On the Integrability of Double Walsh Series with Special Coefficients

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1. Introduction

We study the double Walsh series

(1.1) $$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} w_j(x) w_k(y),$$

where $\{a_{jk}: j, k=0,1,\dots\}$ is a null sequence of real numbers; that is,

(1.2) $$a_{jk} \to 0 \quad \text{as } \max(j,k) \to \infty$$

and $\{w_j(x): j=0,1,\dots\}$ is the well-known Walsh orthonormal system defined on the interval $I=[0,1)$ and taken in the Paley enumeration (see, e.g., [2]). Thus, series (1.1) is considered on the unit square $I^2=[0,1)\times[0,1)$.

The pointwise convergence of series (1.1) is usually meant in Pringsheim's sense. (See, e.g., [5, vol. 2, ch. 17].) In other words, we form the rectangular partial sums

$$s_{mn}(x,y)=\sum_{j=0}^{m-1} \sum_{k=0}^{n-1} a_{jk} w_j(x) w_k(y) \quad (m,n\ge 1),$$

then let both m and n tend to ∞, independently of one another, and assign the limit $f(x,y)$ (if it exists) to the series (1.1) as its sum.

Throughout this paper, we shall use the notations

$$\Delta_{10} a_{jk} = a_{jk} - a_{j+1,k}, \qquad \Delta_{01} a_{jk} = a_{jk} - a_{j,k+1},$$

$$\Delta_{11} a_{jk} = a_{jk} - a_{j+1,k} - a_{j,k+1} + a_{j+1,k+1} \quad (j,k\ge 0).$$

We say that $\{a_{jk}\}$ is a monotone decreasing sequence if a_{jk} is monotone decreasing in both j (for each fixed $k\ge 0$) and k (for each fixed $j\ge 0$), or equivalently

(1.3) $$\Delta_{10} a_{jk} \ge 0 \quad \text{and} \quad \Delta_{01} a_{jk} \ge 0 \quad (j,k\ge 0)$$

We say that $\{a_{jk}\}$ is a sequence of bounded variation if

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(1.4)
$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{11}a_{jk}|<\infty.$$

Observe that condition (1.2) and

(1.5)
$$\Delta_{11}a_{jk}\geq 0 \quad (j,k\geq 0)$$

clearly imply the fulfillment of (1.3) and (1.4), and that $a_{jk}\geq 0$.

2. Results

Móricz [3] proved that if $\{a_{jk}\}$ is a null sequence of bounded variation, then series (1.1) converges to a finite limit $f(x,y)$ for all $0<x,y<1$, and that $|f|^p$ is Lebesgue integrable over I^2 for any $0<p<1$. In order to prove the integrability of $|f|$, we have to assume somewhat more than (1.4).

THEOREM 1. *If a double sequence $\{a_{jk}\}$ is such that condition* (1.2) *is satisfied and*

(2.1)
$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{11}a_{jk}|\ln(j+2)\ln(k+2)<\infty,$$

then

(i) *series* (1.1) *converges to a function $f(x,y)$ for all* $0<x,y<1$,
(ii) *f is Lebesgue integrable over I^2*,
(iii) $\int_0^1\int_0^1|s_{mn}(x,y)-f(x,y)|\,dx\,dy\to 0$ *as* $\min(m,n)\to\infty$,
(iv) *series* (1.1) *is the Walsh–Fourier series of f.*

It is not hard to check that if conditions (1.2) and (1.5) are satisfied, then condition (2.1) is equivalent to the condition

(2.2)
$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_{jk}}{(j+1)(k+1)}<\infty.$$

In fact, a double summation by parts gives

$$\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{a_{jk}}{(j+1)(k+1)}=\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}h_{j+1}h_{k+1}\Delta_{11}a_{jk}+\sum_{j=0}^{m-1}h_{j+1}h_n\Delta_{10}a_{jn}$$
$$+\sum_{k=0}^{n-1}h_m h_{k+1}\Delta_{01}a_{mk}+a_{mn}h_m h_n,$$

where

$$h_m=\sum_{j=1}^{m}\frac{1}{j}\quad (m\geq 1).$$

Since the exact order of magnitude of h_m is $\ln(m+1)$, the equivalence of (2.1) and (2.2) follows easily.

