Some Results about the Space $A^{-\infty}$ of Analytic Functions

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Introduction and Results

In this paper, we prove some results about the class $A^{-\infty}$ of analytic functions on the unit disc $D=\{z\in\mathbf{C}: |z|<1\}$ that satisfy $|f(z)|\le C(1-|z|)^{-n}$ for some $C>0$ and $n\in\mathbf{N}$. This class was studied extensively by Korenblum in [6], where some results about the moduli of the zeros of the functions in the space $A^{-\infty}$ are given. There ([6, p. 202], see also [8, Thm. 6]), a function in $A^{-\infty}$ is constructed whose sequence of zeros $(z_n)_n$ satisfies $\sum_n(1-|z_n|)=+\infty$; so, in general, the Blaschke product cannot be defined. We shall prove that the function $f\in H(D)$, defined by $f(z)=g_3(\tau)$ where $z=e^{2\pi i\tau}$, $\Im\tau>0$ and g_3 is the well-known Eisenstein invariant (see [1, p. 12]), belongs to $A^{-\infty}$ and f also satisfies $\sum_n(1-|a_n|)=+\infty$, where $(a_n)_n$ is its sequence of zeros in D.

It is easy to prove (see [8, p. 224]) that the function

$$(1)\qquad f(z)=\sum_{n\ge 0}a_nz^n \quad \text{belongs to } A^{-\infty} \text{ if and only if } (a_n)_n\in s',$$

where s′ is the space of tempered sequences in which $(a_n)_n\in s'$ if there exist C and $\alpha>0$ such that $|a_n|\le C(n+1)^{\alpha}$. So the boundary values of the functions of the space $A^{-\infty}$ are the distributions on the circle $\mathbf{T}=\{z\in\mathbf{C}: |z|=1\}$ with vanishing negative Fourier coefficients. Moreover, if $f(z)=\sum_{n\ge 0}a_nz^n$ and $u\in D'(\mathbf{T})$ is its boundary value, then $a_n=\langle u, e^{-in\theta}\rangle$.

In the following theorem, we give an analogous identification for the functions in the space $A^{-\infty}$ as some Fourier–Laplace type transforms of the tempered distributions with support contained in $[0,+\infty)$.

THEOREM 1. *A function f belongs to the space $A^{-\infty}$ if and only if there exists a tempered distribution u_f with* $\operatorname{supp}(u_f)\subset[0,+\infty)$ *such that* $f(z)=\langle u_f(t), e^{t(z+1)/(2z-2)}\rangle$ *if* $|z|<1$. *Moreover, if* $f(z)/(1-z)=\sum_n a_nz^n$ *then* $a_n=\langle u_f, L_n(t)e^{-(t/2)}\rangle$, *where the* $L_n(t)$ *are Laguerre polynomials.*

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Using this result, we give a very easy proof of the following classical result (see [10, (3.1), Thm. 1 and (4.3)], also [7, Thm. 1]).

THEOREM A. *Let f be an analytic function on the upper half-plane. Then there exists a tempered distribution u with* $\operatorname{supp}(u) \subset [0, +\infty)$ *such that* $f(w) = \langle u(t), e^{iwt} \rangle$ *when* $\Im w > 0$ *if and only if the function f satisfies:*

$$(2) \qquad |f(w)| \leq \frac{C(1+|w|^2)^m}{(\Im w)^n} \quad \text{for } C > 0,\ n, m \in \mathbf{N}, \text{ and } \Im w > 0.$$

We finally apply the above results to prove the following proposition.

PROPOSITION 2. *There exists a tempered distribution u, with* $\operatorname{supp}(u) \subset [0, +\infty)$, *such that if v is a tempered distribution with* $\operatorname{supp}(v) \subset [0, +\infty)$ *then* $u * v \notin L^p([0, +\infty))$ *when* $1 \leq p \leq 2$.

Notice that if $u \in \mathcal{E}'$ then there exists $v \in \mathcal{E}'$ such that $u * v \in L^p$ for all $p \geq 1$.

