Interpolation from Curves in Pseudoconvex Boundaries

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Introduction

Let D be a smoothly bounded pseudoconvex domain in $\mathbf{C}^n$, and let $A^\infty(D)$ denote those smooth functions on $\bar{D}$ that are holomorphic on D. (Throughout, smooth is synonymous with infinitely differentiable.) Recall that a compact subset K of ∂D is said to be an *interpolation set* for $A^\infty(D)$ if each smooth function on K can be extended to an element of $A^\infty(D)$. In this paper we are interested in conditions on a given smooth submanifold M of ∂D ensuring that each compact subset of M is an interpolation set. For strongly pseudoconvex domains this problem is well understood (see, e.g., [6] and [4]); in this case the natural condition that M be complex-tangential is known to be sufficient for interpolation. (Recall that M is said to be *complex-tangential* if for each $p \in M$ the tangent space $T(M,p)$ is contained in the maximal complex subspace of $T(\partial D, p)$.) In addition to this, in the general case a necessary condition for interpolation is that ∂D satisfy a finite-type condition along M (namely, that complex hypersurfaces have bounded order of contact with ∂D at points of M); for this, see the argument in [10, Ex. 4.1]. We work in $\mathbf{C}^2$ and, in view of the above conditions, assume that M is 1-dimensional. Here are our main results.

THEOREM. *Let D be a smoothly bounded pseudoconvex domain of finite type in $\mathbf{C}^2$, and let $M \subset \partial D$ be a smooth complex-tangential curve.*

- (1.2) *If ∂D is of constant type along M, then every compact subset of M is an interpolation set for $A^\infty(D)$.*
- (2.1) *If ∂D and M are real-analytic, then for each $p \in M$ there exists a neighborhood V of p so that every compact subset of $M \cap V$ is an interpolation set for $A^\infty(D \cap V)$.*

We remark that the formulation of finite type used here can be found in [5, Lecture 28, p. 121].

Our proof of the first result above depends on the following theorem. Recall that a subset K of ∂D is called a *peak set* for $A^\infty(D)$ if there exists

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$f \in A^\infty(D)$ so that $f=1$ on K while $|f|<1$ on $\bar{D}\setminus K$; if for each $p \in K$ there exists a neighborhood V of p so that $K \cap \bar{V}$ is a peak set for $A^\infty(D)$, then we say that K is *locally a peak set* for $A^\infty(D)$.

1.1. THEOREM. *Suppose that D and M are as in* (1.2) *above. Then M is locally a peak set for* $A^\infty(D)$.

We refer the reader to [10] for a brief discussion of earlier interpolation results and to [13] for a survey. Here we comment that most of these earlier results depend on the existence of smooth peak functions. However, Theorem 1.1 is false without the constant type assumption [5, Lecture 29, p. 123], and our proof of Theorem 2.1 necessarily proceeds along a different route (described at the beginning of §2).

1. Peak Functions for Curves of Constant Type

In this section we prove the following theorem.

1.1. THEOREM. *Let D be a smoothly bounded pseudoconvex domain of finite type in* $\mathbf{C}^2$, *and let* $M \subset \partial D$ *be a smooth complex-tangential curve along which* ∂D *is of constant type. Then M is locally a peak set for* $A^\infty(D)$.

Using this theorem and [10, Thm. 2.4], we obtain the following interpolation result.

1.2. COROLLARY. *If D and M are as in the theorem, then every compact subset of M is an interpolation set for* $A^\infty(D)$.

In fact, this corollary was proved in [10] under the additional assumption that M is locally a peak set; in light of Theorem 1.1, this assumption is redundant, and the corollary is proved. We remark that for the proof of both the theorem and its corollary the finite type assumption is required only along M.

With regard to the theorem, it does not follow from the hypotheses that a compact M is (globally) a peak set [9, Ex. 3.11]. A closely related result is proved in [12] under the assumption that ∂D and M are real-analytic.

In the proof of the theorem we make use of an approach due to Bloom [2]. First we use [11, Lemma 2.2] to prove a lemma, for which we need the following terminology. If ϕ is a smooth function defined near a set E in $\mathbf{C}^n$, we say that ϕ is *almost-holomorphic with respect to E* if $\bar{\partial}\phi$ vanishes to infinite order at points of E. We consider coordinates which are almost-holomorphic with respect to a set E, by which we mean that each component of the coordinate tranformation is almost-holomorphic with respect to E in terms of the original holomorphic coordinates.