In this special case Theorem 1 can be reformulated as follows.

COROLLARY. *If conditions* (1.2), (1.5), *and* (2.2) *are satisfied, then the conclusions* (i)–(iv) *of Theorem 1 hold.*

We note that this Corollary is an extension of a theorem by Balašov [1] from one-dimensional to two-dimensional Walsh series.

If we drop condition (2.2), we cannot guarantee the Lebesgue integrability of the sum f of series (1.1). Nevertheless, conditions (1.2) and (1.5) themselves are sufficient to ensure the integrability of f in the sense of the improper Riemann integral and even more, as the following theorem shows.

THEOREM 2. *If conditions* (1.2) *and* (1.5) *are satisfied, then*

(i) *series* (1.1) *converges to a function* $f(x, y)$ *for all* $0<x, y<1$,
(ii) f *is integrable on* I^2 *in the sense of the improper Riemann integral,*
(iii) *series* (1.1) *is the Walsh–Fourier series of* f *in the same sense:*

$$a_{jk}=\lim_{\delta,\epsilon\downarrow 0}\int_{\delta}^{1}\int_{\epsilon}^{1} f(x, y)\, w_j(x)\, w_k(y)\, dx\, dy \quad (j, k\geq 0). \tag{2.3}$$

We note that Theorem 2 is an extension of a theorem by Rubinštein [4] from one-dimensional to two-dimensional Walsh series.

Next we study the asymptotic behavior of the measure of the level sets for f. By $|E|$ we denote the two-dimensional Lebesgue measure of a subset E of I^2.

THEOREM 3. *If conditions* (1.2) *and* (1.5) *are satisfied, then for the sum* f *of series* (1.1) *we have*

$$\mu_N=|\{(x, y)\in I^2\colon |f(x, y)|>N\}|\leq \frac{\epsilon_N}{N}\left(1+\ln^+\frac{N}{\epsilon_N}\right), \tag{2.4}$$

where $\epsilon_N\to 0$ *as* $N\to\infty$.

As usual,

$$\ln^+ t=\begin{cases}\ln t & \text{if } t\geq e,\\ 1 & \text{if } 0\leq t<e,\end{cases}$$

We guess that the right-hand side in (2.4) is exact.

CONJECTURE. Given any monotone decreasing null sequence

$$\{\epsilon_N\colon N=0, 1, \dots\},$$

there exists a double sequence $\{a_{jk}\}$ such that conditions (1.2) and (1.5) are satisfied and

$$\limsup_{N\to\infty}\frac{N\mu_N}{\epsilon_N(1+\ln^+(N/\epsilon_N))}>0.$$

In particular, $N\mu_N\not\to 0$ as $N\to\infty$.

If this conjecture turns out to be true, then the sum f of series (1.1) is not (A)-integrable on I^2 in general, under conditions (1.2) and (1.5). We note that the corresponding one-dimensional statement is true and was proved by Rubinštein [4]. (As to the notion of the (A)-integral, we also refer to [4].)

3. Proofs

Proof of Theorem 1. (i) This part was proved in [3]. For the sake of later references, we sketch the proof briefly.

By a double summation by parts,

$$\begin{aligned} s_{mn}(x,y) &= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} D_{j+1}(x)D_{k+1}(y)\Delta_{11}a_{jk} \\ &+ \sum_{j=0}^{m-1} D_{j+1}(x)D_n(y)\Delta_{10}a_{jn} \qquad (3.1) \\ &+ \sum_{k=0}^{n-1} D_m(x)D_{k+1}(y)\Delta_{01}a_{mk} + a_{mn}D_m(x)D_n(y), \end{aligned}$$

where

$$D_m(x) = \sum_{j=0}^{m-1} w_j(x) \quad (m \geq 1)$$

is the well-known Walsh–Dirichlet kernel. Since

$$|D_m(x)| < 2/x \quad (m \geq 1;\ 0 < x < 1), \tag{3.2}$$

it follows from (1.4) that the series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} D_{j+1}(x)D_{k+1}(y)\Delta_{11}a_{jk}$$

is absolutely convergent for all $0 < x, y < 1$. By (1.2),

$$\Delta_{10}a_{jn} = \sum_{k=n}^{\infty} \Delta_{11}a_{jk}, \tag{3.3}$$

whence

$$\sum_{j=0}^{m-1} |\Delta_{10}a_{jn}| \leq \sum_{j=0}^{m-1}\sum_{k=n}^{\infty} |\Delta_{11}a_{jk}|.$$