Proof of the Results

Given two $\mathbf{R}$-independent complex numbers w_1 and w_2 (i.e., their ratio is not real), they define the lattice $\Omega = \mathbf{Z}w_1 + \mathbf{Z}w_2$. The Eisenstein series of order 6 is defined by $G_6 = \sum_{w \in \Omega,\, w \neq 0} 1/w^6$. We consider the invariant g_3 defined by $g_3 = 140 G_6$ and the function $g_3(\tau) = 140 \sum'_{n,m} 1/(m + n\tau)^6$ where the sum is extended to all pairs of integers except $(0,0)$, which we denote by $\sum'$. This function is defined for $\Im \tau > 0$.

We now prove that this function provides an example of a function in $A^{-\infty}$ whose zeros do not satisfy the Blaschke condition.

PROPOSITION 3. *Let $f \in H(D)$ be the function defined by $f(z) = g_3(\tau)$, where $z = e^{2\pi i \tau}$. Then*

(a) *$f \in A^{-\infty}$, and*
(b) *if $(a_n)_n$ are the zeros of f on D then $\sum_n (1 - |a_n|) = +\infty$.*

Proof. Indeed, by [1, p. 20], $g_3(\tau) = (8\pi^6/27)(1 - 504 \sum_n \sigma_5(n) e^{2\pi i n \tau})$ where $\sigma_5(n) = \sum_{d|n} d^5$, so $f(z) = (8\pi^6/27)(1 - 504 \sum_n \sigma_5(n) z^n)$. By [1, p. 135], $|\sigma_5(n)| \leq C \cdot n^5$ for some constant C. An application of (1) completes the proof of (a).

Consider the unimodular group $H = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \in \mathbf{Z} \text{ and } ad - bc = 1\}$. For simplicity we write $M(a, b, c, d)$ instead of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We prove (b) by using three preliminary steps α, β, and γ.

(α) *$g_3(\tau) = (c\tau + d)^{-6} g_3((a\tau + b)/(c\tau + d))$ for all $M(a, b, c, d) \in H$.* To show this we proceed as follows. $G_6 = G_6(w_1, w_2) = \sum'_{n,m} 1/(nw_1 + mw_2)^6$. Taking $\tau = w_1/w_2$ we obtain $G_6(w_1, w_2) = w_1^{-6} G_6(1, \tau) = (w_1^{-6}/140) g_3(\tau)$. Since G_6 depends only on Ω, by taking another basis w_1', w_2' it follows that $G_6(w_1, w_2) = G_6(w_1', w_2')$.

We consider the lattice $\Omega(1,\tau)$. Given $M(a,b,c,d)\in H$, it is clear that $a\tau+b$ and $c\tau+d$ are bases of $\Omega(1,\tau)$ and that $\Im((a\tau+b)/(c\tau+d))>0$. Thus

$$G_6(1,\tau)=G_6(c\tau+d,a\tau+b)=(c\tau+d)^{-6}G_6\left(1,\frac{a\tau+b}{c\tau+d}\right).$$

Then $g_3(\tau)=(c\tau+d)^{-6}g_3((a\tau+b)/(c\tau+d))$ and (α) is established.

(β) *For all $M(a,b,c,d)\in H$, $g_3((ai+b)/(ci+d))=0$.* To prove this, let $M(0,1,-1,0)$ be in H. By (α), $g_3(i)=(-i)^{-6}g_3(1/(-i))=-g_3(i)$; hence $g_3(i)=0$. As $M(a,b,c,d)\in H$, by (α),

$$0=g_3(i)=(ci+d)^{-6}g_3\left(\frac{ai+b}{ci+d}\right)\quad\text{and}\quad g_3\left(\frac{ai+b}{ci+d}\right)=0.$$

(γ) *If $(\eta_k)_k$ are the zeros of $g_3(\tau)$ then $\sum_{k\in\mathbf{N}}\Im\eta_k/(1+|\eta_k|^2)=\infty$.* By (β),

$$\sum_{k\in\mathbf{N}}\frac{\Im\eta_k}{1+|\eta_k|^2}\ge\sum_{M(a,b,c,d)\in H}\frac{\Im\left(\dfrac{ai+b}{ci+d}\right)}{1+\left|\dfrac{ai+b}{ci+d}\right|^2}$$

$$\ge\sum_{M(a,b,c,d)\in H,\ a,b>0}\frac{\Im\left(\dfrac{(ai+b)(-ci+d)}{c^2+d^2}\right)}{1+\dfrac{a^2+b^2}{c^2+d^2}}$$

$$=\sum_{M(a,b,c,d)\in H,\ a,b>0}\frac{1}{a^2+b^2+c^2+d^2}.$$

Let a and b be relatively prime; that is, let $(a,b)=1$. Then there exist $x_0,y_0\in\mathbf{N}$ satisfying $ax_0-by_0=1$. Hence, the integer solutions of $ax-by=1$ will be $x=x_0+bt$ and $y=y_0+at$ with $t\in\mathbf{Z}$. So there exists a solution satisfying $0\le y\le a-1$. Since $b\in\mathbf{N}$, it follows that

$$0\le x=\frac{1+by}{a}\le\frac{1+b(a-1)}{a}=b+\left(\frac{1-b}{a}\right)\le b.$$

