The Volume/Diameter Ratio for Positively Curved Manifolds

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1. Statement of Main Result

In this paper, a Riemannian manifold (M^n, g) always means a connected C^∞-manifold of dimension n ($n \geq 2$) with a Riemannian metric g. $\mathrm{Vol}(M)$, $i(M)$, $d(M)$, K_M, and Ric_M will denote the volume, the injectivity radius, the diameter, the sectional curvature, and the Ricci curvature of M, respectively.

Gromov [4] showed that if $0 > K_M \geq -1$ ($n \geq 8$) then

$$\mathrm{Vol}(M) \geq C_n(1+d(M)),$$

where the constant $C_n > 0$ depends only on n. Furthermore, the Bishop volume comparison [2] gives $\mathrm{Vol}(M) \leq C_n' d(M)$ when $\mathrm{Ric}_M \geq n-1$. This paper is concerned with a better estimate of this type for $K_M \geq 1$. Our main result can be stated as follows.

THEOREM A. *Let M be a complete Riemannian manifold with $K_M \geq 1$; then*

$$(*) \qquad \frac{\mathrm{Vol}(M)}{d(M)} \leq \frac{\mathrm{Vol}(S^n)}{d(S^n)}.$$

Moreover, the equality holds if and only if M is isometric to S^n or RP^n with constant curvature $+1$.

REMARKS.

1. For $d(M) \leq \pi/2$, $(*)$ is also true when only $\mathrm{Ric}_M \geq n-1$. This follows from the Bishop volume comparison theorem for Ricci curvature. The author does not know if the rigidity (in case equality holds) is true. However, if M is not simply connected, the rigidity is true (cf. §2).

2. For $d(M) > \pi/2$, Theorem A is wrong when only $\mathrm{Ric}_M \geq n-1$. This can be seen using, for example, $M^4 = CP^2$ with metric normalized so that $\mathrm{Ric}_M = 3$.

3. The interesting point is that for $K_M \geq 1$, the same conclusion holds if $d(M) > \pi/2$. For this, one can consider a pair of points at maximal distance which by Berger's lemma are mutually critical for the distance function. Then

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one can sharpen the classical argument (in the proof of the sphere theorem) that $M = B_{\pi/2}(p) \cup B_{\pi/2}(q)$ by combining it with the Grove-Petersen argument for pairs of mutually critical points (cf. §3).

Although the following theorem is well known, it is an immediate consequence of Theorem A and the diameter sphere theorem [6].

THEOREM B. *Let M be a complete Riemannian manifold with $K_M \geq 1$. If* $\mathrm{Vol}(M) \geq \mathrm{Vol}(S^n)/2$, *then*

(1) *M is homeomorphic to S^n, or*
(2) *M is isometric to RP^n with constant curvature* 1.

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2. The Case $d(M) \leq \pi/2$

From now on, let M be a complete Riemannian manifold with $K_M \geq 1$. In this section, we assume $d(M) = l \leq \pi/2$. According to the Bishop volume comparison theorem, one has

$$\mathrm{Vol}(M) \leq \mathrm{Vol}(S^{n-1}) \int_0^l (\sin r)^{n-1}\, dr.$$

Hence

$$\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S^n)} \leq \frac{\int_0^l (\sin r)^{n-1} dr}{\int_0^\pi (\sin r)^{n-1}\, dr} \leq \frac{l}{\pi} = \frac{d(M)}{d(S^n)},$$

that is,

$$\frac{\mathrm{Vol}(M)}{d(M)} \leq \frac{\mathrm{Vol}(S^n)}{d(S^n)}. \tag{1}$$

By the above argument, if the equality in (1) holds then $d(M) = i(M) = \pi/2$ and $K_M \equiv 1$. Let $\tilde{M}$ be the universal covering of M; then, by the Cartan-Ambrose-Hicks theorem, $\tilde{M}$ is isometric to S^n with constant curvature 1, and it is a double covering of M.

CLAIM. *M is isometric to RP^n with constant curvature* 1.

Proof. Let $\Psi: S^n \to M$ be the covering map. For any $m \in M$, $\Psi^{-1}(m) = \{m_1, m_2\}$; one need only show that $d(m_1, m_2) = \pi$, that is, m_2 is the antipodal point of m_1. Let $c(t)$ be a minimal geodesic from m_1 to m_2. Then $L[c] \leq \pi$ and $\Psi \circ c$ is a closed geodesic in M; hence $L[\Psi \circ c] \geq \pi$ for $i(M) = \pi/2$. Since $L[c] = L[\Psi \circ c]$, $L[c] = \pi$; that is, $d(m_1, m_2) = \pi$. This proves Theorem A for the case $d(M) \leq \pi/2$. □

3. The Case $d(M) > \pi/2$

Now we assume $d(M) = l > \pi/2$. The diameter sphere theorem implies that in this case M is homeomorphic to S^n. In particular, M is simply connected.