By (1.4) and (3.2), for all $0 < x, y < 1$ we have

$$\sum_{j=0}^{m-1} D_{j+1}(x)D_n(y)\Delta_{10}a_{jn} \to 0 \quad \text{as } n \to \infty$$

uniformly in m. Analogously, for all $0 < x, y < 1$,

$$\sum_{k=0}^{n-1} D_m(x)D_{k+1}(y)\Delta_{01}a_{mk} \to 0 \quad \text{as } m \to \infty$$

uniformly in n. Finally, by (1.2), for all $0 < x, y < 1$ we have

$$a_{mn}D_m(x)D_n(y) \to 0 \quad \text{as } \max(m,n) \to \infty.$$

To sum up, under conditions (1.2) and (1.4), series (1.1) converges for all $0 < x, y < 1$ to the function

$$f(x, y)=\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} D_{j+1}(x) D_{k+1}(y) \Delta_{11} a_{jk}. \tag{3.4}$$

(ii) We apply the well-known estimate

$$\int_0^1 |D_m(x)|\, dx \leq \ln(m+1) \quad (m \geq 1). \tag{3.5}$$

By (3.4) and (3.5),

$$\int_0^1 \int_0^1 |f(x, y)|\, dx\, dy \leq \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{11} a_{jk}| \ln(j+2) \ln(k+2),$$

and this is finite due to (2.1).

(iii) By (3.1) and (3.4),

$$\begin{aligned} f(x, y)-s_{mn}(x, y)= & \sum_{(j, k) \in R_{mn}}\sum D_{j+1}(x) D_{k+1}(y) \Delta_{11} a_{jk} \\ & -\sum_{j=0}^{m-1} D_{j+1}(x) D_n(y) \Delta_{10} a_{jn} \\ & -\sum_{k=0}^{n-1} D_m(x) D_{k+1}(y) \Delta_{01} a_{mk}-a_{mn} D_m(x) D_n(y), \end{aligned} \tag{3.6}$$

where $R_{mn}=\{(j, k): j$ and k are nonnegative integers such that either $j \geq m$ or $k \geq n$ or both$\}$.

By (2.1) and (3.5),

$$\begin{aligned} & \int_0^1 \int_0^1 \left| \sum_{(j, k) \in R_{mn}}\sum D_{j+1}(x) D_{k+1}(y) \Delta_{11} a_{jk}\right| dx\, dy \\ & \quad \leq \sum_{(j, k) \in R_{mn}}\sum |\Delta_{11} a_{jk}| \ln(j+2) \ln(k+2) \rightarrow 0 \quad \text{as } \min(m, n) \rightarrow \infty. \end{aligned} \tag{3.7}$$

By (2.1), (3.3), and (3.5),

$$\begin{aligned} & \int_0^1 \int_0^1 \left| \sum_{j=0}^{m-1} D_{j+1}(x) D_n(y) \Delta_{10} a_{jn}\right| dx\, dy \\ & \quad \leq \sum_{j=0}^{m-1} |\Delta_{10} a_{jn}| \ln(j+2) \ln(n+1) \\ & \quad \leq \sum_{j=0}^{m-1} \sum_{k=n}^{\infty} |\Delta_{11} a_{jk}| \ln(j+2) \ln(k+1) \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned} \tag{3.8}$$

uniformly in m. Analogously,

$$\int_0^1 \int_0^1 \left| \sum_{k=0}^{n-1} D_m(x) D_{k+1}(y) \Delta_{01} a_{mk}\right| dx\, dy \rightarrow 0 \quad \text{as } m \rightarrow \infty \tag{3.9}$$

uniformly in n. By (1.2),

$$a_{mn}=\sum_{j=m}^{\infty} \sum_{k=n}^{\infty} \Delta_{11} a_{jk}. \tag{3.10}$$

Thus, by (2.1) and (3.5),

$$\int_0^1\int_0^1 |a_{mn}D_m(x)D_n(y)|\,dx\,dy \tag{3.11}$$
$$\le \sum_{j=m}^{\infty}\sum_{k=n}^{\infty}|\Delta_{11}a_{jk}|\ln(j+1)\ln(k+1)\to 0 \quad \text{as } \max(m,n)\to\infty.$$

Combining (3.6)–(3.11) yields the statement in (iii).