1.3. LEMMA. *Suppose that D is a smoothly bounded domain in $\mathbf{C}^2$ and that $M \subset \partial D$ is a smooth complex-tangential curve. Then for each fixed $p \in M$ there exist a neighborhood W of p and C^∞ coordinates (z, w) on W in which p is the origin so that, with $z = x+iy$ and $w = u+iv$, we have:*

(a) *(z, w) is almost-holomorphic with respect to $\{(z, w) : y = u = 0\}$;*
(b) *$M = \{(z, w) \in W : y = w = 0\}$; and*
(c) *D has a defining function r of the form*

$$r(z, w) = u + A(z) + B(z)v + O(v^2), \tag{1.3.1}$$

where $A(x) \equiv B(x) \equiv 0$ and $\nabla A(x) \equiv \nabla B(x) \equiv 0$.

Proof. Near M we let $S \subset \partial D$ denote the union of the integral curves, through points of M, of the vector field defined by the outer unit normal to ∂D multiplied by i. Since M is complex-tangential, S is a smooth surface whose tangent space at each point contains no complex line. Now we can apply [11, Lemma 2.2] to get coordinates (z, w) in which (a) holds with $S = \{(z, w) : y = u = 0\}$, and so that $\partial/\partial u$ is an outward-pointing normal to ∂D along S. It is obvious from the proof in [11] that, since M is complex-tangential, we can choose (z, w) so that (b) holds. Clearly, D has a defining function r of the form (1.3.1); further, $A(x) \equiv 0$ since $M \subset \partial D$ and $B(x) \equiv 0$ because $S \subset \partial D$. Finally, since $T(\partial D, q) = \{u = 0\}$ if $q = (z, w) \in S$, an easy computation gives that $\nabla A(x) \equiv \nabla B(x) \equiv 0$. □

Proof of Theorem 1.1. Fix $p \in M$ and apply Lemma 1.3. If $2k$ denotes the type of ∂D along M, we can write

$$A(z) = a_m(x) y^m + o(y^m),$$

with $a_m \not\equiv 0$ near 0 and $m \le 2k$ (see Remark 1.4 below). We claim that near 0 we have that, for some $c > 0$,

$$A(z) \ge c y^{2k}. \tag{1.1.1}$$

To see this, first choose x_0 so that $a_m(x_0) \neq 0$. Note that the initial homogeneous term of A at x_0 is $a_m(x_0) y^m$; but $m \ge 2$ by Lemma 1.3(c), so the type of ∂D at $(x_0, 0)$ is m. Hence $m = 2k$. A similar argument involving the type now shows that a_{2k} never vanishes. Also, along M the pseudoconvexity of D implies that $a_{2k}(x) y^{2k}$ is subharmonic. Hence $a_{2k} > 0$, and (1.1.1) follows.

We claim that there exists $C > 0$ so that, near 0,

$$B^2(z) \le C A(z). \tag{1.1.2}$$

In view of (1.1.1), it suffices to prove that, if $B(z) = b_j(x) y^j + o(y^j)$ with $b_j \not\equiv 0$ near 0, then $j \ge k$. If $b_j(x_0) \neq 0$, the initial homogeneous term of B at x_0 is $b_j(x_0) y^j$. Now a Levi form calculation (given by Bloom in [2, Lemma 3.7]) implies that, if the initial homogeneous term in the expansion of B about a point is not harmonic, then this term is of order at least half the order of vanishing of A at that point. By Lemma 1.3(c), $j \geq 2$, and so $b_j(x_0)y^j$ is not harmonic; thus (1.1.1) and the aforementioned calculation imply that $j \geq k$, as desired.