Since $ax-by=1$, we have $M(a,b,y,x)\in H$. Consequently, given the relatively prime $a,b\in\mathbf{N}$, there exist $d,c\in\mathbf{N}$ satisfying $d\le b$, $c\le a-1\le a$, and $M(a,b,c,d)\in H$. Hence

$$\frac{1}{a^2+b^2+c^2+d^2}\ge\frac{1}{a^2+b^2+a^2+b^2}=\frac{1}{2}\left(\frac{1}{a^2+b^2}\right).$$

Then

$$\sum_{k\in\mathbf{N}}\frac{\Im\eta_k}{1+|\eta_k|^2}\ge\frac{1}{2}\sum_{(a,b)=1}\frac{1}{a^2+b^2}.$$

We shall prove that the last series diverges. Indeed,

$$\frac{\pi^2}{6}\sum_{(a,b)=1}\frac{1}{a^2+b^2}=\left(\sum_{d=1}^{\infty}\frac{1}{d^2}\right)\left(\sum_{(a,b)=1}\frac{1}{a^2+b^2}\right)$$
$$\geq\sum_{m,n=1,m\neq n}^{\infty}\frac{1}{m^2+n^2}=\sum_{m,n=1}^{\infty}\frac{1}{m^2+n^2}-\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m^2}$$
$$=\sum_{m,n=1}^{\infty}\frac{1}{m^2+n^2}-\frac{\pi^2}{12}\geq\int_1^{\infty}\int_1^{\infty}\frac{dx\,dy}{x^2+y^2}-\frac{\pi^2}{12}$$
$$=\int_1^{\infty}\frac{dx}{x}\int_1^{\infty}\frac{d(y/x)}{1+y^2/x^2}-\frac{\pi^2}{12}=\int_1^{\infty}\arctan\left(\frac{y}{x}\right)\Big|_1^{\infty}\frac{dx}{x}-\frac{\pi^2}{12}$$
$$=\int_1^{\infty}\left(\frac{\pi}{2}-\arctan\left(\frac{1}{x}\right)\right)\frac{dx}{x}-\frac{\pi^2}{12}$$
$$=\int_1^{\infty}\arctan x\frac{dx}{x}-\frac{\pi^2}{12}=\infty.$$

This proves (γ).

We now use (γ) to establish part (b) of Proposition 3. If $(a_n)_n$ are the zeros of $f(z)$, it is clear (by the definition of f) that the zeros of $g_3(\tau)$ are $\eta_{k,n}=(1/(2\pi i))\log a_k=n+\arg(a_k)/(2\pi)-i\log|a_k|/(2\pi)$. Hence

$$\sum_{n,k=1}^{\infty}\frac{\Im\eta_{k,n}}{1+|\eta_{k,n}|^2}=\sum_k\sum_{n\in\mathbf{Z}}\frac{-(\log|a_k|/2\pi)}{1+(n+\arg(a_k)/2\pi)^2+(\log|a_k|/2\pi)^2}$$
$$\leq\sum_k\sum_{n\in\mathbf{Z}}\frac{-(\log|a_k|/2\pi)}{1+(n+\arg(a_k)/2\pi)^2}.$$

As $-1\leq\arg(a_k)/(2\pi)\leq 1$, we obtain

$$\sum_{n,k=1}^{\infty}\frac{\Im\eta_{k,n}}{1+|\eta_{k,n}|^2}\leq\sum_k-\frac{\log|a_k|}{2\pi}\left(\sum_{n\geq1}\frac{1}{1+(n+1)^2}+\sum_{n\leq-1}\frac{1}{1+(n-1)^2}+\frac{1}{2}\right)$$
$$\leq C\sum_k-\log|a_k|$$

for some constant C. By (γ), $\sum_{n,k=1}^{\infty}\Im\eta_{k,n}/(1+|\eta_{k,n}|^2)$ diverges, hence $\sum_k-\log|a_k|$ diverges too, and so $\sum_k(1-|a_k|)=\infty$. This establishes part (b) of Proposition 3. □