(iv) It is enough to take into account that norm convergence (so-called strong convergence) implies weak convergence. □

Proof of Theorem 2. Conditions (1.2) and (1.5) imply that a_{jk} is monotone decreasing in both j and k. Consequently, series (1.1) converges uniformly on every rectangle $[\delta,1)\times[\epsilon,1)$ where $0<\delta,\epsilon<1$. In order to prove (2.3) we fix a pair (j,k) of nonnegative integers. By interchanging integration and summation, we have

$$\int_\delta^1\int_\epsilon^1 f(x,y)\,w_j(x)\,w_k(y)\,dx\,dy$$
$$= a_{jk} - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\iint_{Q_{\delta\epsilon}} w_j(x)\,w_m(x)\,w_k(y)\,w_n(y)\,dx\,dy,$$

where $Q_{\delta\epsilon}=I^2\sim[\delta,1)\times[\epsilon,1)$. Thus, it suffices to show

$$\lim_{\delta,\epsilon\downarrow 0}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\iint_{Q_{\delta\epsilon}} w_j(x)\,w_m(x)\,w_k(y)\,w_n(y)\,dx\,dy = 0. \tag{3.12}$$

The integral here breaks into three pieces, two thin rectangles along the axes and one small rectangle near the origin. According to this decomposition the sum in (3.12) can be written in the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\left\{\int_0^\delta\int_0^1+\int_0^1\int_0^\epsilon+\int_0^\delta\int_0^\epsilon\right\} w_j(x)\,w_m(x)\,w_k(y)\,w_n(y)\,dx\,dy \tag{3.13}$$
$$= S_1(\delta)+S_2(\epsilon)+S_3(\delta,\epsilon), \quad \text{say.}$$

Clearly,

$$S_1(\delta) = \sum_{m=0}^{\infty} a_{mk}\int_0^\delta w_j(x)\,w_m(x)\,dx \tag{3.14}$$
$$= \sum_{m=0}^{\infty} a_{mk}\int_0^\delta w_m(x)\,dx$$

provided $\delta<1/2j$, which we assume in the following. (We remind the reader that j is fixed and $\delta\downarrow 0$.) Similarly, we can write

$$S_2(\epsilon) = \sum_{n=0}^{\infty} a_{jn}\int_0^\epsilon w_n(y)\,dy \tag{3.15}$$

provided that $\epsilon<1/2k$. Finally,

(3.16) $$S_3(\delta,\epsilon)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}\int_0^{\delta}\int_0^{\epsilon}w_m(x)\,w_n(y)\,dx\,dy$$

whenever both $\delta<1/2j$ and $\epsilon<1/2k$.

Since $0<\delta,\epsilon<1$, we can choose two nonnegative integers r and s in such a way that

(3.17) $$2^{-r-1}\le\delta<2^{-r}\quad\text{and}\quad 2^{-s-1}\le\epsilon<2^{-s}.$$

To estimate $S_1(\delta)$, we begin with the observations that, for $p\ge1$ and $0\le q<2^{r+1}$,