If R is a large positive constant, define

$$f(z, w) = -w - Rw^2.$$

We claim that there exist $\lambda > 0$ and a neighborhood $W' \subset W$ of p so that

$$\operatorname{Re} f(q) \geq \lambda d^{2k}(q, M) \tag{1.1.3}$$

for all $q \in \bar{D} \cap W'$, where d denotes Euclidean distance. To see this, fix $q = (z, w) \in \bar{D}$ near p, and assume first that $u \leq 0$. Then

$$\begin{aligned} \operatorname{Re} f(q) &= -u - Ru^2 + Rv^2 \\ &\geq -\tfrac{1}{2}u + \tfrac{1}{2}[A(z) + B(z)v + O(v^2)] - Ru^2 + Rv^2 \qquad \text{(by (1.3.1))} \\ &= -\tfrac{1}{2}u + \tfrac{1}{2}[\tfrac{1}{2}A(z) + B(z)v + O(v^2) + Rv^2] + \tfrac{1}{4}A(z) - Ru^2 + \tfrac{1}{2}Rv^2. \end{aligned}$$

If R is large enough, the term in brackets is nonnegative because of (1.1.2). Then, using (1.1.1) and Lemma 1.3(b), we get the claim. If $u > 0$ then we first use the inequality $-u - Ru^2 \geq -2u$, valid if $Ru \leq 1$; next we apply (1.3.1) as above. The result is the estimate

$$\operatorname{Re} f(q) \geq \delta(y^{2k} + v^2), \tag{1.1.4}$$

valid for some $\delta > 0$. To conclude the proof of (1.1.3) for $u > 0$, observe that the right-hand side of (1.1.4) dominates $d^{2k}(q, M)$ because, in light of (1.3.1), it dominates u.

Choose s $(0 < s \ll 1)$ and let $I = \{(x, 0) : |x| \leq s\}$. We will show that I is a peak set for $A^\infty(D)$. Let $\psi(x)$ be the even function determined by $\psi(x) = \exp[-1/(x - s)]$ if $x > s$ and $\psi(x) = 0$ if $|x| \leq s$, and let $\phi(z)$ be a smooth extension of ψ which is almost-holomorphic with respect to M (use, e.g., [7, Lemma 1.6]). Straightforward estimates (given in the proof of Lemma 4.7 in [9]) show that for each $\epsilon > 0$ there exists a neighborhood of $\{x : |x| \leq s\}$ on which

$$\operatorname{Re} \phi(z) \geq -\epsilon y^{2k}. \tag{1.1.5}$$

Define $g(z, w) = f(z, w) + \phi(z)$. By (1.1.3) and (1.1.5) there exists a neighborhood of I in which $g^{-1}(0) = I$ and

$$\operatorname{Re} g(q) \geq \gamma d^{2k}(q, M)$$

for some $\gamma > 0$. Now e^{-g} has the properties of a local peak function except that it is not holomorphic; it is only almost-holomorphic with respect to M. The final step in the proof is then to solve a $\bar{\partial}$-problem as in [4, Prop. 10] to obtain a peak function for I. □

1.4. REMARK. In the preceding proof we have drawn conclusions from certain differential properties (such as pseudoconvexity) of the defining function

r that depend only on Taylor expansions of r to finite order; the conclusions clearly would be valid in holomorphic coordinates, but some justification is needed in the almost-holomorphic coordinates (z, w). In each case the justification required is accomplished easily: Simply note that the properties are invariant under holomorphic coordinate changes, and that the (z, w) coordinates differ from holomorphic ones by terms vanishing to arbitrarily high order along the set in question.

2. Local Interpolation from Curves

In this section we prove a local result on interpolation from curves. The techniques bear a resemblance to those used by Burns and Stout in [3]; the idea of the proof is as follows. First we extend a given function on the curve to a subset of the complexification of the curve; this is essentially a one-variable procedure. Then we extend from this subset to the domain by solving a $\bar{\partial}$-problem; the procedure here is an adaptation of the method given by Amar in [1, Thm. 2.1].

