implies the differentiability of $(h_t)_{t\geq 0}$.

(3) (1.3) *holds.* Part (2) shows that, for every $z\in\mathbf{D}$, the mapping $t\mapsto h_t(z)$ is a solution of (1.4). According to the theory of ordinary differential equations, the unique solution is given by (1.3). □

3. Proof of Theorem 2 and Discussion of Continuity of the Generator

The following lemma is a generalization of [6, Thm. 1]. That proof applies here with small changes.

LEMMA 3.1. *Let p be in $(0,\infty)$, Φ be a semiflow, and $(h_t)_{t\geq 0}$ be a semicocycle for Φ. If $\limsup_{t\to 0+}\|h_t\|_\infty\leq 1$, then $\lim_{t\to 0+}\|h_t\cdot(f\circ\Phi_t)-f\|_p=0$ for every $f\in H^p$.*

In the situation of Theorem 2 we have, for $t\geq 0$,

$$\|h_t\|_\infty\leq\sup_{z\in\mathbf{D}}\exp\left(\int_0^t\operatorname{Re}g(\Phi_s(z))\,ds\right)\leq e^{tM},$$

so Lemma 3.1 implies the strong continuity of $(T_t)_{t\geq 0}$. The rest of the proof of Theorem 2 is a generalization and reformulation of [6, Thms. 2 and 3]. □

An advantage of the form (1.3) of a semicocycle is that every multiplication semigroup on H^p can be seen as a special case of a weighted composition semigroup. We will now discuss the continuity of the generator of $(T_t)_{t\geq 0}$. The following generalization of [6, Cor. 1] is also applicable to multiplication semigroups.

COROLLARY 3.2. *Let the assumptions of Theorem 2 be satisfied. Then the following two conditions are equivalent:*

(i) *A is continuous,*

(ii) *$G \equiv 0$, and g is bounded in $\mathbf{D}$.*

The proof follows from [6, Cor. 1], together with the following lemma.

LEMMA 3.3. *Let p be in $[1, \infty)$, Φ be a Möbius transformation of $\mathbf{D}$, and h be analytic in $\mathbf{D}$ so that $A: H^p \to H^p$, $Af := h \cdot (f \circ \Phi)$, is well defined. Then h is bounded in $\mathbf{D}$.*

Proof. A is a closed operator on H^p; therefore A is continuous. By induction we obtain $h^n \in H^p$ for every $n \in \mathbf{N}$. Use Littlewood's subordination theorem [3, p. 29] to see that $h^n \circ \Phi^{-1} \in H^p$ if $h^n \in H^p$. So we have $h \in H^q$ for every $q \in [1, \infty)$. It follows that, for every $n \in \mathbf{N}$,

$$\|h\|_{np} = \left(\|A(h^{n-1} \circ \Phi^{-1})\|_p\right)^{1/n} \le \cdots \le \left(\|A\|^n \left(\frac{1+|\Phi^{-1}(0)|}{1-|\Phi^{-1}(0)|}\right)^{n/p}\right)^{1/n} =: M.$$

For $\alpha > 0$, put $E_\alpha := \{\theta \in [0, 2\pi] : |h(e^{i\theta})| > \alpha\}$. By λ we denote the 1-dimensional Lebesgue measure on $\partial\mathbf{D}$. If $\lambda(E_\alpha) > 0$, then a straightforward computation yields $\alpha^q \lambda(E_\alpha)/(2\pi) \le \|h\|_q^q$ for $q \ge 1$, and therefore

$$M \ge \|h\|_q \ge \alpha\left(\frac{\lambda(E_\alpha)}{2\pi}\right)^{1/q} \uparrow \alpha \quad \text{as } q \uparrow \infty.$$

It follows that $M \ge \alpha$. We have now deduced that $\operatorname{ess\,sup}_{\theta \in [0, 2\pi]} |h(e^{i\theta})| \le M$. We write $h = s \cdot F$ as in [4, p. 63] with an inner function s and

$$F(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|h(e^{i\theta})|\, d\theta\right) \quad \text{for } z \in \mathbf{D}.$$

Using the Poisson integral we obtain, for $z \in \mathbf{D}$,

$$|h(z)| \le \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log M\, d\theta\right) = M,$$

and the conclusion follows. □

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References

1. E. Berkson and H. Porta, *Semigroups of analytic functions and composition operators,* Michigan Math. J. 25 (1978), 101–115.
2. N. Dunford and H. Schwarz, *Linear operators I,* Wiley-Interscience, New York, 1958.
3. P. L. Duren, *Theory of H^p spaces,* Academic Press, New York, 1970.

4. K. Hoffman, *Banach spaces of analytic functions,* Prentice-Hall, Englewood Cliffs, N.J., 1962.
5. A. Siskakis, *Semigroups of composition operators and the Cesaro operator on* H^p, Thesis, Univ. of Illinois, 1985.
6. ———, *Weighted composition semigroups on Hardy spaces,* Linear Algebra Appl. 84 (1986), 359–371.

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