(3.18) $$\int_0^{2^{-r-1}}w_{p2^{r+1}+q}(x)\,dx=0,$$

and for $t\ge1$, $0\le q<2^r$, and $2^{-r-1}\le x<2^{-r}$,

(3.19) $$w_{2t2^r+q}(x)=-w_{(2t+1)2^r+q}(x).$$

By (3.14) and (3.17)-(3.19),

$$\begin{aligned}S_1(\delta)&=\sum_{t=0}^{\infty}\sum_{q=0}^{2^r-1}\sum_{e=0}^{1}a_{(2t+e)2^r+q,k}\int_0^{\delta}w_{(2t+e)2^r+q}(x)\,dx\\&\le\sum_{q=0}^{2^{r+1}-1}a_{qk}\int_0^{\delta}w_q(x)\,dx\\&\quad+\sum_{t=1}^{\infty}\sum_{q=0}^{2^r-1}(a_{2t2^r+q,k}-a_{(2t+1)2^r+q,k})\int_{2^{-r-1}}^{\delta}w_{2t2^r+q}(x)\,dx,\end{aligned}$$

whence

$$|S_1(\delta)|\le2^{-r}\sum_{q=0}^{2^{r+1}-1}a_{qk}+2^{-r}\sum_{t=1}^{\infty}\sum_{q=0}^{2^r-1}(a_{2t2^r+q,k}-a_{(2t+1)2^r+q,k}).$$

Using the monotone property of the coefficients a_{qk}, we conclude that

$$|S_1(\delta)|\le2^{-r}\sum_{q=0}^{2^{r+1}+2^r-1}a_{qk}.$$

In particular,

(3.20) $$S_1(\delta)\to0\ \text{as}\ r\to\infty\quad\text{or equivalently}\ \delta\downarrow0.$$

A similar argument gives

(3.21) $$S_2(\epsilon)\to0\ \text{as}\ s\to\infty\quad\text{or equivalently}\ \epsilon\downarrow0.$$

Now we turn to the estimation of $S_3(\delta,\epsilon)$. By (3.16),

(3.22) $$\begin{aligned}S_3(\delta,\epsilon)=&\sum_{t_1=0}^{\infty}\sum_{q_1=0}^{2^r-1}\sum_{e_1=0}^{1}\sum_{t_2=0}^{\infty}\sum_{q_2=0}^{2^s-1}\sum_{e_2=0}^{1}a_{(2t_1+e_1)2^r+q_1,(2t_2+e_2)2^s+q_2}\\&\times\int_0^{\delta}\int_0^{\epsilon}w_{(2t_1+e_1)2^r+q_1}(x)\,w_{(2t_2+e_2)2^s+q_2}(y)\,dx\,dy.\end{aligned}$$

We split the sum in (3.22) into four pieces according to (i) $t_1=t_2=0$; (ii) $t_1=0$, $t_2\ge 1$; (iii) $t_1\ge 1$, $t_2=0$; and (iv) $t_1\ge 1$, $t_2\ge 1$; denoting the corresponding subsums by $S_3^i(\delta,\epsilon)$ for $i=1,2,3,4$. Making use of equalities (3.18) and (3.19) and their counterparts for $w(y)$, we obtain

$$|S_3^1(\delta,\epsilon)|\le 2^{-r-s}\sum_{q_1=0}^{2^{r+1}-1}\sum_{q_2=0}^{2^{s+1}-1}a_{q_1,q_2}, \tag{3.23}$$

$$\begin{aligned}|S_3^2(\delta,\epsilon)| &\le \sum_{q_1=0}^{2^{r+1}-1}\sum_{t_2=1}^{\infty}\sum_{q_2=0}^{2^s-1}(a_{q_1,2t_22^s+q_2}-a_{q_1,(2t_2+1)2^s+q_2}) \\ &\quad\times\left|\int_0^\delta\int_{2^{-s-1}}^{\epsilon} w_{q_1}(x)\,w_{2t_22^s+q_2}(y)\,dx\,dy\right| \\ &\le 2^{-r-s}\sum_{q_1=0}^{2^{r+1}-1}\sum_{q_2=2^{s+1}}^{2^{s+1}+2^s-1}a_{q_1,q_2}\end{aligned} \tag{3.24}$$

and similarly

$$|S_3^3(\delta,\epsilon)|\le 2^{-r-s}\sum_{q_1=2^{r+1}}^{2^{r+1}+2^r-1}\sum_{q_2=0}^{2^{s+1}-1}a_{q_1,q_2}. \tag{3.25}$$