2.1. THEOREM. *Suppose $D \subset\subset \mathbf{C}^2$ is a pseudoconvex domain with real-analytic boundary, and that $M \subset \partial D$ is a complex-tangential real-analytic curve. Then for each $p \in M$ there exists a neighborhood V of p so that every compact subset of $M \cap V$ is an interpolation set for $A^\infty(D \cap V)$.*

Proof. Choose holomorphic coordinates (z, w) near p in which p is the origin and M is the $(\operatorname{Re} z)$-axis. We write $q = (z, w)$ and $z = x + iy$, $w = u + iv$. We may assume that, in (z, w) coordinates, D has a real-analytic defining function of the form

$$u + A(z) + O(|zw| + |w|^2). \tag{2.1.1}$$

First we study the intersection of $\{w = 0\}$ with $\bar{D}$. Note that $A(x) \equiv 0$ since $M \subset \partial D$, and that $A(z) \not\equiv 0$ since ∂D contains no disc. Near 0 we write

$$A(z) = \sum_{j=m}^{\infty} a_j(x) y^j,$$

with $a_m \not\equiv 0$. Note that $m \geq 2$ since M is complex-tangential. As in the proof of Theorem 1.1, a type argument and the pseudoconvexity of D yield that m is even and $a_m \geq 0$. Since a_m vanishes to finite order at 0, there exist an even natural number n and $\epsilon > 0$ so that

$$a_m(x) \geq \epsilon x^n. \tag{2.1.2}$$

Now, if δ satisfies $0 < \delta \ll \epsilon$, put

$$S = \{z : |y| > \delta x^n\}.$$

Then if $z \in \mathbf{C} \setminus S$ near 0 we have

$$\begin{aligned} A(z) &= a_m(x)y^m + O(y^{m+1}) \\ &\geq \epsilon x^n y^m + O(y^{m+1}) \qquad \text{(by (2.1.2))} \\ &\geq \epsilon\delta^{-1}|y|^{m+1} + O(y^{m+1}), \end{aligned}$$

so

$$A(z) \geq y^{m+2}. \tag{2.1.3}$$

In particular, note that if z is near 0 and $(z,0) \in \bar{D}$ then $z \in S \cup \mathbf{R}$.

By a result due to Amar [1, Prop. 1.1], there exist arbitrarily small neighborhoods of p, each of which intersects D in a smoothly bounded pseudoconvex domain. Choose such a neighborhood V in which the preceding estimates are valid, let $\Omega = D \cap V$, and fix a compact subset K of $M \cap V$. We will show that K is an interpolation set for $A^\infty(\Omega)$.

Given a function $f \in C^\infty(K)$, extend f to a smooth function on M with support in $M \cap V$. As in [10, Prop. 3.4], by Whitney's extension theorem and a one-variable interpolation result we can further extend f to a function $g \in C^\infty(V)$ so that g depends only on z, g is holomorphic along S, and $\bar{\partial} g$ vanishes to infinite order along M. Set $N = (S \cup \mathbf{R}) \times \mathbf{C}$, and define $H(q)$ to be $\bar{\partial} g(q)/w$ if $q \in \bar{\Omega} \setminus N$ and 0 otherwise.

We claim that H is smooth on $\bar{\Omega}$. First we note that, by our choice of g, for every j there exists $c_j > 0$ so that

$$|\bar{\partial} g(q)| \leq c_j d^j(q, N) \tag{2.1.4}$$

for all q in $\bar{\Omega}$; here d denotes Euclidean distance. Next we show that there exists $c > 0$ so that if $q \in \bar{\Omega}$ then

$$d^{m+2}(q, N) \leq c|w|. \tag{2.1.5}$$

To prove (2.1.5), observe that

$$d(q, N) \leq d(z, S \cup \mathbf{R}) \leq |y|; \tag{2.1.6}$$

also, if $q \in \bar{\Omega}$ then by (2.1.1) we have

$$O(w) + A(z) \leq 0$$

and thus, for some constant c,

$$A(z) \leq c|w|. \tag{2.1.7}$$

Now (2.1.5) follows from (2.1.3), (2.1.6), and (2.1.7). From (2.1.4) and (2.1.5) we have that $H(q) \to 0$ as $d(q, N) \to 0$, and the desired continuity of H follows. Since the estimate of (2.1.4) is valid for all derivatives of $\bar{\partial} g$, the same argument yields smoothness of H; we omit the details.

By the above claim and a fundamental theorem due to Kohn [8], there exists $\alpha \in C^\infty(\bar{\Omega})$ so that $\bar{\partial}\alpha = H$. Then the function $g - \alpha w$ belongs to $A^\infty(\Omega)$ and agrees with f on K. □

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