3.1. *Volume comparison related to nets in spheres* (see [5]). We will use the following notation. For $\Gamma \subset S^{n-1}$ and $0 \le \theta \le \pi$, set

$$\Gamma(\theta) = \{u \in S^{n-1} : \sphericalangle(u, \Gamma) < \theta\} \quad \text{and} \quad \Gamma^C(\theta) = \{u \in S^{n-1} : \sphericalangle(u, \Gamma) \ge \theta\}.$$

Let $V_\alpha = \{u, v\} \subset S^{n-1}$ with $\sphericalangle(u, v) = \pi - 2\alpha$, $0 \le \alpha \le \pi/2$. A subset Γ of S^{n-1} is said to be a *θ-net* provided $\Gamma(\theta) = S^{n-1}$. The following lemma is shown in the appendix of [5].

LEMMA 1. *Let Γ be a finite $(\pi/2+\alpha)$-net in S^{n-1}, $0 \le \alpha \le \pi/2$. Then*

$$\operatorname{Vol}(\Gamma(\theta)) \ge \operatorname{Vol}(V_\alpha(\theta)) \quad \textit{for } 0 \le \theta \le \pi/2.$$

LEMMA 2. *Let Γ be a $(\pi/2+\alpha)$-net in S^{n-1} for all $\alpha \in (0, \pi/2)$. Then*

$$\operatorname{Vol}(\Gamma(\theta)) \ge \operatorname{Vol}(V_0(\theta)) \quad \textit{for } \theta \ge 0.$$

Proof. For any $\alpha \in (0, \pi/2)$, since Γ is a $(\pi/2+\alpha)$-net in S^{n-1}, by compactness of S^{n-1} we can extract a finite subset Γ_α from Γ such that Γ_α is still a $(\pi/2+\alpha)$-net in S^{n-1}. For a fixed $\theta \ge 0$,

$$\operatorname{Vol}(\Gamma(\theta)) \ge \operatorname{Vol}(\Gamma_\alpha(\theta))$$

and, by Lemma 1,

$$\operatorname{Vol}(\Gamma_\alpha(\theta)) \ge \operatorname{Vol}(V_\alpha(\theta)).$$

Hence,

$$\operatorname{Vol}(\Gamma(\theta)) \ge \operatorname{Vol}(V_\alpha(\theta)).$$

Now let $\alpha \to 0$ to obtain $\operatorname{Vol}(\Gamma(\theta)) \ge \operatorname{Vol}(V_0(\theta))$. □

3.2. *The number l_α.* Let $\bar{p}$ be the north pole of S^2. Pick $\bar{q} \in S^2$ with $d(\bar{p}, \bar{q}) = l$ and $\bar{c}$ a minimal geodesic from $\bar{q}$ to $\bar{p}$. For any $\alpha \in [0, \pi/2]$, choose a geodesic σ_α starting from $\bar{q}$ with $\sphericalangle(\bar{c}'(0), \sigma'_\alpha(0)) = \alpha$. Let x_α be the first intersection point of σ_α with the equator.

Now define $l_\alpha = d(\bar{q}, x_\alpha)$ for $0 \le \alpha \le \pi/2$.

On S^n, choose $\bar{q} \in S^n$, $v_0 \in SS^n_{\bar{q}} \simeq S^{n-1}$, and $V_0 = \{v_0, -v_0\} \subset S^{n-1}$. Define $\alpha(w)$ and U by

$$\alpha(w) = \sphericalangle(w, V_0) \quad \text{for } w \in S^{n-1}$$

and

$$U = \{\exp_{\bar{q}}(rw) : w \in SS^n_{\bar{q}},\ 0 < r < l_{\alpha(w)}\}.$$

By the definition of l_α and U, one can see that

$$\operatorname{Vol}(U) \le \frac{l - \pi/2}{\pi} \operatorname{Vol}(S^n). \tag{2}$$

Now, let $\bar{A}_r = \{v \in SS^n_{\bar{q}} : l_{\alpha(v)} \ge r\}$; then

$$\operatorname{Vol}(U) = \int_0^{\pi/2} \int_{\bar{A}_r} (\sin r)^{n-1}\, dv\, dr. \tag{3}$$

3.3. *Estimating the volume of $B_p^C(\pi/2)$.* Choose $p, q \in M$ so that $d(p, q) = d(M)$. Let $B_p(\pi/2)$ be the closed geodesic ball of radius $\pi/2$ in M about p

and let $B_p^C(\pi/2)$ be the complement of $B_p(\pi/2)$ in M. In [3, p. 115], it is shown that $B_p^C(\pi/2)$ is convex. By the argument in Section 2, we have

$$\mathrm{Vol}\left(B_p\left(\frac{\pi}{2}\right)\right) \leq \frac{1}{2}\,\mathrm{Vol}(S^n). \tag{4}$$

In order to estimate the volume of $B_p^C(\pi/2)$, we need the following lemma of Berger [3, p. 106].