We now prove Theorem 1. In [2], we write the set of all tempered distributions with support contained in $[0,+\infty)$ as the dual of the space

$$S^+=\{\psi:[0,+\infty)\to\mathbf{C}\mid\psi(t)=\phi(t)\text{ for }t\geq0\text{ and some }\phi\text{ in }S\}.$$

This space is a Fréchet space with the seminorms $\|\psi\|_{k,n}=\sup_{t\geq0}t^k|\psi^{(n)}(t)|$, where $k,n\in\mathbf{N}$. Since $\Re((z+1)/(2z-2))<0$ when $|z|<1$, it follows that $e^{t(z+1)/(2z-2)}\in S^+$. Also $L_n(t)e^{-(t/2)}\in S^+$; so if $u\in(S^+)'$ (i.e., if $u\in S'$ and $\operatorname{supp}(u)\subset[0,+\infty)$) then $\langle u(t),e^{t(z+1)/(2z-2)}\rangle$ and $\langle u,L_n(t)e^{-(t/2)}\rangle$ are well defined.

We will use the following result, which can be found in [2, Thm. 2.9] and in [5, p. 550].

THEOREM B. *The mapping* $\mathcal{L}: (S^+)' \to s'$ *defined by*

$$\mathcal{L}(u) = (\langle u, L_n(t)e^{-(t/2)}\rangle)_n$$

is an isomorphism from $(S^+)'$ *onto* s'. *So, if* $u \in (S^+)'$, *then*

$$u = \sum_n \langle u, L_n(t)e^{-(t/2)}\rangle L_n(t)e^{-(t/2)}$$

in the weak topology of $(S^+)'$.

Proof of Theorem 1. We first prove that the first condition implies the second. It is clear that $f(z)/(1-z) \in A^{-\infty}$ if $f \in A^{-\infty}$. By (1), $f(z)/(1-z) = \sum_{n\geq 0} a_n z^n$ with $(a_n)_n \in s'$. By Theorem B, there exists $u \in S'$ with $\operatorname{supp}(u) \subset [0, +\infty)$ such that $u = \sum_n a_n L_n(t)e^{-(t/2)}$. As the series converges in the weak topology of $(S^+)'$, it follows that

$$\begin{aligned}\langle u(t), e^{t(z+1)/(2z-2)}\rangle &= \sum_{n\geq 0} a_n \int_0^\infty L_n(t)e^{-(t/2)}e^{t(z+1)/(2z-2)}\,dt\\ &= \sum_{n\geq 0} a_n z^n(1-z),\end{aligned}$$

where the second equality follows from [3, p. 191, (3.2)]. Hence $f(z) = \langle u(t), e^{t(z+1)/(2z-2)}\rangle$.

We now prove that the second condition implies the first. Let $u \in S'$ with $\operatorname{supp}(u) \subset [0, +\infty)$. Proceeding analogously we have

$$\langle u(t), e^{t(z+1)/(2z-2)}\rangle = (1-z) \sum_{n\geq 0} \langle u, L_n(t)e^{-(t/2)}\rangle z^n.$$

From Theorem B and (1), it follows that $\langle u(t), e^{t(z+1)/(2z-2)}\rangle \in A^{-\infty}$. □

Using Theorem 1, we give a very easy proof of Theorem A.

Proof of Theorem A. We consider the bilinear transformation defined by $\mathcal{Z}(w) = (w-i)/(w+i)$, which transforms the upper half-plane on the unit disc. It is clear that its inverse is $\mathcal{W}(z) = (iz+i)/(1-z)$. Now, by using the above transformation, given an analytic function f on the upper half-plane we obtain an analytic function g_f on the unit disc by the formula $g_f(z) = f((iz+i)/(1-z))$; reciprocally, if g is an analytic function on the unit disc then the function $f_g(w) = g((w-i)/(w+i))$ is an analytic function on the upper half-plane.