Finally, we can write

$$\begin{aligned}S_3^4(\delta,\epsilon) &= \sum_{t_1=1}^{\infty}\sum_{q_1=0}^{2^r-1}\sum_{t_2=1}^{\infty}\sum_{q_2=0}^{2^s-1}\sum_{e_1=0}^{1}\sum_{e_2=0}^{1}(-1)^{e_1+e_2}a_{(2t_1+e_1)2^r+q_1,(2t_2+e_2)2^s+q_2} \\ &\quad\times\int_{2^{-r-1}}^{\delta}\int_{2^{-s-1}}^{\epsilon} w_{(2t_1+e_1)2^r+q_1}(x)\,w_{(2t_2+e_2)2^s+q_2}(y)\,dx\,dy\end{aligned}$$

whence, by (3.17),

$$\begin{aligned}|S_3^4(\delta,\epsilon)| \le 2^{-r-s}\sum_{t_1=1}^{\infty}\sum_{q_1=0}^{2^r-1}\sum_{t_2=1}^{\infty}\sum_{q_2=0}^{2^s-1}\big|&a_{2t_12^r+q_1,2t_22^s+q_2}-a_{(2t_1+1)2^r+q_1,2t_22^s+q_2} \\ &-a_{2t_12^r+q_1,(2t_2+1)2^s+q_2} \\ &+a_{(2t_1+1)2^r+q_1,(2t_2+1)2^s+q_2}\big|.\end{aligned}$$

Using properties (1.2) and (1.5) it follows that

$$|S_3^4(\delta,\epsilon)|\le 2^{-r-s}\sum_{q_1=2^{r+1}}^{2^{r+1}+2^r-1}\sum_{q_2=2^{s+1}}^{2^{s+1}+2^s-1}a_{q_1,q_2}. \tag{3.26}$$

Combining (3.22)–(3.26) gives

$$S_3(\delta,\epsilon)\to 0 \text{ as } r,s\to\infty \quad\text{or equivalently } \delta,\epsilon\downarrow 0. \tag{3.27}$$

Collecting (3.13), (3.20), (3.21), and (3.27) yields (3.12), and this proves (2.3).

In particular, putting $j=k=0$ in (2.3), we find that f is integrable on I^2 in the sense of the improper Riemann integral. The proof of Theorem 2 is complete. □

Proof of Theorem 3. Given $N\ge 1$, let $m=[\sqrt{N/4a_{00}}]$ where $[\cdot]$ means the integral part. Then

$$\left|\sum_{j=0}^{m-1}\sum_{k=0}^{m-1} a_{jk} w_j(x) w_k(y)\right| \le \sum_{j=0}^{m-1}\sum_{k=0}^{m-1} a_{jk} \le m^2 a_{00} \le \frac{N}{4}.$$

Thus, we can estimate μ_N by

$$\begin{aligned}\mu_N \le & \left|\left\{(x,y)\in I^2 : \left|\sum_{j=m}^{\infty}\sum_{k=0}^{m-1} a_{jk} w_j(x) w_k(y)\right| > \frac{N}{4}\right\}\right| \\ & + \left|\left\{(x,y)\in I^2 : \left|\sum_{j=0}^{m-1}\sum_{k=m}^{\infty} a_{jk} w_j(x) w_k(y)\right| > \frac{N}{4}\right\}\right| \\ & + \left|\left\{(x,y)\in I^2 : \left|\sum_{j=m}^{\infty}\sum_{k=m}^{\infty} a_{jk} w_j(x) w_k(y)\right| > \frac{N}{4}\right\}\right|.\end{aligned} \tag{3.28}$$

Applying a single summation by parts gives

$$\begin{aligned}&\sum_{k=0}^{m-1}\left(\sum_{j=m}^{\infty} a_{jk} w_j(x)\right) w_k(y) \\ &\quad = \sum_{k=0}^{m-1} D_{k+1}(y) \sum_{j=m}^{\infty} w_j(x)\Delta_{01} a_{jk} + D_m(y) \sum_{j=m}^{\infty} a_{jm} w_j(x).\end{aligned} \tag{3.29}$$