LEMMA 3. *Let M be a compact Riemannian manifold. Suppose $p, q \in M$ are such that $d(p,q) = d(M)$. Then, for any vector $v \in TM_q$, there exists a minimal geodesic c from q to p with $\measuredangle(c'(0), v) \leq \pi/2$.*

REMARK. The point q is called a *critical point of p* if there exists, for any vector $v \in TM_q$, a minimal geodesic c from q to p with $\measuredangle(c'(0), v) \leq \pi/2$.

Let Γ_{qp} denote the set of unit vectors in TM_q corresponding to the set of minimal geodesics from q to p. By Lemma 3, Γ_{qp} is a $(\pi/2+\alpha)$-net in $SM_q \simeq S^{n-1}$ for all $\alpha \in (0, \pi/2)$.

If $v \in SM_q$ with $\measuredangle(v, \Gamma_{qp}) = \alpha$, then $\alpha \leq \pi/2$. By Toponogov's comparison theorem, it is easy to see that $\exp_q(rv) \in B_p(\pi/2)$ if $l_\alpha \leq r \leq l$.

Define A_r by

$$A_r = \{v \in SM_q : \exp_q(rv) \in B_p^C(\pi/2) \text{ and } d(q, \exp_q(rv)) = r\}.$$

Note that:

(a) A_r is an open subset of $SM_q \simeq S^{n-1}$,
(b) $A_r = S^{n-1}$ if $0 \leq r \leq l - \pi/2$, and
(c) $A_r = \emptyset$ if $r \geq \pi/2$.

We have

$$\begin{aligned} \mathrm{Vol}\left(B_p^C\left(\frac{\pi}{2}\right)\right) &= \int_0^{\pi/2}\int_{A_r} r^{n-1}\|d(\exp_q)_{rv}\|\,dv\,dr \\ &\leq \int_0^{\pi/2}\int_{A_r} (\sin r)^{n-1}\,dv\,dr. \end{aligned} \tag{5}$$

For $l - \pi/2 \leq r \leq \pi/2$, let θ be such that $l_\theta = r$. Again by Toponogov's comparison theorem, one has $A_r \subset \Gamma_{qp}^C(\theta)$, and hence by Lemma 2

$$\mathrm{Vol}(A_r) \leq \mathrm{Vol}(\Gamma_{qp}^C(\theta)) \leq \mathrm{Vol}(V_0^C(\theta)) = \mathrm{Vol}(\bar{A}_r). \tag{6}$$

For $0 \leq r \leq l - \pi/2$, it is trivial that $\mathrm{Vol}(A_r) = \mathrm{Vol}(\bar{A}_r)$.

Now since $K_M \geq 1$, by (2), (3), (5), and (6) we obtain

$$\mathrm{Vol}\left(B_p^C\left(\frac{\pi}{2}\right)\right) \leq \frac{l - \pi/2}{\pi}\,\mathrm{Vol}(S^n). \tag{7}$$

Therefore

$$\mathrm{Vol}(M) = \mathrm{Vol}\left(B_p\left(\frac{\pi}{2}\right)\right) + \mathrm{Vol}\left(B_p^C\left(\frac{\pi}{2}\right)\right) \tag{8}$$

and

$$\text{(9)} \qquad \operatorname{Vol}(M) \le \frac{1}{2}\operatorname{Vol}(S^n) + \frac{l - \pi/2}{\pi}\operatorname{Vol}(S^n).$$

Hence,

$$\text{(10)} \qquad \operatorname{Vol}(M) \le \frac{l}{\pi}\operatorname{Vol}(S^n).$$

That is,

$$(*) \qquad \frac{\operatorname{Vol}(M)}{d(M)} \le \frac{\operatorname{Vol}(S^n)}{d(S^n)}.$$

Thus we have proved the first part of Theorem A for the case $d(M) > \pi/2$.

CLAIM. *The equality in* $(*)$ *holds if and only if* M *is isometric to* S^n *with constant curvature* 1.

Proof. Since $d(M) > \pi/2$, M is simply connected. In addition, by (4) and (8)-(10), we must have

$$\operatorname{Vol}\left(B_p\left(\frac{\pi}{2}\right)\right) = \frac{1}{2}\operatorname{Vol}(S^n)$$

and $K_M \equiv 1$ on $B_p(\pi/2)$. If we exchange the roles of p and q, then (similarly) $K_M \equiv 1$ on $B_q(\pi/2)$. By Lemma 3 and Toponogov's comparison theorem, $B_p(\pi/2) \cup B_q(\pi/2) = M$. Thus $K_M \equiv 1$ on M. Once again, by the Cartan-Ambrose-Hicks theorem, M is isometric to S^n with constant curvature 1, and this completes the proof of Theorem A. □

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