Since $1-|z|^2 = (2\Im w)/(1+|w|^2+2\Im w)$, it follows that

$$\frac{1+|w|^2}{2\Im w} \leq \frac{1}{1-|z|} \leq \frac{2(1+|w|^2)}{2\Im w}. \tag{3}$$

By (3), we deduce that if f is an analytic function on the upper half-plane, then f satisfies (2) if and only if g_f belongs to $A^{-\infty}$. Applying Theorem 1, we conclude that f satisfies (2) if and only if there exists a tempered distribution u with $\operatorname{supp}(u) \subset [0, +\infty)$ and $f(w) = \langle u(t), e^{(itw)/2} \rangle$ if $\Im w > 0$. □

To finish, we prove Proposition 2. We need the following lemma.

LEMMA 4. *Let $f \in L^p([0, +\infty))$ with $1 \leq p \leq 2$. Then the analytic function on the upper half-plane defined by $\tilde{f}(w) = \mathfrak{F}_1(f(t)e^{-\Im wt})(\Re w)$ belongs to H^q with $1/p + 1/q = 1$, where $\mathfrak{F}_1$ is the Fourier transform on $L^1([0, +\infty))$ and H^q is the Hardy space in the upper half-plane* (*see* [4, II]).

Proof of Lemma 4. If $f \in L^1([0, +\infty))$, it is clear that $\tilde{f}(w)$ is bounded, so $\tilde{f} \in H^\infty$. Let $1 < p \leq 2$. If $w = x + iy$ with $y > 0$, then

$$f(t)e^{-yt} \in L^1([0, +\infty)) \cap L^p([0, +\infty)).$$

Hence $\mathfrak{F}_1(f(t)e^{-yt}) = \mathfrak{F}_p(f(t)e^{-yt})$ a.e., where $\mathfrak{F}_p$ is the Fourier transform on $L^p([0, +\infty))$. By [9, (4.1.2)], $\tilde{f}(x+iy) \in L^q(\mathbf{R})$ as a function of x, and

$$\int_{\mathbf{R}} |\tilde{f}(x+iy)|^q \, dx \leq C \cdot \left(\int_{\mathbf{R}} |f(t)e^{-yt}|^p \, dt \right)^{1/(1-p)} \leq C(\|f\|_p^p)^{1/(1-p)};$$

that is, $\tilde{f} \in H^q$. □

Proof of Proposition 2. Let h be the analytic function on the upper half-plane defined by $h(w) = f((w-i)/(w+i))$, where f is a function of the space $A^{-\infty}$ satisfying $\sum_n (1 - |a_n|) = +\infty$ and $(a_n)_n$ are the zeros of f on the unit disc. As $f \in A^{-\infty}$, h satisfies (2) (see the proof of Theorem A). By Theorem A, there exists $u \in S'$ with $\operatorname{supp}(u) \subset [0, +\infty))$ such that $h(w) = \langle u(t), e^{iwt} \rangle$ if $\Im w > 0$.

By the definition of the function h, its zeros are $\beta_n = (-i - ia_n)/(a_n - 1)$. By the choice of f, $\sum_n (1 - |a_n|) = +\infty$, so

$$\sum_n \frac{\Im \beta_n}{1 + |\beta_n|^2} = +\infty. \tag{4}$$

We assume that there exist $p \in [1, 2]$ and $v \in S'$ with $\operatorname{supp}(v) \subset [0, +\infty)$, such that $u * v \in L^p([0, +\infty))$. From Lemma 4 it follows that the function φ defined on the upper half-plane by $\varphi(w) = \langle (u * v)(t), e^{iwt} \rangle$ belongs to the space H^q for q satisfying $1/p + 1/q = 1$.

It is clear that $\varphi(w) = \langle u(t), e^{iwt} \rangle \langle v(t), e^{iwt} \rangle = h(w) \langle v(t), e^{iwt} \rangle$ and so $\varphi(\beta_n) = 0$. By (4), if $(\gamma_n)_n$ are the zeros of φ then $\sum_n \Im \gamma_n / (1 + |\gamma_n|^2) = +\infty$. But φ belongs to H^q with $2 \leq q < +\infty$, so by [4, p. 55] its zeros must satisfy $\sum_n \Im \gamma_n / (1 + |\gamma_n|^2) < +\infty$. Thus, if $v \in S'$ with $\operatorname{supp}(v) \subset [0, +\infty)$, it follows that $u * v \notin L^p([0, +\infty))$ $(1 \leq p \leq 2)$, and Proposition 2 is proved. □

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