Performing further single summations by parts, while using (1.2), yields

$$\sum_{j=m}^{\infty} w_j(x)\Delta_{01} a_{jk} = -D_m(x)\Delta_{01} a_{mk} + \sum_{j=m}^{\infty} D_{j+1}(x)\Delta_{11} a_{jk}$$

and

$$\sum_{j=m}^{\infty} a_{jm} w_j(x) = -a_{mm} D_m(x) + \sum_{j=m}^{\infty} D_{j+1}(x)\Delta_{10} a_{jm}.$$

Hence, by (3.2) and (1.5),

$$\left|\sum_{j=m}^{\infty} w_j(x)\Delta_{01} a_{jk}\right| \le \frac{4}{x}\Delta_{01} a_{mk}$$

and

$$\left|\sum_{j=m}^{\infty} a_{jm} w_j(x)\right| \le \frac{4}{x} a_{mm} \quad (0<x<1).$$

Combining these with (3.29) and again using (3.2) gives

$$\begin{aligned}\left|\sum_{j=m}^{\infty}\sum_{k=0}^{m-1} a_{jk} w_j(x) w_k(y)\right| &\le \frac{8}{xy}\left(\sum_{k=0}^{m-1}\Delta_{01} a_{mk} + a_{mm}\right) \\ &= \frac{8}{xy} a_{m0} \quad (0<x, y<1).\end{aligned} \tag{3.30}$$

Analogously, one can also verify

$$\left|\sum_{j=0}^{m-1}\sum_{k=m}^{\infty} a_{jk} w_j(x) w_k(y)\right| \le \frac{8}{xy} a_{0m} \quad (0<x, y<1). \tag{3.31}$$

Applying a double summation by parts and using (1.2) yields

$$\sum_{j=m}^{\infty}\sum_{k=m}^{\infty} a_{jk}w_j(x)w_k(y) = \sum_{j=m}^{\infty}\sum_{k=m}^{\infty} D_{j+1}(x)D_{k+1}(y)\Delta_{11}a_{jk} - \sum_{j=m}^{\infty} D_{j+1}(x)D_m(y)\Delta_{10}a_{jm} - \sum_{k=m}^{\infty} D_m(x)D_{k+1}(y)\Delta_{01}a_{mk} + a_{mm}D_m(x)D_m(y).$$

Hence, by (3.2) and (1.5),

$$\left|\sum_{j=m}^{\infty}\sum_{k=m}^{\infty} a_{jk}w_j(x)w_k(y)\right| \leq \frac{4}{xy}\left(\sum_{j=m}^{\infty}\sum_{k=m}^{\infty}\Delta_{11}a_{jk} + \sum_{j=m}^{\infty}\Delta_{10}a_{jm} + \sum_{k=m}^{\infty}\Delta_{01}a_{mk} + a_{mn}\right) \tag{3.32}$$
$$= \frac{16}{xy}a_{mm} \quad (0 < x, y < 1).$$

Combining (3.28) and (3.30)–(3.32) results in the following:

$$\mu_N \leq \frac{32}{N}a_{m0}\left(1+\ln^+\frac{N}{32a_{m0}}\right) + \frac{32}{N}a_{0m}\left(1+\ln^+\frac{N}{32a_{0m}}\right) + \frac{64}{N}a_{mm}\left(1+\ln^+\frac{N}{64a_{mm}}\right). \tag{3.33}$$

We consider the auxiliary function

$$h(t) = t\left(1+\ln^+\frac{1}{t}\right).$$

It is easy to see that $h(t)$ is monotone increasing for $t \geq 0$. Moreover, (1.2) and (1.5) imply $a_{mm} \leq a_{m0}, a_{0m}$. Therefore, it follows from (3.33) that

$$\mu_N \leq \frac{\bar{\epsilon}_N}{N}\left(1+\ln^+\frac{4N}{\bar{\epsilon}_N}\right)$$

for $\bar{\epsilon}_N = 128\max(a_{m0}, a_{0m})$. Using once again the fact that $h(t)$ is monotone, we conclude that (2.4) holds for $\epsilon_N = (1+\ln 4)\bar{\epsilon}_N$. Since $\epsilon_N \to 0$ as $m \to \infty$ (equivalently, $N \to \infty$), the proof of Theorem 3 is complete. □

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